ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

ANALYTICAL AND NUMERICAL SOLUTIONS OF EXTENSIONS OF LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

ABSTRACT. In this paper, we consider extensions of linear Volterra integro-differential equations of the first and the second kinds and apply the Kamal transform to solve their analytical solutions on convolution type kernels. We also present numerical solutions of the extensions irrelevant to convolution type kernels using Touchard polynomials.

1. Introduction

In this research, we consider the following linear *Volterra integro-differential equations* (or only VIDEs) with initial conditions. The linear VIDEs of the first kind are given by

$$(1)\quad \int_0^x k_1(x,t)u^{(n)}(t)dt = f(x) + \int_0^x k_2(x,t)u(t)dt, u(0) = a_0, u^{'}(0) = a_1, \ldots, u^{(n-1)}(0) = a_{n-1}$$

when $k_1(x,t) \neq k_2(x,t)$ or $n \neq 0$ and the linear VIDEs of the second kind are expressed by

$$(2)\qquad u^{(n)}(x) = g(x) + \int_0^x k_3(x,t)u(t)dt, u(0) = b_0, u^{'}(0) = b_1, \ldots, u^{(n-1)}(0) = b_{n-1},$$

where $k_1(x,t), k_2(x,t)$ and $k_3(x,t)$ are called *kernels* of the linear VIDEs.

The *Kamal transform* of a function $F(x)$ is defined by

$$K\Big\{F(x)\Big\} = \int_0^\infty F(x)e^{-x/v}dx = G(v), x \geq 0,$$

where K is called the *Kamal transform operator*. $F(x)$ is said to be the *inverse Kamal transform* of $G(v)$, denoted by $F(x) = K^{-1}\Big\{G(v)\Big\}$, where K^{-1} is called the *inverse Kamal transform operator*. Let us recall some useful results in [3] that shall be used in the next hereinafter.

1. The Kamal transform of some functions:

$K\Big\{1\Big\} = v,$ $\quad K\Big\{x\Big\} = v^2,$ $\quad K\Big\{x^2\Big\} = 2!v^3,$

$K\Big\{x^n\Big\} = n!v^{n+1}, n \geq 0,$ $\quad K\Big\{e^{ax}\Big\} = \frac{v}{1-av},$ $\quad K\Big\{\sin ax\Big\} = \frac{av^2}{1+a^2v^2},$

$K\Big\{\cos ax\Big\} = \frac{v}{1+a^2v^2},$ $\quad K\Big\{\sinh ax\Big\} = \frac{av^2}{1-a^2v^2},$ $\quad K\Big\{\cosh ax\Big\} = \frac{v}{1-a^2v^2}.$

...

The author would like to thank the academic referees for the careful reading and helpful comments for improving this paper.

2020 *Mathematics Subject Classification.* 45D05, 65L05.

Key words and phrases. Volterra integro-differential equation, Kamal transform, Touchard polynomial.

2. If $K\Big\{F(x)\Big\} = G(v)$ then $K\Big\{F^{'}(x)\Big\} = \frac{1}{v}G(v) - F(0)$.

3. If $K\Big\{F(x)\Big\} = G(v)$ then $K\Big\{F^{''}(x)\Big\} = \frac{1}{v^2}G(v) - \frac{1}{v}F(0) - F^{'}(0)$.

4. If $K\Big\{F(x)\Big\} = G(v)$ then $K\Big\{F^{(n)}(x)\Big\} = \frac{1}{v^n}G(v) - \frac{1}{v^{n-1}}F(0) - \frac{1}{v^{n-2}}F^{'}(0) - \cdots - F^{(n-1)}(0)$.

5. The *convolution* of two functions $F(x)$ and $H(x)$, denoted by $F(x) * H(x)$, is defined by $F(x) * H(x) = \int_0^x F(t)H(x-t)dt = \int_0^x F(x-t)H(t)dt$. If $K\Big\{F(x)\Big\} = G(v)$ and $K\Big\{H(x)\Big\} = I(v)$ then $K\Big\{F(x) * H(x)\Big\} = K\Big\{F(x)\Big\}K\Big\{H(x)\Big\} = G(v)I(v)$.

6. The inverse Kamal transform of some functions:

$$K^{-1}\Big\{v\Big\} = 1, \qquad K^{-1}\Big\{v^2\Big\} = x, \qquad K^{-1}\Big\{v^3\Big\} = \frac{1}{2!}x^2,$$

$$K^{-1}\Big\{v^{n+1}\Big\} = \frac{1}{n!}x^n,\ n \geq 0, \qquad K^{-1}\Big\{\frac{v}{1-av}\Big\} = e^{ax}, \qquad K^{-1}\Big\{\frac{v^2}{1+a^2v^2}\Big\} = \frac{\sin ax}{a},$$

$$K^{-1}\Big\{\frac{v}{1+a^2v^2}\Big\} = \cos ax, \qquad K^{-1}\Big\{\frac{v^2}{1-a^2v^2}\Big\} = \frac{\sinh ax}{a}, \qquad K^{-1}\Big\{\frac{v}{1-a^2v^2}\Big\} = \cosh ax.$$

The *Touchard polynomial* is a polynomial function given by

$$T_\alpha(x) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!},$$

where α and k are called the *degree* and the *index* of the Touchard polynomial, respectively. Some important results on the Touchard polynomials that shall be referred in the next as the following:

1. $T_\alpha^{'}(x) = \frac{d}{dx}\Big[\sum_{k=0}^{\alpha}\binom{\alpha}{k}x^k\Big] = \sum_{k=1}^{\alpha}\binom{\alpha}{k}kx^{k-1}$,

2. $T_\alpha^{''}(x) = \frac{d^2}{dx^2}\Big[\sum_{k=0}^{\alpha}\binom{\alpha}{k}x^k\Big] = \sum_{k=2}^{\alpha}\binom{\alpha}{k}k(k-1)x^{k-2}$,

3. $T_\alpha^{'''}(x) = \frac{d^3}{dx^3}\Big[\sum_{k=0}^{\alpha}\binom{\alpha}{k}x^k\Big] = \sum_{k=3}^{\alpha}\binom{\alpha}{k}k(k-1)(k-2)x^{k-3}$,

4. $T_\alpha^{(n)}(x) = \frac{d^n}{dx^n}\Big[\sum_{k=0}^{\alpha}\binom{\alpha}{k}x^k\Big] = \sum_{k=n}^{\alpha}\binom{\alpha}{k}k(k-1)(k-2)\cdots(k-n+1)x^{k-n}, n \leq \alpha$.

Volterra integro-differential equations are typically mathematical models in many areas of science and engineering. Solutions of these equations play vital roles in a number of processes and phenomena such as nuclear reactors, circuit analyses, wave propagation, glass forming processes, nano-hydrodynamics, visco elasticity, biological populations, etc. Therefore, there are many researchers who have been interested in the VIDEs and founded numerous methods to solve the analytical and numerical solutions of VIDEs up to the present as follows. Estimated solutions of nonlinear VIDEs of fractional order were investigated applying the Laplace transform and Adomian polynomials by C. Yang and J. Hou in 2013, see [1]. Moreover, the Legendre polynomial approximation was used to find numerical solutions of nonlinear VIDEs of the second kind by M. Gachpazan, M. Erfanian and H. Beiglo in 2014, see [2]. In addition, analytical solutions of linear VIDEs of the second kind were solved using the Kamal transform by S. Aggarwal and A.R. Gupta in 2019, see [3]. The modified

Adomian decomposition method was utilized to explain exact solutions of linear VIDEs of the second kind by J.O. Okai, D.O. Ilejimi and M. Ibrahim in 2019, see [4]. Furthermore, approximate solutions of nonlinear VIDEs involving delay were found taking a new higher order method by A. Jhinga, J. Patade and V.D. Gejji in 2020, see [5]. Other than those findings, the Sadik transform was applied to figure out exact solutions of the first kind VIDEs on convolution type kernels by S. Aggarwal, A. Vyas and S.D. Sharma in 2020, see [6]. Numerical solutions of linear VIDEs were estimated using Laguerre and Touchard polynomials by J.T. Abdullah and H.S. Ali in 2020, see [7]. Sofar, some asymtotic behavior of exact solutions of the nonlinear VIDEs has been studied by M. Cakir, B. Gunes and H. Duru in 2021, see [8]. The quasilinearization technique to different scheme also has been applied to solve estimated solutions of VIDEs in [8]. Recently, the asymtotic behavior of the analytical solutions of the singularly perturbed nonlinear VIDEs has been established by F. Cakir, M. Cakir and H.G. Cakir in 2022, see [9]. The uniform difference scheme on a Bakhvalov-Shishkin mesh points according to the boundary layer conditions has been introduced to find numerical solutions of VIDEs as well in [9]. Exact solutions of the Faltung type VIDEs for the first kind have been solved applying Kushare transform by D.P. Patil, P.S. Nikam and P.D. Shinde in 2022, see [10].

In this study, we extend linear Volterra integro-differential equations of the first and the second kinds. Then, we use the Kamal transform to solve exact solutions of these extended equations on convolution type kernels. Besides, we apply Touchard polynomials to figure out numerical solutions of the extensions disconnected to convolution type kernels.

2. Analytical Solutions of Extended VIDEs for the First Kind

For this section, we investigate analytical solutions of an extension of linear VIDEs with initial conditions (1) on convolution type kernels given by

$$\begin{aligned} &\int_0^x k_1(x-t)u^{(n)}(t)dt = f(x) + \int_0^x k_2(x-t)u^{(m)}(t)dt, \\ &u(0)=a_0, u'(0)=a_1, \ldots, u^{(n-1)}(0)=a_{n-1} \end{aligned} \tag{3}$$

when $k_1(x-t) \neq k_2(x-t)$ or $n \neq m$. For the case $m=0$, we can see [6, 10] for more vital results. In this extension, we will apply the Kamal transform to figure out the problem as follows: Taking the Kamal transform to (3), we have

$$K\Big\{\int_0^x k_1(x-t)u^{(n)}(t)dt\Big\} = K\Big\{f(x)\Big\} + K\Big\{\int_0^x k_2(x-t)u^{(m)}(t)dt\Big\}. \tag{4}$$

Working a convolution of the Kamal tranform on (4), we then obtain

$$K\Big\{k_1(x)\Big\}K\Big\{u^{(n)}(x)\Big\} = K\Big\{f(x)\Big\} + K\Big\{k_2(x)\Big\}K\Big\{u^{(m)}(x)\Big\}. \tag{5}$$

Using the Kamal transform of derivatives on (5) with initial conditions, we also get

$$\begin{aligned} &K\Big\{k_1(x)\Big\}\Big[\frac{1}{v^n}K\Big\{u(x)\Big\} - \frac{a_0}{v^{n-1}} - \frac{a_1}{v^{n-2}} - \cdots - a_{n-1}\Big] \\ &= K\Big\{f(x)\Big\} + K\Big\{k_2(x)\Big\}\Big[\frac{1}{v^m}K\Big\{u(x)\Big\} - \frac{a_0}{v^{m-1}} - \frac{a_1}{v^{m-2}} - \cdots - a_{m-1}\Big] \end{aligned}$$

and we receive

$$\begin{aligned} &\Big[K\Big\{k_1(x)\Big\} - v^{n-m}K\Big\{k_2(x)\Big\}\Big]K\Big\{u(x)\Big\} = K\Big\{k_1(x)\Big\}\Big(a_0v + a_1v^2 + \cdots + a_{n-1}v^n\Big) \\ &+v^nK\Big\{f(x)\Big\} - K\Big\{k_2(x)\Big\}\Big(a_0v^{n-m+1} + a_1v^{n-m+2} + \cdots + a_{m-1}v^n\Big). \end{aligned} \tag{6}$$

Applying the inverse Kamal transform on (6), we have the solution of initial-value problem (3) as the following.

$$\begin{aligned} u(x) &= K^{-1}\left\{\frac{K\Big\{k_1(x)\Big\}}{K\Big\{k_1(x)\Big\} - v^{n-m}K\Big\{k_2(x)\Big\}}\Big(a_0v + a_1v^2 + \cdots + a_{n-1}v^n\Big)\right\} \\ &+ K^{-1}\left\{\frac{v^nK\Big\{f(x)\Big\}}{K\Big\{k_1(x)\Big\} - v^{n-m}K\Big\{k_2(x)\Big\}}\right\} \\ &- K^{-1}\left\{\frac{K\Big\{k_2(x)\Big\}}{K\Big\{k_1(x)\Big\} - v^{n-m}K\Big\{k_2(x)\Big\}}\Big(a_0v^{n-m+1} + a_1v^{n-m+2} + \cdots + a_{m-1}v^n\Big)\right\}. \end{aligned}$$

Example 2.1. *Solve the Volterra integro-differential problem:* $\int_0^x \Big[1 - (x-t) + \frac{1}{2}(x-t)^2\Big]u'''(t)dt = \frac{1}{2}x^2 + \int_0^x \Big[\frac{1}{2}(x-t)^2\Big]u''(t)dt, u(0) = -1, u'(0) = 2, u''(0) = 1.$

Solution. First, applying the Kamal transform to the problem, we have

$$K\Big\{\int_0^x \Big[1 - (x-t) + \frac{1}{2}(x-t)^2\Big]u'''(t)dt\Big\} = K\Big\{\frac{1}{2}x^2\Big\} + K\Big\{\int_0^x \Big[\frac{1}{2}(x-t)^2\Big]u''(t)dt\Big\}.$$

Then, using a convolution of the Kamal transform, we immediately get

$$K\Big\{1 - x + \frac{1}{2}x^2\Big\}K\Big\{u'''(x)\Big\} = K\Big\{\frac{1}{2}x^2\Big\} + K\Big\{\frac{1}{2}x^2\Big\}K\Big\{u''(x)\Big\}.$$

After that, taking the Kamal transform of derivatives, we obtain

$$(v - v^2 + v^3)\Big[\frac{1}{v^3}K\Big\{u(x)\Big\} - \frac{1}{v^2}u(0) - \frac{1}{v}u'(0) - u''(0)\Big] = v^3 + v^3\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{1}{v}u(0) - u'(0)\Big]$$

and using initial conditions, we also have

$$(v - v^2 + v^3)\Big[\frac{1}{v^3}K\Big\{u(x)\Big\} + \frac{1}{v^2} - \frac{2}{v} - 1\Big] = v^3 + v^3\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} + \frac{1}{v} - 2\Big].$$

Next, rearranging the equation, we certainly receive

$$(v - v^2 + v^3)K\Big\{u(x)\Big\} - v^4K\Big\{u(x)\Big\} = (v - v^2 + v^3)(-v + 2v^2 + v^3) + v^6 + v^5 - 2v^6$$

and we get $K\Big\{u(x)\Big\} = \frac{-v^2 + 3v^3 - 2v^4 + 2v^5}{v - v^2 + v^3 - v^4} = \frac{-v + 3v^2 - 2v^3 + 2v^4}{(1-v)(1+v^2)}.$

Finally, taking the inverse Kamal transform of the equation, we suddenly have an analytical solution

$$\begin{aligned} u(x) &= K^{-1}\Big\{\frac{-v+3v^2-2v^3+2v^4}{(1-v)(1+v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v^2-v^3}{(1-v)(1+v^2)}\Big\}+K^{-1}\Big\{\frac{v+v^3}{(1-v)(1+v^2)}\Big\}+K^{-1}\Big\{\frac{-2v+2v^2-2v^3+2v^4}{(1-v)(1+v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v^2}{1+v^2}\Big\}+K^{-1}\Big\{\frac{v}{1-v}\Big\}-2K^{-1}\Big\{v\Big\} = \sin x + e^x - 2. \end{aligned}$$

Example 2.2. *Solve the Volterra integro-differential problem:* $\int_0^x \left[\frac{1}{2}(x-t)^2\right]u'''(t)dt = \frac{1}{3!}x^3+\frac{2}{4!}x^4+\frac{1}{5!}x^5+\int_0^x\left[\frac{1}{4!}(x-t)^4\right]u'(t)dt, u(0)=2, u'(0)=0, u''(0)=-2.$

Solution. First, applying the Kamal transform to the problem, we have

$$K\Big\{\int_0^x\left[\frac{1}{2}(x-t)^2\right]u'''(t)dt\Big\} = K\Big\{\frac{1}{3!}x^3+\frac{2}{4!}x^4+\frac{1}{5!}x^5\Big\}+K\Big\{\int_0^x\left[\frac{1}{4!}(x-t)^4\right]u'(t)dt\Big\}.$$

Then, using a convolution of the Kamal transform, we immediately get

$$K\Big\{\frac{1}{2}x^2\Big\}K\Big\{u'''(x)\Big\} = K\Big\{\frac{1}{3!}x^3+\frac{2}{4!}x^4+\frac{1}{5!}x^5\Big\}+K\Big\{\frac{1}{4!}x^4\Big\}K\Big\{u'(x)\Big\}.$$

After that, taking the Kamal transform of derivatives, we obtain

$$v^3\left[\frac{1}{v^3}K\Big\{u(x)\Big\}-\frac{1}{v^2}u(0)-\frac{1}{v}u'(0)-u''(0)\right] = v^4+2v^5+v^6+v^5\left[\frac{1}{v}K\Big\{u(x)\Big\}-u(0)\right]$$

and using initial conditions, we also have

$$v^3\left[\frac{1}{v^3}K\Big\{u(x)\Big\}-\frac{2}{v^2}+2\right] = v^4+2v^5+v^6+v^5\left[\frac{1}{v}K\Big\{u(x)\Big\}-2\right].$$

Next, rearranging the equation, we certainly receive

$$K\Big\{u(x)\Big\}-v^4K\Big\{u(x)\Big\} = 2v-2v^3+v^4+2v^5+v^6-2v^5$$

and we get $K\Big\{u(x)\Big\} = \frac{2v-2v^3+v^4+v^6}{1-v^4} = \frac{2v-2v^3+v^4+v^6}{(1-v^2)(1+v^2)}$.

Finally, taking the inverse Kamal transform of the equation, we suddenly have an exact solution

$$\begin{aligned} u(x) &= K^{-1}\Big\{\frac{2v-2v^3+v^4+v^6}{(1-v^2)(1+v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v^2+v^4}{(1-v^2)(1+v^2)}\Big\}+K^{-1}\Big\{\frac{2v-2v^3}{(1-v^2)(1+v^2)}\Big\}-K^{-1}\Big\{\frac{v^2-v^6}{(1-v^2)(1+v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v^2}{1-v^2}\Big\}+2K^{-1}\Big\{\frac{v}{1+v^2}\Big\}-K^{-1}\Big\{v^2\Big\} = \sinh x + 2\cos x - x. \end{aligned}$$

3. Analytical Solutions of Extended VIDEs for the Second Kind

In this section, we study about analytical solutions of an extension of linear VIDEs with initial conditions (2) on convolution type kernels expressed by

$$u^{(n)}(x) = g(x) + \int_0^x k_3(x-t)u^{(m)}(t)dt, u(0) = b_0, u'(0) = b_1, ..., u^{(n-1)}(0) = b_{n-1}. \tag{7}$$

For the case $m = 0$, we can see [3, 4] for more comprehensive findings. In this extension, we will utilize the Kamal transform to solve the problem as the following: Applying the Kamal transform to (7), we get

$$K\Big\{u^{(n)}(x)\Big\} = K\Big\{g(x)\Big\} + K\Big\{\int_0^x k_3(x-t)u^{(m)}(t)dt\Big\}. \tag{8}$$

Using a convolution of the Kamal transform to (8), we then obtain

$$K\Big\{u^{(n)}(x)\Big\} = K\Big\{g(x)\Big\} + K\Big\{k_3(x)\Big\}K\Big\{u^{(m)}(x)\Big\}. \tag{9}$$

Taking the Kamal transform of derivatives on (9) with initial conditions, we also have

$$\begin{aligned} &\frac{1}{v^n}K\Big\{u(x)\Big\} - \frac{b_0}{v^{n-1}} - \frac{b_1}{v^{n-2}} - \cdots - b_{n-1} \\ &= K\Big\{g(x)\Big\} + K\Big\{k_3(x)\Big\}\Big[\frac{1}{v^m}K\Big\{u(x)\Big\} - \frac{b_0}{v^{m-1}} - \frac{b_1}{v^{m-2}} - \cdots - b_{m-1}\Big] \end{aligned}$$

and we obtain

$$\begin{aligned} &\Big[1 - v^{n-m}K\Big\{k_3(x)\Big\}\Big]K\Big\{u(x)\Big\} = b_0v + b_1v^2 + \cdots + b_{n-1}v^n \\ &+v^nK\Big\{g(x)\Big\} - K\Big\{k_3(x)\Big\}\Big(b_0v^{n-m+1} + b_1v^{n-m+2} + \cdots + b_{m-1}v^n\Big). \end{aligned} \tag{10}$$

Operating the inverse Kamal transform on (10), we receive the solution of initial-value problem (7) as follows.

$$\begin{aligned} u(x) &= K^{-1}\left\{\frac{1}{1 - v^{n-m}K\Big\{k_3(x)\Big\}}\Big(b_0v + b_1v^2 + \cdots + b_{n-1}v^n\Big)\right\} \\ &+ K^{-1}\left\{\frac{v^nK\Big\{g(x)\Big\}}{1 - v^{n-m}K\Big\{k_3(x)\Big\}}\right\} \\ &- K^{-1}\left\{\frac{K\Big\{k_3(x)\Big\}}{1 - v^{n-m}K\Big\{k_3(x)\Big\}}\Big(b_0v^{n-m+1} + b_1v^{n-m+2} + \cdots + b_{m-1}v^n\Big)\right\}. \end{aligned}$$

Example 3.1. *Solve the Volterra integro-differential problem:* $u^{(4)}(x) = -32 + \int_0^x 16(x-t)u''(t)dt, u(0) = -2, u'(0) = 0, u''(0) = 16, u'''(0) = 0.$

Solution. First, applying the Kamal transform to the problem, we have

$$K\Big\{u^{(4)}(x)\Big\} = -K\Big\{32\Big\} + K\Big\{\int_0^x 16(x-t)u''(t)dt\Big\}.$$

Then, using a convolution of the Kamal transform, we get

$$K\Big\{u^{(4)}(x)\Big\} = -K\Big\{32\Big\} + K\Big\{16x\Big\}K\Big\{u''(x)\Big\}.$$

After that, taking the Kamal transform of derivatives, we also have

$$\frac{1}{v^4}K\Big\{u(x)\Big\} - \frac{1}{v^3}u(0) - \frac{1}{v^2}u'(0) - \frac{1}{v}u''(0) - u'''(0) = -32v + 16v^2\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{1}{v}u(0) - u'(0)\Big]$$

and using initial conditions, we obtain

$$\frac{1}{v^4}K\Big\{u(x)\Big\} + \frac{2}{v^3} - \frac{16}{v} = -32v + 16v^2\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} + \frac{2}{v}\Big].$$

Next, rearranging the equation, we certainly receive

$$K\Big\{u(x)\Big\} - 16v^4K\Big\{u(x)\Big\} = -2v + 16v^3 - 32v^5 + 32v^5$$

and we get $K\Big\{u(x)\Big\} = \dfrac{-2v+16v^3}{1-16v^4} = \dfrac{-2v+16v^3}{(1-4v^2)(1+4v^2)}$.

Finally, taking the inverse Kamal transform of the equation, we suddenly have an analytical solution

$$\begin{aligned} u(x) &= K^{-1}\Big\{\frac{-2v+16v^3}{(1-4v^2)(1+4v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v+4v^3}{(1-4v^2)(1+4v^2)}\Big\} - 3K^{-1}\Big\{\frac{v-4v^3}{(1-4v^2)(1+4v^2)}\Big\} \\ &= K^{-1}\Big\{\frac{v}{1-4v^2}\Big\} - 3K^{-1}\Big\{\frac{v}{1+4v^2}\Big\} = \cosh 2x - 3\cos 2x. \end{aligned}$$

Example 3.2. *Solve the Volterra integro-differential problem:* $u''(x) = 9 + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \int_0^x 3u''(t)dt, u(0) = 2, u'(0) = 3.$

Solution. First, applying the Kamal transform to the problem, we have

$$K\Big\{u''(x)\Big\} = K\Big\{9 + \frac{1}{2}x^2 - \frac{1}{2}x^3\Big\} + K\Big\{\int_0^x 3u''(t)dt\Big\}.$$

Then, using a convolution of the Kamal transform, we get

$$K\Big\{u''(x)\Big\} = K\Big\{9 + \frac{1}{2}x^2 - \frac{1}{2}x^3\Big\} + K\Big\{3\Big\}K\Big\{u''(x)\Big\}.$$

After that, taking the Kamal transform of derivatives, we also have

$$\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{1}{v}u(0) - u'(0) = 9v + v^3 - 3v^4 + 3v\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{1}{v}u(0) - u'(0)\Big]$$

and using initial conditions, we obtain

$$\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{2}{v} - 3 = 9v + v^3 - 3v^4 + 3v\Big[\frac{1}{v^2}K\Big\{u(x)\Big\} - \frac{2}{v} - 3\Big].$$

Next, rearranging the equation, we certainly receive

$$K\Big\{u(x)\Big\} - 3vK\Big\{u(x)\Big\} = 2v + 3v^2 + 9v^3 + v^5 - 3v^6 - 6v^2 - 9v^3$$

and we get $K\Big\{u(x)\Big\} = \dfrac{2v - 3v^2 + v^5 - 3v^6}{1-3v}$.

Finally, taking the inverse Kamal transform of the equation, we suddenly have an exact solution

$$\begin{aligned} u(x) &= K^{-1}\Big\{\frac{2v - 3v^2 + v^5 - 3v^6}{1-3v}\Big\} \\ &= K^{-1}\Big\{\frac{v}{1-3v}\Big\} + K^{-1}\Big\{\frac{v^5 - 3v^6}{1-3v}\Big\} + K^{-1}\Big\{\frac{v - 3v^2}{1-3v}\Big\} \\ &= K^{-1}\Big\{\frac{v}{1-3v}\Big\} + K^{-1}\Big\{v^5\Big\} + K^{-1}\Big\{v\Big\} = e^{3x} + \frac{1}{4!}x^4 + 1. \end{aligned}$$

4. Numerical Solutions of Extended VIDEs for the First Kind

For this section, we apply Touchard polynomials to approximate solutions of an extension of the linear Volterra integro-differential problem (1) on kernels, which are not convolution types, given by

$$\begin{aligned} &\int_0^x k_1(x,t)u^{(n)}(t)dt = f(x) + \int_0^x k_2(x,t)u^{(m)}(t)dt, \\ &u(0) = a_0, u'(0) = a_1, ..., u^{(n-1)}(0) = a_{n-1} \end{aligned} \tag{11}$$

when $k_1(x,t) \neq k_2(x,t)$ or $n \neq m$. The approximation using the Touchard polynomials is below: Suppose that the function $u_\alpha(x)$ is an approximate solution of (11) defined by

$$u_\alpha(x) = \sum_{k=0}^{\alpha} c_k T_k(x) = c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots + c_\alpha T_\alpha(x), m, n \leq \alpha, 0 \leq x \leq \beta, \tag{12}$$

where $T_k(x)$ are Touchard polynomials and c_k are unknown constants, $k = 0, 1, ..., \alpha$, and β is a known constant. Writing equation (12) as a dot product, we then have

$$u_\alpha(x) = \begin{bmatrix} T_0(x) & T_1(x) & T_2(x) & \ldots & T_\alpha(x) \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix}. \tag{13}$$

Rearranging the equation (13) in a matrix formula, we also have

$$u_\alpha(x) = \begin{bmatrix} 1 & x & x^2 & x^3 & \ldots & x^\alpha \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix},$$

where b_{ij} are known constants. Finding the derivatives of $u_\alpha(x)$, we have as follows:

$$u_\alpha'(x) = \begin{bmatrix} 0 & 1! & 2x & 3x^2 & \dots & \alpha x^{\alpha-1} \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix},$$

$$u_\alpha''(x) = \begin{bmatrix} 0 & 0 & 2! & 6x & \dots & \alpha(\alpha-1)x^{\alpha-2} \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix},$$

$$u_\alpha'''(x) = \begin{bmatrix} 0 & 0 & 0 & 3! & \dots & \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix},$$

$$\begin{aligned} u_\alpha^{(n)}(x) &= \begin{bmatrix} 0 & 0 & 0 & \dots & n! & \dots & \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n} \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{bmatrix}. \end{aligned} \tag{14}$$

Substituting the equation (14) into the equation (11), we receive

$$
\begin{aligned}
&\int_0^x k_1(x,t)\Big\{\Big[0\ \ 0\ \ 0\ \ \ldots\ \ n!\ \ \ldots\ \ \alpha(\alpha-1)\ldots(\alpha-n+1)t^{\alpha-n}\Big]\cdot\\
&\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha}\\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha}\\ 0 & 0 & b_{22} & \cdots & b_{2\alpha}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha}\end{bmatrix}\cdot\begin{bmatrix} c_0\\ c_1\\ c_2\\ \vdots\\ c_\alpha\end{bmatrix}\Big\}dt\\
&=f(x)+\int_0^x k_2(x,t)\Big\{\Big[0\ \ 0\ \ 0\ \ \ldots\ \ m!\ \ \ldots\ \ \alpha(\alpha-1)\ldots(\alpha-m+1)t^{\alpha-m}\Big]\cdot\\
&\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha}\\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha}\\ 0 & 0 & b_{22} & \cdots & b_{2\alpha}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha}\end{bmatrix}\cdot\begin{bmatrix} c_0\\ c_1\\ c_2\\ \vdots\\ c_\alpha\end{bmatrix}\Big\}dt.
\end{aligned}
\tag{15}
$$

Simplifying and integrating the equation (15), we then have the new equation with unknown constants $c_0,c_1,\ldots,c_\alpha$. In order to determine $c_0,c_1,\ldots,c_\alpha$, using n initial conditions and selecting $x_i\in[0,\beta], i=1,2,\ldots,\alpha-n+1$, with substituting in the new equation, we get a system of linear algrbraic equations of $\alpha+1$ unknown constants. Solving this system by a program, we have the values of the unknown constants, that is, the numerical solution of the initial-value problem (11) is obtained.

In order to guarantee the convergence of this method, we will verify as follows. Let $u(x)$ be an analytical solution of initial-value problem (11) that has derivatives of all orders at $x=0$. Then the Taylor series of $u(x)$ at $x=0$ is defined by

$$u(x)=u(0)+u^{'}(0)x+\frac{1}{2!}u^{''}(0)x^2+\cdots+\frac{1}{\alpha!}u^{(\alpha)}(0)x^\alpha+\cdots.$$

Thus, by the definition and process to find $u_\alpha(x)$, we obtain that

$$|u(x)-u_\alpha(x)|\leq\Big|\frac{1}{(\alpha+1)!}u^{(\alpha+1)}(0)x^{\alpha+1}\Big|+\Big|\frac{1}{(\alpha+2)!}u^{(\alpha+2)}(0)x^{\alpha+2}\Big|+\cdots.$$

Here, it is sufficient to show that $\frac{1}{\alpha!}x^\alpha$ converges to 0 as $\alpha\to\infty$ to confirm that $|u(x)-u_\alpha(x)|$ converges to 0 as $\alpha\to\infty$. Since $e\big(\frac{\alpha}{e}\big)^\alpha\leq\alpha!\leq e\big(\frac{\alpha+1}{e}\big)^{\alpha+1}$, we get $\frac{1}{\alpha+1}\big(\frac{ex}{\alpha+1}\big)^\alpha\leq\frac{1}{\alpha!}x^\alpha\leq\frac{1}{e}\big(\frac{ex}{\alpha}\big)^\alpha$. It is easy to determine that $\frac{1}{\alpha+1}\big(\frac{ex}{\alpha+1}\big)^\alpha$ and $\frac{1}{e}\big(\frac{ex}{\alpha}\big)^\alpha$ converge to 0 as as $\alpha\to\infty$. This means that $\frac{1}{\alpha!}x^\alpha$ converges to 0 as $\alpha\to\infty$ to confirm the convergence.

Example 4.1. *Approximate the solution of the linear Volterra integro-differential problem using $u_4(x)$:*
$\int_0^x(x-t)u^{''}(t)dt=-x^2e^{2x}+\frac{1}{2}xe^{2x}+e^{2x}+\frac{1}{2}x^3-\frac{5}{2}x-1+\int_0^x xtu^{'}(t)dt, u(0)=1, u^{'}(0)=1, 0\leq x\leq 1.$
An exact solution is $u(x)=e^{2x}-x$.

Solution. First, suppose that a function $u_4(x)$ is an estimated solution of this problem, that is,

$$\begin{aligned} u_4(x) &= c_0T_0(x)+c_1T_1(x)+c_2T_2(x)+c_3T_3(x)+c_4T_4(x) \\ &= c_0+c_1(1+x)+c_2(1+2x+x^2)+c_3(1+3x+3x^2+x^3) \\ &+ c_4(1+4x+6x^2+4x^3+x^4). \end{aligned}$$

Next, finding derivatives of $u_4(x)$, we have as follows:

$$u_4'(x) = c_1+c_2(2+2x)+c_3(3+6x+3x^2)+c_4(4+12x+12x^2+4x^3),$$

$$u_4''(x) = c_2(2)+c_3(6+6x)+c_4(12+24x+12x^2).$$

After that, substituting the derivatives into the problem, we obtain

$$\begin{aligned} &\int_0^x (x-t)\Big[c_2(2)+c_3(6+6t)+c_4(12+24t+12t^2)\Big]dt \\ &= -x^2e^{2x}+\frac{1}{2}xe^{2x}+e^{2x}+\frac{1}{2}x^3-\frac{5}{2}x-1 \\ &+\int_0^x xt\Big[c_1+c_2(2+2t)+c_3(3+6t+3t^2)+c_4(4+12t+12t^2+4t^3)\Big]dt. \end{aligned}$$

Then, simplifying and integrating the equation, we receive the new equation. Selecting $x_1=0.25, x_2=0.5, x_3=1$ to substitute in the new equation with using 2 initial conditions, we get the following system:

$$\begin{aligned} &c_0+c_1+c_2+c_3+c_4=1, \\ &c_1+2c_2+3c_3+4c_4=1, \\ &-7.8125c_1+44.270833c_2+171.142578c_3+391.40625c_4=134.578851, \\ &-6.25c_1+8.333333c_2+53.90625c_3+145.625c_4=53.078182, \\ &-30c_1-40c_2-15c_3+72c_4=30e^2-180. \end{aligned}$$

Finally, solving the system by a program, we have

$$c_0=2.406612, c_1=-5.499561, c_2=7.110917, c_3=-4.349601, c_4=1.331632.$$

Therefore, the numerical solution is

$$\begin{aligned} u_4(x) &= 2.406612-5.499561(1+x)+7.110917(1+2x+x^2) \\ &- 4.349601(1+3x+3x^2+x^3)+1.331632(1+4x+6x^2+4x^3+x^4). \end{aligned}$$

TABLE 1. Values of exact and approximate $u_4(x)$ solutions for the example 4.1.

x	Exact solution	Approximate solution	Absolute error
0.0	1.000000	0.999999	0.000001
0.1	1.121403	1.121628	0.000225
0.2	1.291825	1.292021	0.000196
0.3	1.522119	1.521833	0.000286
0.4	1.825541	1.824916	0.000625
0.5	2.218282	2.218317	0.000035
0.6	2.720117	2.722280	0.002163
0.7	3.355200	3.360242	0.005042
0.8	4.153032	4.158840	0.005808
0.9	5.149647	5.147905	0.001742
1.0	6.389056	6.360462	0.028594

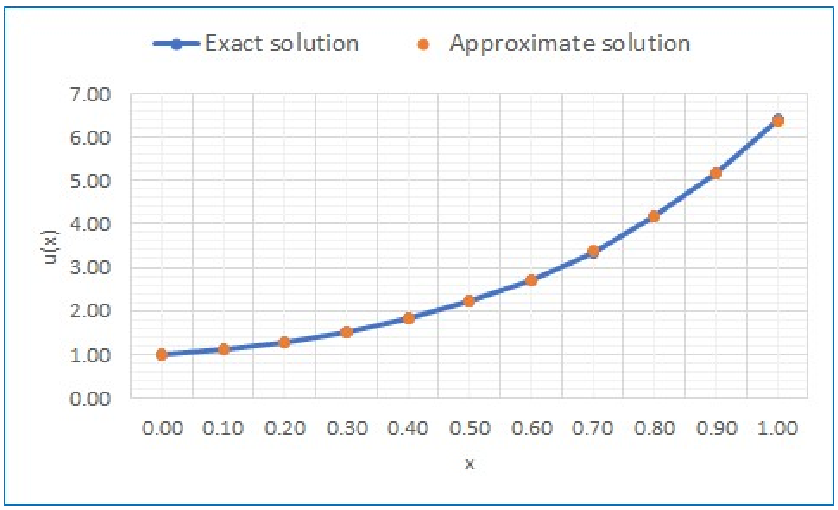

FIGURE 1. Graphs of exact and approximate $u_4(x)$ solutions for the example 4.1.

5. Numerical Solutions of Extended VIDEs for the Second Kind

In this section, we utilize Touchard polynomials to estimate solutions of an extension of the linear Volterra integro-differential problem (2) on kernels, which are not convolution types, expressed by

$$u^{(n)}(x) = g(x) + \int_0^x k_3(x,t)u^{(m)}(t)dt, u(0) = b_0, u^{'}(0) = b_1, ..., u^{(n-1)}(0) = b_{n-1}. \tag{16}$$

For the case $m = 0, n = 1$, we can see [7] for more important results. The estimation taking by the Touchard polynomials is the same as the first kind of VIDEs, that is, substituting an equation (14) into

(16), we receive

$$\begin{bmatrix}0 & 0 & 0 & \dots & n! & \dots & \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}\end{bmatrix}\cdot\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha}\end{bmatrix}\cdot\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha\end{bmatrix}$$

$$= g(x) + \int_0^x k_3(x,t)\Big\{\begin{bmatrix}0 & 0 & 0 & \dots & m! & \dots & \alpha(\alpha-1)\dots(\alpha-m+1)t^{\alpha-m}\end{bmatrix}\cdot$$

$$\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\alpha} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\alpha} \\ 0 & 0 & b_{22} & \cdots & b_{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\alpha\alpha}\end{bmatrix}\cdot\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha\end{bmatrix}\Big\}dt. \tag{17}$$

Simplifying and integrating the equation (17), we then have the new equation with unknown constants $c_0, c_1, \dots, c_\alpha$. Determining $c_0, c_1, \dots, c_\alpha$ and replacing into the equation (12), the numerical solution of the initial-value problem (16) is obtained.

Example 5.1. *Estimate the solution of the linear Volterra integro-differential problem using $u_5(x)$ and $u_7(x)$: $u^{(4)}(x) = \sin x + e^x(\sin x - x\cos x + x^2) + \int_0^x te^x u''(t)dt, u(0) = 0, u'(0) = 1, u''(0) = -2, u'''(0) = -1, 0 \le x \le \pi$. An exact solution is $u(x) = \sin x - x^2$.*

Solution. First, suppose that a function $u_5(x)$ is an approximate solution of this problem, that is,

$$\begin{aligned} u_5(x) &= c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + c_3T_3(x) + c_4T_4(x) + c_5T_5(x) \\ &= c_0 + c_1(1+x) + c_2(1+2x+x^2) + c_3(1+3x+3x^2+x^3) \\ &+ c_4(1+4x+6x^2+4x^3+x^4) + c_5(1+5x+10x^2+10x^3+5x^4+x^5). \end{aligned}$$

Next, finding derivatives of $u_5(x)$, we have as follows:

$$\begin{aligned} u_5'(x) &= c_1 + c_2(2+2x) + c_3(3+6x+3x^2) + c_4(4+12x+12x^2+4x^3) \\ &+ c_5(5+20x+30x^2+20x^3+5x^4), \end{aligned}$$

$$u_5''(x) = c_2(2) + c_3(6+6x) + c_4(12+24x+12x^2) + c_5(20+60x+60x^2+20x^3),$$

$$u_5'''(x) = c_3(6) + c_4(24+24x) + c_5(60+120x+60x^2),$$

$$u_5^{(4)}(x) = c_4(24) + c_5(120+120x).$$

After that, substituting the derivatives into the problem, we receive

$$\begin{aligned} &c_4(24) + c_5(120+120x) = \sin x + e^x(\sin x - x\cos x + x^2) \\ &+ \int_0^x te^x\Big[c_2(2) + c_3(6+6t) + c_4(12+24t+12t^2) + c_5(20+60t+60t^2+20t^3)\Big]dt. \end{aligned}$$

Then, simplifying and integrating the equation, we obtain the new equation. Selecting $x_1 = 0, x_2 = 0.5$ to substitute in the new equation with using 4 initial conditions, we get the following system:

$$\begin{aligned}
&c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = 0,\\
&c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 = 1,\\
&2c_2 + 6c_3 + 12c_4 + 20c_5 = -2,\\
&6c_3 + 24c_4 + 60c_5 = -1,\\
&-24c_4 - 120c_5 = 0,\\
&4.121803c_2 + 16.487212c_3 - 195.690615c_4 - 1700.046272c_5 = -9.586004.
\end{aligned}$$

Finally, solving the system by a program, we have

$$c_0 = -1.841322, c_1 = 2.539947, c_2 = -0.579894, c_3 = -0.086716, c_4 = -0.039947, c_5 = 0.007989.$$

Therefore, the numerical solution is

$$\begin{aligned}
u_5(x) &= -1.841322 + 2.539947(1+x) - 0.579894(1+2x+x^2)\\
&- 0.086716(1+3x+3x^2+x^3) - 0.039947(1+4x+6x^2+4x^3+x^4)\\
&+ 0.007989(1+5x+10x^2+10x^3+5x^4+x^5).
\end{aligned}$$

For the approximate solution $u_7(x)$, we put

$$\begin{aligned}
u_7(x) &= c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + c_3T_3(x) + c_4T_4(x) + c_5T_5(x) + c_6T_6(x) + c_7T_7(x)\\
&= c_0 + c_1(1+x) + c_2(1+2x+x^2) + c_3(1+3x+3x^2+x^3)\\
&+ c_4(1+4x+6x^2+4x^3+x^4) + c_5(1+5x+10x^2+10x^3+5x^4+x^5)\\
&+ c_6(1+6x+15x^2+20x^3+15x^4+6x^5+x^6)\\
&+ c_7(1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7).
\end{aligned}$$

Then, finding derivatives of $u_7(x)$, we have as follows:

$$\begin{aligned}
u_7'(x) &= c_1 + c_2(2+2x) + c_3(3+6x+3x^2) + c_4(4+12x+12x^2+4x^3)\\
&+ c_5(5+20x+30x^2+20x^3+5x^4) + c_6(6+30x+60x^2+60x^3+30x^4+6x^5)\\
&+ c_7(7+42x+105x^2+140x^3+105x^4+42x^5+7x^6),
\end{aligned}$$

$$\begin{aligned}
u_7''(x) &= c_2(2) + c_3(6+6x) + c_4(12+24x+12x^2)\\
&+ c_5(20+60x+60x^2+20x^3) + c_6(30+120x+180x^2+120x^3+30x^4)\\
&+ c_7(42+210x+420x^2+420x^3+210x^4+42x^5),
\end{aligned}$$

$$\begin{aligned}
u_7'''(x) &= c_3(6) + c_4(24+24x) + c_5(60+120x+60x^2)\\
&+ c_6(120+360x+360x^2+120x^3) + c_7(210+840x+1260x^2+840x^3+210x^4),
\end{aligned}$$

$$u_7^{(4)}(x) = c_4(24) + c_5(120+120x) + c_6(360+720x+360x^2) + c_7(840+2520x+2520x^2+840x^3).$$

Next, substituting the derivatives into the problem, we receive

$$
\begin{aligned}
& c_4(24)+c_5(120+120x)+c_6(360+720x+360x^2)+c_7(840+2520x+2520x^2+840x^3) \\
& = \sin x + e^x(\sin x - x\cos x + x^2) + \int_0^x te^x\Big[c_2(2)+c_3(6+6t)+c_4(12+24t+12t^2)\Big]dt \\
& + \int_0^x te^x\Big[c_5(20+60t+60t^2+20t^3)+c_6(30+120t+180t^2+120t^3+30t^4)\Big]dt \\
& + \int_0^x te^x\Big[c_7(42+210t+420t^2+420t^3+210t^4+42t^5)\Big]dt.
\end{aligned}
$$

After that, simplifying and integrating the equation, we receive the new equation. Selecting $x_1 = 0, x_2 = 0.5, x_3 = 1.5, x_4 = 2.5$ to substitute in the new equation with using 4 initial conditions, we get the following system:

$$
\begin{aligned}
& c_0+c_1+c_2+c_3+c_4+c_5+c_6+c_7 = 0, \\
& c_1+2c_2+3c_3+4c_4+5c_5+6c_6+7c_7 = 1, \\
& 2c_2+6c_3+12c_4+20c_5+30c_6+42c_7 = -2, \\
& 6c_3+24c_4+60c_5+120c_6+210c_7 = -1, \\
& -24c_4-120c_5-360c_6-840c_7 = 0, \\
& 4.121803c_2+16.487212c_3-195.690615c_4-1700.046273c_5-7895.713129c_6 \\
& -27957.91347c_7 = -9.586004, \\
& 10.0838c_2+60.502802c_3+225.57406c_4+579.811585c_5+599.303853c_6 \\
& -4367.219344c_7 = -15.076224, \\
& 76.140587c_2+609.124698c_3+3383.29128c_4+16045.402c_5+69184.63636c_6 \\
& +277522.4207c_7 = -108.42976.
\end{aligned}
$$

At last, solving the system by a program, we have

$$
\begin{aligned}
& c_0 = -1.842363, c_1 = 2.545356, c_2 = -0.591238, c_3 = -0.074767, c_4 = -0.046290, \\
& c_5 = 0.009142, c_6 = 0.000251, c_7 = -0.000091.
\end{aligned}
$$

Therefore, the numerical solution is

$$
\begin{aligned}
u_7(x) &= -1.842363+2.545356(1+x)-0.591238(1+2x+x^2) \\
&- 0.074767(1+3x+3x^2+x^3)-0.046290(1+4x+6x^2+4x^3+x^4) \\
&+ 0.009142(1+5x+10x^2+10x^3+5x^4+x^5) \\
&+ 0.000251(1+6x+15x^2+20x^3+15x^4+6x^5+x^6) \\
&- 0.000091(1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7).
\end{aligned}
$$

TABLE 2. Values of exact and approximate $u_5(x)$ solutions for the example 5.1.

x	Exact solution	Approximate solution	Absolute error
0.000000	0.000000	0.000057	0.000057
0.261799	0.190280	0.190393	0.000113
0.523599	0.225844	0.226031	0.000187
0.785398	0.090257	0.090506	0.000249
1.047198	−0.230597	−0.230288	0.000309
1.308997	−0.747547	−0.746921	0.000626
1.570796	−1.467401	−1.465247	0.002154
1.832596	−2.392481	−2.385225	0.007256
2.094395	−3.520465	−3.499740	0.020725
2.356194	−4.844546	−4.793425	0.051121
2.617994	−6.353892	−6.241480	0.112412
2.879793	−8.034390	−7.808496	0.225894
3.141593	−9.869604	−9.447272	0.422332

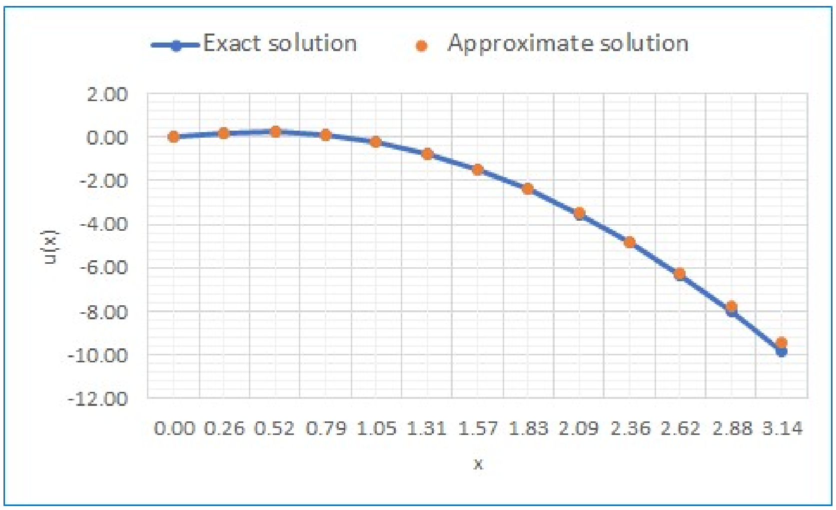

FIGURE 2. Graphs of exact and approximate $u_5(x)$ solutions for the example 5.1.

TABLE 3. Values of exact and approximate $u_7(x)$ solutions for the example 5.1.

x	Exact solution	Approximate solution	Absolute error
0.000000	0.000000	0.000000	0.000000
0.261799	0.190280	0.190280	0.000000
0.523599	0.225844	0.225850	0.000006
0.785398	0.090257	0.090299	0.000042
1.047198	−0.230597	−0.230467	0.000130
1.308997	−0.747547	−0.747284	0.000263
1.570796	−1.467401	−1.466998	0.000403
1.832596	−2.392481	−2.391979	0.000502
2.094395	−3.520465	−3.519919	0.000546
2.356194	−4.844546	−4.843944	0.000602
2.617994	−6.353892	−6.353094	0.000798
2.879793	−8.034390	−8.033200	0.001190
3.141593	−9.869604	−9.868194	0.001410

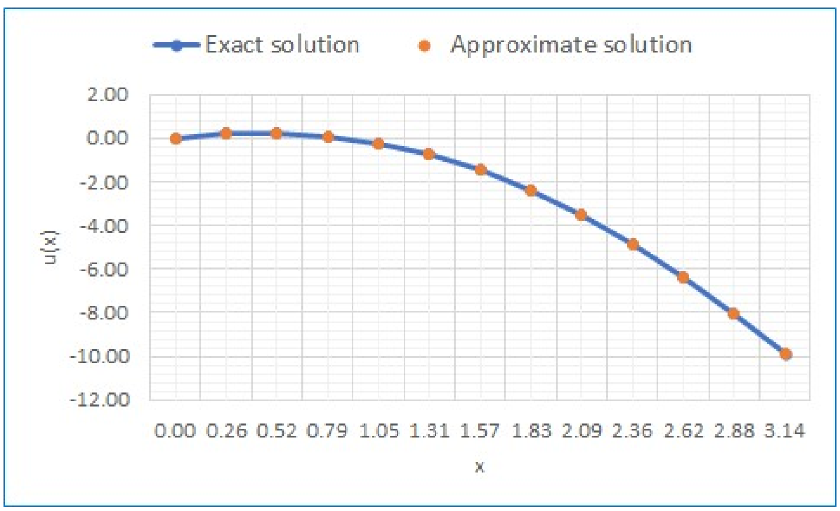

FIGURE 3. Graphs of exact and approximate $u_7(x)$ solutions for the example 5.1.

6. Conclusion

In this paper, the extensions of linear VIDEs of the first and the second kinds have been introduced already. In general, all results show that the Kamal transform has been effective to solve analytical solutions of the extensions of both kinds on convolution type kernels repeatedly and the Touchard polynomials have been successful to figure out numerical solutions of the extensions of both kinds unrelated to convolution type kernels several times. However, Laplace transform is another method that can be analytically solved on convolution types of the first and second extensions similarly. Moreover, the main advantage of this analytical method is the fact that it gives the exact solutions in just few processes and uses very less computational work. We also suggest that this numerical method can be applicable to singularly perturbed linear VIDEs to obtain accurate approximate solutions.

References

[1] C. Yang and J. Hou, Numerical solution of integro-differential equations of fractional order by Laplace decomposition method, *Wseas Transactions on Mathematics* **12(12)** (2013), 1173-1183.

[2] M. Gachpazan, M. Erfanian and H. Beiglo, Solving nonlinear Volterra integro-differential equation by using Legendre polynomial approximations, *Iranian Journal of Numerical Analysis and Optimization* **4(2)** (2014), 73-83.

[3] S. Aggarwal and A.R. Gupta, Solution of linear Volterra integro-differential equations of second kind using Kamal transform, *Journal of Emerging Technologies and Innovative Research* **6(1)** (2019), 741-747.

[4] J.O. Okai, D.O. Ilejimi and M. Ibrahim, Solution of linear Volterra integro-differential equation of the second kind using the modified Adomian decomposition method, *Global Scientific Journals* **7(5)** (2019), 288-294.

[5] A. Jhinga, J. Patade and V.D. Gejji, Solving Volterra integro-differential equations involving delay: a new higher order numerical method, *ResearchGate* (2020), 1-10.

[6] S. Aggarwal, A. Vyas and S.D. Sharma, Analytical solution of first kind Volterra integro-differential equation using Sadik transform, *International Journal of Research and Innovation in Applied Science* **5(8)** (2020), 73-80.

[7] J.T. Abdullah and H.S. Ali, Laguerre and Touchard polynomials for linear Volterra integral and integro differential equations, *Journal of Physics: Conference Series* **1591** (2020), 1-17.

[8] M. Cakir, B. Gunes and H. Duru, A novel computational method for solving nonlinear Volterra integro-differential equation, *Kuwait J. Sci.* **48(1)** (2021), 1-9.

[9] F. Cakir, M. Cakir and H.G. Cakir, A robust numerical technique for solving non-linear Volterra integro-differential equations with boundary layer, *Commun. Korean Math. Soc.* **37(3)** (2022), 939-955.

[10] D.P. Patil, P.S. Nikam and P.D. Shinde, Kushare transform in solving Faltung type Volterra integro-differential equation of first kind, *International Advanced Research Journal in Science, Engineering and Technology* **9(10)** (2022), 84-91.

99 Moo 10, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Chiangrai, Phan, Chiang Rai, 57120, Thailand

Email address: phaisat@rmutl.ac.th