Close links of Bernoulli and Euler numbers and polynomials with symmetric functions

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Abstract

In this paper, we give some new convolution sums of Bernoulli and Euler numbers and polynomials with symmetric functions, by make use of the elementary methods including exponential generating functions. From these convolutions we deduce several new identities of Bernoulli and Euler numbers and polynomials with special numbers.

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1. Introduction and preliminaries

The bivariate Mersenne and bivariate Mersenne-Lucas sequences $\{M_n(x,y)\}_{n\geq 0}$ and $\{m_n(x,y)\}_{n\geq 0}$ are the polynomials sequences that have been studied by many researchers and constructed with the same recurrence relation but different initial values as follows (see [3, 23]):

$$M_n(x,y) = 3yM_{n-1}(x,y) - 2xM_{n-2}(x,y), \text{ with } M_0(x,y)=0 \text{ and } M_1(x,y)=1, \quad (1.1)$$

$$m_n(x,y) = 3ym_{n-1}(x,y) - 2xm_{n-2}(x,y), \text{ with } m_0(x,y)=2 \text{ and } m_1(x,y)=3y. \quad (1.2)$$

Particular cases of $\{M_n(x,y)\}_{n\geq 0}$ and $\{m_n(x,y)\}_{n\geq 0}$ are (see [10]):

$$M_n(1,k) = M_{k,n} := \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n=1 \\ 3kM_{k,n-1} - 2M_{k,n-2}, & \text{if } n\geq 2 \end{cases}, \quad (1.3)$$

and

$$m_n(1,k) = m_{k,n} := \begin{cases} 2, & \text{if } n=0 \\ 3k, & \text{if } n=1 \\ 3km_{k,n-1} - 2m_{k,n-2} & \text{if } n\geq 2 \end{cases}, \quad (1.4)$$

which are, respectively, called k-Mersenne and k-Mersenne-Lucas numbers. The Binet's formulas for k-Mersenne and k-Mersenne-Lucas numbers are, respectively, given by

$$M_{k,n} = \frac{x_1^n - x_2^n}{x_1 - x_2} \text{ and } m_{k,n} = x_1^n + x_2^n,$$

where $x_1 = \frac{3k+\sqrt{9k^2-8}}{2}$ and $x_2 = \frac{3k-\sqrt{9k^2-8}}{2}$ are roots of the characteristic equation $x^2 - 3kx + 2 = 0$. Note that, we have

$$x_1 + x_2 = 3k,\ x_1 - x_2 = \sqrt{9k^2-8} \text{ and } x_1x_2 = 2.$$

In the literature, we have seen several articles that are interested in studying the other k-numbers, for example Falcon and Plaza in [13] defined and studied the k-Fibonacci numbers which are defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \text{with } F_{k,0} = 0 \text{ and } F_{k,1} = 1. \quad (1.5)$$

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After that, Falcon in [12] presented some results of the k-Lucas numbers which are defined as follows:

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \text{with } L_{k,0} = 2 \text{ and } L_{k,1} = k. \tag{1.6}$$

Also, the authors in [25, 26], calculated the ordinary generating functions of the products of k-Fibonacci and k-Lucas numbers with certain numbers.

If we put $k = 1$ in the Eqs. (1.3), (1.4), (1.5) and (1.6) we get the recurrence relations of Mersenne, Mersenne-Lucas, Fibonacci and Lucas numbers which are, respectively, given by (see [9, 16, 20, 22])

$$\begin{aligned} M_n &= 3M_{n-1} - 2M_{n-2}, \text{with } M_0 = 0 \text{ and } M_1 = 1, \\ m_n &= 3m_{n-1} - 2m_{n-2}, \text{with } m_0 = 2 \text{ and } m_1 = 3, \\ F_n &= F_{n-1} + F_{n-2}, \text{with } F_0 = 0 \text{ and } F_1 = 1, \\ L_n &= L_{n-1} + L_{n-2}, \text{with } L_0 = 2 \text{ and } L_1 = 1. \end{aligned}$$

The Bernoulli and Euler numbers $\{B_n\}_{n\geqslant 0}$ and $\{E_n\}_{n\geqslant 0}$ are, respectively, defined by the following exponential generating functions [11]

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{\exp(z) - 1}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{2\exp(z)}{\exp(2z) + 1}. \tag{1.8}$$

The Bernoulli and Euler polynomials $\{B_n(x)\}_{n\geqslant 0}$ and $\{E_n(x)\}_{n\geqslant 0}$ are, respectively, defined by the following exponential generating functions [11]

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z\exp(xz)}{\exp(z) - 1}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2\exp(xz)}{\exp(2z) + 1}. \tag{1.10}$$

Several authors have focused on the relations between Bernoulli/Euler numbers (polynomials) and other important numbers sequences (see [2, 14, 17, 18, 19, 21]).

Next, we recall some properties of the symmetric functions and the exponential generating functions that we will need in the sequel (see for example [4, 5]).

Definition 1. [1] *Let A and E be any two alphabets. We define $S_n(A - E)$ by the following form:*

$$\frac{\prod_{e\in E}(1 - ez)}{\prod_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - E)z^n, \tag{1.11}$$

with the condition $S_n(A - E) = 0$ for $n < 0$.

Equation (1.11) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A - E)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-E)z^n\right),$$

where

$$S_n(A - E) = \sum_{j=0}^{n} S_{n-j}(-E)S_j(A).$$

Remark 1. *Taking $A = \{0\}$ in (1.11) gives*

$$\sum_{n=0}^{\infty} S_n(-E)z^n = \prod_{e\in E}(1 - ez).$$

Definition 2. [24] *Let n be a positive integer and $E=\{e_1,e_2\}$ be a set of given variables. Then, the n^{th} symmetric function $S_n(e_1+e_2)$ is defined by*

$$S_n(E)=S_n(e_1+e_2)=\frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2},\ \textit{for } n\geq 0. \tag{1.12}$$

Theorem 1. *Let n be a positive integer. The following equalities hold*

$$S_n(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2)=e_1^n+e_2^n, \tag{1.13}$$

$$\frac{1}{2}\left(S_n(e_1+e_2)+(e_1-e_2)S_{n-1}(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2)\right)=e_1^n, \tag{1.14}$$

$$\frac{1}{2}\left(S_n(e_1+e_2)-(e_1-e_2)S_{n-1}(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2)\right)=e_2^n. \tag{1.15}$$

Proof. From (1.12), we have

$$\begin{aligned}S_n(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2) &= \frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2}-e_1e_2\frac{e_1^{n-1}-e_2^{n-1}}{e_1-e_2}\\ &= \frac{e_1^{n+1}-e_2^{n+1}-e_1^ne_2+e_1e_2^n}{e_1-e_2}\\ &= e_1^n+e_2^n,\end{aligned}$$

which is (1.13). Other equations can be proved similarly. □

The exponential function $\exp(z)$ appears in studying radioactive decay, bacterial growth, compound interest and probability theory. We are concerned with the important property of $\exp(z)$

$$\exp(z)=1+\frac{z}{1!}+\frac{z^2}{2!}+...+\frac{z^n}{n!}+....=\sum_{n=0}^{\infty}\frac{z^n}{n!}, \tag{1.16}$$

which is the exponential generating function of the sequence of number $(1,\cdots)$.

Theorem 2. *The exponential generating functions of $S_{n-1}(e_1+e_2)$ and $S_n(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2)$ are respectively given by*

$$\sum_{n=0}^{\infty}S_{n-1}(e_1+e_2)\frac{z^n}{n!}=\frac{1}{e_1-e_2}\left(\exp(e_1z)-\exp(e_2z)\right), \tag{1.17}$$

$$\sum_{n=0}^{\infty}(S_n(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2))\frac{z^n}{n!}=\exp(e_1z)+\exp(e_2z). \tag{1.18}$$

Proof. By using (1.12), we get

$$\begin{aligned}\sum_{n=0}^{\infty}S_{n-1}(e_1+e_2)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\frac{e_1^n-e_2^n}{e_1-e_2}\frac{z^n}{n!}\\ &= \frac{1}{e_1-e_2}\left(\sum_{n=0}^{\infty}e_1^n\frac{z^n}{n!}-\sum_{n=0}^{\infty}e_2^n\frac{z^n}{n!}\right).\end{aligned}$$

From (1.16), we obtain

$$\sum_{n=0}^{\infty}S_{n-1}(e_1+e_2)\frac{z^n}{n!}=\frac{1}{e_1-e_2}\left(\exp(e_1z)-\exp(e_2z)\right),$$

which is the first equation. The second equation can be proved similarly. □

The authors in [7, 8, 15, 27], established some combinatorial identities involving Bernoulli and Euler numbers and polynomials with Fibonacci, Lucas, balancing, Lucas-balancing numbers and polynomials.

In [6], the authors obtained a more results linking Bernoulli and Euler numbers with k-Jacobsthal, k-Jacobsthal-Lucas numbers, bivariate Fibonacci, bivariate Lucas, bivariate Pell and bivariate Pell-Lucas polynomials.

The purpose of the present paper is to derive some connection formulas between symmetric functions and Bernoulli and Euler numbers and polynomials. From these results we will be able to obtain some interesting combinatorial identities involving Bernoulli and Euler numbers and polynomials with some well-known k-numbers including k-Fibonacci, k-Lucas, k-Mersenne and k-Mersenne-Lucas numbers.

2. Main results of the Bernoulli and Euler numbers

In this section, we firstly prove some new theorems by using the Bernoulli and Euler numbers and the symmetric functions. Secondly, we study some special cases.

2.1. New theorems.

In this part, we are now in a position to provide three new theorems.

Theorem 3. *Given an alphabet $E=\{e_1,e_2\}$, then for any positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l S_{n-l-1}(e_1+e_2)B_l = ne_2^{n-1}, \tag{2.1}$$

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l(2^l-1)\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l = -(e_1-e_2)ne_2^{n-1}. \tag{2.2}$$

Proof. Using the change of variable $z=(e_1-e_2)z$ in (1.7), we get

$$\sum_{l=0}^{\infty}(e_1-e_2)^l B_l\frac{z^l}{l!} = \frac{(e_1-e_2)z}{\exp((e_1-e_2)z)-1}, \tag{2.3}$$

and multiplying (1.17) by (2.3), we obtain

$$\left(\sum_{n=0}^{\infty}S_{n-1}(e_1+e_2)\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(e_1-e_2)^l B_l\frac{z^l}{l!}\right) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l B_l S_{n-l-1}(e_1+e_2)\frac{z^n}{n!}.$$

Then,

$$\frac{1}{e_1-e_2}\left(\exp(e_1z)-\exp(e_2z)\right)\frac{(e_1-e_2)z}{\exp((e_1-e_2)z)-1} = z\exp(e_2z) = \sum_{n=0}^{\infty}ne_2^{n-1}\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

Similarly, we use the change of variable $z=2(e_1-e_2)z$ in (1.7), we obtain

$$\sum_{l=0}^{\infty}(2(e_1-e_2))^l B_l\frac{z^l}{l!} = \frac{2(e_1-e_2)z}{\exp(2(e_1-e_2)z)-1}, \tag{2.4}$$

and multiplying (1.18) by (2.4), we get

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}(S_n(e_1+e_2)-e_1e_2S_{n-2}(e_1+e_2))\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(2(e_1-e_2))^l B_l\frac{z^l}{l!}\right)\\ =\ &\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^l(e_1-e_2)^l\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l\frac{z^n}{n!}.\end{aligned}$$

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Then,

$$\begin{aligned}(\exp(e_1z)+\exp(e_2z))\frac{2(e_1-e_2)z}{\exp(2(e_1-e_2)z)-1} &= \exp(e_2z)\frac{2(e_1-e_2)z}{\exp((e_1-e_2)z)-1}\\ &= \left(\sum_{n=0}^{\infty}2e_2^n\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(e_1-e_2)^lB_l\frac{z^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2e_2^{n-l}(e_1-e_2)^lB_l\frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}2^l(e_1-e_2)^l\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l=\sum_{l=0}^{n}\binom{n}{l}2e_2^{n-l}(e_1-e_2)^lB_l. \tag{2.5}$$

Using (1.15), we can write (2.5) as follows

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l(2^l-1)\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l=-(e_1-e_2)\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^lS_{n-l-1}(e_1+e_2)B_l.$$

Equivalently

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l(2^l-1)\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l=-(e_1-e_2)ne_2^{n-1}.$$

Hence the result. □

Theorem 4. *Given an alphabet $E=\{e_1,e_2\}$. For any positive integer n, the following results hold*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(e_1-e_2)^{2l}S_{n-2l-1}(e_1+e_2)B_{2l}=\frac{n(S_{n-1}(e_1+e_2)-e_1e_2S_{n-3}(e_1+e_2))}{2},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(e_1-e_2)^{2l}(2^{2l}-1)\left(S_{n-2l}(e_1+e_2)-e_1e_2S_{n-2l-2}(e_1+e_2)\right)B_{2l}=\frac{(e_1-e_2)^2nS_{n-2}(e_1+e_2)}{2}.$$

Proof. We have

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^lS_{n-l-1}(e_1+e_2)B_l=ne_2^{n-1}.$$

Equivalently

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^{n-l}S_{l-1}(e_1+e_2)B_{n-l}=\frac{n(S_{n-1}(e_1+e_2)-(e_1-e_2)S_{n-2}(e_1+e_2)-e_1e_2S_{n-3}(e_1+e_2))}{2},$$

with $B_0=1$, $B_1=-\frac{1}{2}$.

For $l\geqslant 0$, we have $B_{2l+1}=0$, then we get

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(e_1-e_2)^{2l}S_{n-2l-1}(e_1+e_2)B_{2l}=\frac{n(S_{n-1}(e_1+e_2)-e_1e_2S_{n-3}(e_1+e_2))}{2}.$$

Similarly, by (2.2), we get

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l(2^l-1)\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)B_l=$$
$$\frac{-(e_1-e_2)n(S_{n-1}(e_1+e_2)-(e_1-e_2)S_{n-2}(e_1+e_2)-e_1e_2S_{n-3}(e_1+e_2))}{2},$$

with $B_0 = 1$, $B_1 = -\frac{1}{2}$.

For $l \geqslant 0$, we have $B_{2l+1} = 0$, then we obtain

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} (e_1 - e_2)^{2l} (2^{2l} - 1) \left(S_{n-2l}(e_1 + e_2) - e_1 e_2 S_{n-2l-2}(e_1 + e_2)\right) B_{2l} = \frac{(e_1 - e_2)^2 n S_{n-2}(e_1 + e_2)}{2}.$$

Hence the desired result. □

Theorem 5. *Given an alphabet $E = \{e_1, e_2\}$, then for any positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 2(4)^{n-2l-1} (e_1 - e_2)^{2l+1} S_{n-2l-1}(e_1 + e_2) E_{2l} = (3e_1 + e_2)^n - (e_1 + 3e_2)^n,$$

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} \left(\frac{e_1 - e_2}{2}\right)^{2l} \left(S_{n-2l}(e_1 + e_2) - e_1 e_2 S_{n-2l-2}(e_1 + e_2)\right) E_{2l} = 2^{1-n} (e_1 + e_2)^n.$$

Proof. By using the change of variable $z = 4z$ in (1.17) and $z = (e_1 - e_2)z$ in (1.8), we obtain

$$\sum_{n=0}^{\infty} 4^n S_{n-1}(e_1 + e_2) \frac{z^n}{n!} = \frac{\exp(4e_2 z)}{(e_1 - e_2)} (\exp(2(e_1 - e_2)z) - 1)(\exp(2(e_1 - e_2)z) + 1), \tag{2.6}$$

$$\sum_{l=0}^{\infty} (e_1 - e_2)^l E_l \frac{z^l}{l!} = \frac{2 \exp((e_1 - e_2)z)}{\exp(2(e_1 - e_2)z) + 1}, \tag{2.7}$$

and multiplying (2.6) by (2.7), as follows

$$\left(\sum_{n=0}^{\infty} 4^n S_{n-1}(e_1 + e_2) \frac{z^n}{n!}\right) \left(\sum_{l=0}^{\infty} (e_1 - e_2)^l E_l \frac{z^l}{l!}\right) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} 4^{n-l} (e_1 - e_2)^l S_{n-l-1}(e_1 + e_2) E_l \frac{z^n}{n!},$$

then, we get

$$\begin{aligned}
\frac{2 \exp(4e_2 z)}{(e_1 - e_2)} \exp((e_1 - e_2)z)(\exp(2(e_1 - e_2)z) - 1) &= \frac{2}{e_1 - e_2} \left(\exp((3e_1 + e_2)z) - \exp((e_1 + 3e_2)z)\right) \\
&= \frac{2}{e_1 - e_2} \sum_{n=0}^{\infty} \Big((3e_1 + e_2)^n - (e_1 + 3e_2)^n\Big) \frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n} \binom{n}{l} 2(4)^{n-l-1} (e_1 - e_2)^{l+1} S_{n-l-1}(e_1 + e_2) E_l = (3e_1 + e_2)^n - (e_1 + 3e_2)^n.$$

For $l \geqslant 0$, we obtain $E_{2l+1} = 0$ and the desired result.

Similarly, we use the change of variable $z = \frac{e_1 - e_2}{2} z$ in (1.8), we obtain

$$\sum_{l=0}^{\infty} \left(\frac{e_1 - e_2}{2}\right)^l E_l \frac{z^l}{n!} = \frac{2 \exp(\frac{e_1 - e_2}{2} z)}{\exp((e_1 - e_2)z) + 1}, \tag{2.8}$$

and multiplying (1.18) by (2.8), we get

$$\left(\sum_{n=0}^{\infty} (S_n(e_1 + e_2) - e_1 e_2 S_{n-2}(e_1 + e_2)) \frac{z^n}{n!}\right) \left(\sum_{l=0}^{\infty} \left(\frac{e_1 - e_2}{2}\right)^l E_l \frac{z^l}{l!}\right) =$$

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{e_1 - e_2}{2}\right)^l \left(S_{n-l}(e_1 + e_2) - e_1 e_2 S_{n-l-2}(e_1 + e_2)\right) E_l \frac{z^n}{n!},$$

and

$$\frac{2\exp(\frac{e_1-e_2}{2}z)}{(\exp(e_1-e_2)z)+1}\left(\exp(e_1z)+\exp(e_2z)\right)=2\sum_{n=0}^{\infty}\left(\frac{e_1+e_2}{2}\right)^n\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}\left(\frac{e_1-e_2}{2}\right)^l\left(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2)\right)E_l=2^{1-n}(e_1+e_2)^n,$$

For $k\geqslant 0$, we have $E_{2k+1}=0$ and the desired result. □

2.2. **Some applications.** We, now, consider the Theorems 3−5 to derive the following two cases.

Case 1. Let $e_1=\frac{k+\sqrt{k^2+4}}{2}$ and $e_2=\frac{k-\sqrt{k^2+4}}{2}$, then we have the following results of Bernoulli and Euler numbers with k-Fibonacci and k-Lucas numbers.

Corollary 1. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+4}\right)^l F_{k,n-l}B_l=n\left(\frac{k-\sqrt{k^2+4}}{2}\right)^{n-1},$$

or, equivalently,

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(k^2+4\right)^l F_{k,n-2l}B_{2l}=\frac{nL_{k,n-1}}{2}.$$

Corollary 2. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+4}\right)^l(2^l-1)L_{k,n-l}B_l=-\sqrt{k^2+4}n\left(\frac{k-\sqrt{k^2+4}}{2}\right)^{n-1},$$

or, equivalently,

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(k^2+4\right)^l(2^{2l}-1)L_{k,n-2l}B_{2l}=\frac{(k^2+4)nF_{k,n-1}}{2}.$$

Corollary 3. *Let n be a positive integer, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}\sqrt{k^2+4}^{2l+1}F_{k,n-2l}E_{2l}=(2k+\sqrt{k^2+4})^n-(2k-\sqrt{k^2+4})^n,$$

and

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(\frac{k^2+4}{4}\right)^l L_{k,n-2l}E_{2l}=2^{1-n}(k)^n.$$

- If we put $k=1$ in the Corollaries 1−3, we get the following results (see [7, 8])

$$\sum_{l=0}^{n}\binom{n}{l}5^{\frac{l}{2}}F_{n-l}B_l=n\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\quad\text{or, equivalently}\quad\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}5^lF_{n-2l}B_{2l}=\frac{nL_{n-1}}{2},$$

$$\sum_{l=0}^{n}\binom{n}{l}5^{\frac{l}{2}}(2^l-1)L_{n-l}B_l=-\sqrt{5}n\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\quad\text{or, equivalently}\quad\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(20^l-5^l)L_{n-2l}B_{2l}=\frac{5nF_{n-1}}{2},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-4l-1}\sqrt{5}^{2l+1}F_{n-2l}E_{2l}=(2+\sqrt{5})^n-(2-\sqrt{5})^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(\frac{5}{4}\right)^{l}L_{n-2l}E_{2l}=2^{1-n}.$$

Case 2. Let $e_1=\frac{3k+\sqrt{9k^2-8}}{2}$ and $e_2=\frac{3k-\sqrt{9k^2-8}}{2}$, then we have the following results of Bernoulli and Euler numbers with k-Mersenne and k-Mersenne-Lucas numbers.

Corollary 4. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{9k^2-8}\right)^{l}M_{k,n-l}B_l=n\left(\frac{3k-\sqrt{9k^2-8}}{2}\right)^{n-1}, \tag{2.9}$$

or, equivalently,

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(9k^2-8\right)^{l}M_{k,n-2l}B_{2l}=\frac{nm_{k,n-1}}{2}. \tag{2.10}$$

Corollary 5. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{9k^2-8}\right)^{l}(2^l-1)m_{k,n-l}B_l=-\sqrt{9k^2-8}n\left(\frac{3k-\sqrt{9k^2-8}}{2}\right)^{n-1},$$

or, equivalently,

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(9k^2-8\right)^{l}(2^{2l}-1)m_{k,n-2l}B_{2l}=\frac{(9k^2-8)nM_{k,n-1}}{2}.$$

Corollary 6. *Let n be a positive integer, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}\sqrt{9k^2-8}^{\,2l+1}M_{k,n-2l}E_{2l}=(6k+\sqrt{9k^2-8})^n-(6k-\sqrt{9k^2-9})^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(\frac{9k^2-8}{4}\right)^{l}m_{k,n-2l}E_{2l}=2^{1-n}(3k)^n.$$

• By putting $k=1$ in the Corollaries 4−6, we have the following Mersenne-Bernoulli, Mersenne-Lucas-Bernoulli, Mersenne-Euler and Mersenne-Lucas-Euler identities

$$\sum_{l=0}^{n}\binom{n}{l}M_{n-l}B_l=n \text{ or, equivalently } \sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}M_{n-2l}B_{2l}=\frac{nm_{n-1}}{2},$$

$$\sum_{l=0}^{n}\binom{n}{l}(2^l-1)m_{n-l}B_l=-n \text{ or, equivalently } \sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(2^{2l}-1)m_{n-2l}B_{2l}=\frac{nM_{n-1}}{2},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}M_{n-2l}E_{2l}=7^n-5^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(\frac{1}{4}\right)^{l}m_{n-2l}E_{2l}=2^{1-n}3^n.$$

3. Main results on the Bernoulli and Euler polynomials

By using the Bernoulli and Euler polynomials and the symmetric functions we prove some new theorems and we give some applications.

3.1. **New theorems.** In this part, we are now in a position to provide two new theorems.

Theorem 6. *Given an alphabet $E=\{e_1,e_2\}$, then for any positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l S_{n-l-1}(e_1+e_2)B_l(x)=n(e_2+(e_1-e_2)x)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l}(e_1-e_2)^l S_{n-l-1}(e_1+e_2)B_l(x)=n(2e_1+(e_1-e_2)x)^{n-1}+n(e_1+e_2+x(e_1-e_2))^{n-1},$$

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}3^{n-l}(e_1-e_2)^l S_{n-l-1}(e_1+e_2)B_l(x)&=n(2e_1+e_2+(e_1-e_2)x)^{n-1}\\&\quad+n(e_1+2e_2+x(e_1-e_2))^{n-1}+n(3e_2+x(e_1-e_2))^{n-1}.\end{aligned}$$

Proof. We use the change of variable $z=(e_1-e_2)z$ in (1.9), we obtain

$$\sum_{l=0}^{\infty}(e_1-e_2)^l B_l(x)\frac{z^l}{l!}=\frac{(e_1-e_2)z\exp((e_1-e_2)xz)}{\exp((e_1-e_2)z)-1},\tag{3.1}$$

and multiplying (1.17) by (3.1), we get

$$\left(\sum_{n=0}^{\infty}S_{n-1}(e_1+e_2)\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(e_1-e_2)^l B_l(x)\frac{z^l}{l!}\right)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^l S_{n-l-1}(e_1+e_2)B_l(x)\frac{z^n}{n!}.$$

Equivalently

$$z\exp((e_1-e_2)xz)\exp(e_2z)=z\exp((e_2+(e_1-e_2)x)z)=\sum_{n=0}^{\infty}n(e_2+x(e_1-e_2))^{n-1}\frac{z^n}{n!},$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

For the second equation, using the change of variable $z=2z$ in (1.17), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}2^nS_{n-1}(e_1-e_2)\frac{z^n}{n!}&=\frac{1}{e_1-e_2}\left(\exp(2e_1z)-\exp(2e_2z)\right)\\&=\frac{\exp(2e_2)}{e_1-e_2}(\exp((e_1-e_2)z)-1)(\exp((e_1-e_2)z)+1).\end{aligned}\tag{3.2}$$

By multiplying (3.1) by (3.2), we get

$$\left(\sum_{n=0}^{\infty}2^nS_{n-1}(e_1-e_2)\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(e_1-e_2)^l B_l(x)\frac{z^l}{l!}\right)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^{n-l}(e_1-e_2)^l B_l(x)S_{n-l-1}(e_1+e_2)\frac{z^n}{n!}.$$

Then, we obtain

$$z\exp(2e_2z)(\exp((e_1-e_2)z)+1)\exp(x(e_1-e_2)z)=\sum_{n=0}^{\infty}\left(n(2e_2+x(e_1-e_2))^{n-1}+n(e_1+e_2+x(e_1-e_2)^{n-1}\right)\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

Similarly, using the change of variable $z=3z$ in (1.17), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}3^nS_{n-1}(e_1-e_2)\frac{z^n}{n!}&=\frac{1}{e_1-e_2}\left(\exp(3e_1z)-\exp(3e_2z)\right)\\&=\frac{\exp(3e_2)}{e_1-e_2}(\exp((e_1-e_2)z)-1)(\exp(2(e_1-e_2)z)+\exp((e_1-e_2)z)+1),\end{aligned}\tag{3.3}$$

and multiplying (3.1) by (3.3), we get

$$\left(\sum_{n=0}^{\infty} 3^n S_{n-1}(e_1+e_2)\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}(e_1-e_2)^l B_l(x)\frac{z^l}{l!}\right) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}3^{n-l}(e_1-e_2)^l B_l(x) S_{n-l-1}(e_1+e_2)\frac{z^n}{n!}.$$

Then, we obtain

$$\begin{aligned}
&z\exp(3e_2z)(\exp(2(e_1-e_2)z)+\exp((e_1-e_2)z)+1)\exp((e_1-e_2)xz)\\
&= z(\exp((2e_1+e_2+(e_1-e_2)x)z)+\exp((e_1+2e_2+(e_1-e_2)x)z)+\exp((3e_2+(e_1-e_2)x)z))\\
&= \sum_{n=0}^{\infty} n(2e_1+e_2+(e_1-e_2)x)^{n-1}\frac{z^n}{n!}+\sum_{n=0}^{\infty} n(e_1+2e_2+(e_1-e_2)x)^{n-1}\frac{z^n}{n!}+\sum_{n=0}^{\infty} n(3e_2+(e_1-e_2)x)^{n-1}\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result. □

Theorem 7. *Given an alphabet $E=\{e_1,e_2\}$, the following formulas hold for any positive integer n*

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l-1}(e_1-e_2)^{l+1}S_{n-l-1}(e_1+e_2)E_l(x) = (e_1+e_2+(e_1-e_2)x)^n-(2e_2+(e_1-e_2)x)^n,$$

$$\sum_{l=0}^{n}\binom{n}{l}(e_1-e_2)^{l-1}(S_{n-l}(e_1+e_2)-e_1e_2S_{n-l-2}(e_1+e_2))E_l(x) = 2(e_2+(e_1-e_2)x)^n.$$

Proof. We use the change of variable $z=(e_1-e_2)z$ in (1.10), we obtain

$$\sum_{l=0}^{\infty}(e_1-e_2)^l E_l(x)\frac{z^l}{l!} = \frac{2\exp((e_1-e_2)xz)}{\exp((e_1-e_2)z)+1}, \tag{3.4}$$

and multiplying (3.2) by (3.4), we obtain

$$\begin{aligned}
&\frac{2\exp(2e_2z)}{e_1-e_2}\exp((e_1-e_2)xz)(\exp((e_1-e_2)z)-1)\\
&= \frac{2}{e_1-e_2}\exp((2e_2+(e_1-e_2)x)z)(\exp((e_1-e_2)z)-1)\\
&= \frac{2}{e_1-e_2}(\exp((e_1+e_2+(e_1-e_2)x)z)-\exp((2e_2+(e_1-e_2)x)z))\\
&= \frac{2}{e_1-e_2}\sum_{n=0}^{\infty}\Big((e_1+e_2+(e_1-e_2)x)^n-(2e_2+(e_1-e_2)x)^n\Big)\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

For provide the equation (7), multiplying (3.4) by (1.18), we obtain

$$2\exp((e_1-e_2)xz)\exp(e_2z) = 2\exp((e_2+(e_1-e_2)x)z) = \sum_{n=0}^{\infty}2(e_2+(e_1-e_2)x)^n\frac{z^n}{n!}.$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain the result. □

3.2. **Some special cases.** We now consider the Theorems 6 and 7 to derive the following two cases.

Case 1. Let $e_1=\frac{k+\sqrt{k^2+4}}{2}$ and $e_2=\frac{k-\sqrt{k^2+4}}{2}$, then we have the following k-Fibonacci-Bernoulli, k-Fibonacci-Euler and k-Lucas-Euler polynomials identities.

Corollary 7. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(k^2+4)^{\frac{l}{2}}F_{k,n-l}B_l(x) = n\left(\frac{k+\sqrt{k^2+4}(2x-1)}{2}\right)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l}(k^2+4)^{\frac{l}{2}}F_{k,n-l}B_l(x)=n(k+\sqrt{k^2+4}(x+1))^{n-1}+n(k+\sqrt{k^2+4}x)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}3^{n-l}(k^2+4)^{\frac{l}{2}}F_{k,n-l}B_l(x)=n\left(\frac{3k+\sqrt{k^2+4}(2x+1)}{2}\right)^{n-1}+n\left(\frac{3k+\sqrt{k^2+4}(2x-1)}{2}\right)^{n-1}+n\left(\frac{3k+\sqrt{k^2+4}(2x-3)}{2}\right)^{n-1}.$$

Corollary 8. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l-1}\sqrt{k^2+4}^{\,l+1}F_{k,n-l}E_l(x)=(k+\sqrt{k^2+4}x)^n-(k+\sqrt{k^2+4}(x-1))^n, \tag{3.5}$$

$$\sum_{l=0}^{n}\binom{n}{l}\sqrt{k^2+4}^{\,l-1}L_{k,n-l}E_l(x)=2\left(\frac{k+\sqrt{k^2+4}(2x-1)}{2}\right)^n. \tag{3.6}$$

- If we put $k=1$ in the Corollaries 7 and 8, we get the following results

$$\begin{aligned}
\sum_{l=0}^{n}\binom{n}{l}5^{\frac{l}{2}}F_{n-l}B_l(x) &= n\left(\frac{1-\sqrt{5}}{2}+\sqrt{5}x\right)^{n-1} \textit{(see [15])},\\
\sum_{l=0}^{n}\binom{n}{l}2^{n-l}5^{\frac{l}{2}}F_{n-l}B_l(x) &= n(1+\sqrt{5}+\sqrt{5}x)^{n-1}+n(1+\sqrt{5}x)^{n-1},\\
\sum_{l=0}^{n}\binom{n}{l}3^{n-l}5^{\frac{l}{2}}F_{n-l}B_l(x) &= n\left(\frac{3+\sqrt{5}}{2}+\sqrt{5}x\right)^{n-1}+n\left(\frac{3-\sqrt{5}}{2}+\sqrt{5}x\right)^{n-1}+n\left(\frac{3-3\sqrt{5}}{2}+\sqrt{5}x\right)^{n-1},\\
\sum_{l=0}^{n}\binom{n}{l}2^{n-l-1}\sqrt{5}^{\,l+1}F_{n-l}E_l(x) &= (1+\sqrt{5}x)^n-(1-\sqrt{5}+\sqrt{5}x)^n,\\
\sum_{l=0}^{n}\binom{n}{l}\sqrt{5}^{\,l-1}L_{n-l}E_l(x) &= 2\left(\frac{1-\sqrt{5}}{2}+\sqrt{5}x\right)^{n} \textit{(see [15])}.
\end{aligned}$$

Case 2. Let $e_1=\frac{3k+\sqrt{9k^2-8}}{2}$ and $e_2=\frac{3k-\sqrt{9k^2-8}}{2}$, then we have the following k-Mersenne-Bernoulli, k-Mersenne-Euler and k-Mersenne-Lucas-Euler polynomials identities.

Corollary 9. *Let n be a positive integer, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(9k^2-8)^{\frac{l}{2}}M_{k,n-l}B_l(x)=n\left(\frac{3k+\sqrt{9k^2-8}(2x-1)}{2}\right)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l}(9k^2-8)^{\frac{l}{2}}M_{k,n-l}B_l(x)=n(3k+\sqrt{9k^2-8}(x+1))^{n-1}+n(3k+\sqrt{9k^2-8}x)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}3^{n-l}(9k^2-8)^{\frac{l}{2}}M_{k,n-l}B_l(x)=n\left(\frac{9k+\sqrt{9k^2-8}(2x+1)}{2}\right)^{n-1}+n\left(\frac{9k+\sqrt{9k^2-8}(2x-1)}{2}\right)^{n-1}+n\left(\frac{9k+\sqrt{9k^2-8}(2x-3)}{2}\right)^{n-1}.$$

Corollary 10. *Let n be an positive integer, we have*

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}2^{n-l-1}\sqrt{9k^2-8}^{\,l+1}M_{k,n-l}E_l(x) &= (3k+\sqrt{9k^2-8}x)^n-(3k+\sqrt{9k^2-8}(x-1))^n,\\ \sum_{l=0}^{n}\binom{n}{l}\sqrt{9k^2-8}^{\,l-1}m_{k,n-l}E_l(x) &= 2\left(\frac{3k+\sqrt{9k^2-8}(2x-1)}{2}\right)^n.\end{aligned}$$

• If we put $k=1$ in the Corollaries 9 and 10, we get the following Mersenne-Bernoulli, Mersenne-Euler and Mersenne-Lucas-Euler polynomials identities.

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}M_{n-l}B_l(x) &= n(1+x)^{n-1},\\ \sum_{l=0}^{n}\binom{n}{l}2^{n-l}M_{n-l}B_l(x) &= n(4+x)^{n-1}+n(3+x)^{n-1},\\ \sum_{l=0}^{n}\binom{n}{l}3^{n-l}M_{n-l}B_l(x) &= n(5+x)^{n-1}+n(4+x)^{n-1}+n(3+x)^{n-1},\\ \sum_{l=0}^{n}\binom{n}{l}2^{n-l}M_{n-l}E_l(x) &= 2(3+x)^n-2(2+x)^n,\\ \sum_{l=0}^{n}\binom{n}{l}m_{n-l}E_l(x) &= 2(1+x)^n.\end{aligned}$$

4. Conclusion

In this paper, by using the symmetric functions we introduced several new convolution sums formulas of Bernoulli and Euler numbers and polynomials. From these formulas we deduce some special cases such as, k-Lucas-Euler polynomials identity, k-Fibonacci-Euler numbers identity and k-Mersenne -Lucas-Bernoulli polynomials identity. For prospects, we can apply the same approach to the Genocchi numbers and polynomials.

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