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A MIXED FORMULATION FOR A FLEXURAL NAGHDI SHELL WITH OBSTACLE

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ABSTRACT. We consider a linear elastic Naghdi shell incorporating shear, membrane, bending dominated effects and an obstacle. The asymptotic analysis shows that the numerical locking is expected. We make use of the theory proposed by Arnold and Brezzi [1] to propose a locking free non-standard mixed variational formulation for a flexural Naghdi shell with obstacle.

1. Introduction

Many problems in physics and engineering involve the model of a thin elastic shell in contact with an obstacle or another deformable body. The modeling of such problems is done by variational inequalities in a convenient functional space. Most of the papers describe and analyze the contact problem of a thin shell with an obstacle as an unilateral contact problem and with Signorini boudary conditions. In [2], the authors propose and analyze a mixed formulation with double Lagrange multipliers modeling the unilateral contact of a Naghdi shell with an obstacle in cartesian coordinates.
By incorporating the inequality constraints associated to the three dimensional contact problem model, the asymptotic analysis when the thickness vanishes is similar to the the case of thin shell without contact for a confinement Koiter shell problem [10] and for a Signorini problem with unilateral contact shell problem [5][13]. In [5], the authors establish an extension of the error estimate between the solution of the three dimensional problem and the two dimensional membrane shell one to the case of membrane shell with unilateral conditions on the contact boundary.
The proposed contact model in this paper is different from the Signorini problem where only the undeformed shell and the lower face of the deformed shell are confined [10]. The asymptotic analysis shows that the membrane locking is expected for a totally confined deformed shell in a half space viewed as an obstacle. Then we propose a free locking mixed formulation modelling a bending dominated Naghdi shell with an obstacle.

2. Geometrical notations

Greek indices and exponents take their values in the set $\{1,2\}$ and Latin indices and exponents take their values in $\{1,2,3\}$. The convention of repeated indices and exponents is used. Let ω be a bounded open set of $\mathbb{R}^2$. We consider a shell whose midsurface is given by $S = \vec{\psi}(\bar{\omega})$ where $\vec{\psi} \in C^3(\omega, \mathbb{R}^3)$ is a one to one mapping such that the vectors $\vec{a}_\alpha = \vec{\psi}_{,\alpha}$ are linearly independent at each point $x =$

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$(x_1,x_2) \in \bar{\omega}$. We let $\vec{a}_3 = \frac{\vec{a}_1\wedge\vec{a}_2}{\|\vec{a}_1\wedge\vec{a}_2\|}$ be the unit normal vector on the midsurface at point $\vec{\psi}(x_1,x_2)$. The contravariant basis $\vec{a}^i$ is defined by the relation $\vec{a}^i.\vec{a}_j = \delta^i_j$ where δ^i_j is the Kronecker symbol. We let $a(x) = \|\vec{a}_1\wedge\vec{a}_2\|^2$ so that $\sqrt{a(x)}$ is the area element of the midsurface in the chart $\vec{\psi}$. The first and second fundamental forms of the surface are given in covariant components by:

$$a_{\alpha\beta} = \vec{a}_\alpha.\vec{a}_\beta,\ b_{\alpha\beta} = \vec{a}_3.\vec{a}_{\alpha,\beta} = -\vec{a}_\alpha.\vec{a}_{3,\beta}.$$

The contravariant components of the first fundamental form and the Christoffel symbol are respectively given by $a^{\alpha\beta} = \vec{a}^\alpha.\vec{a}^\beta$ and $\Gamma^\gamma_{\alpha\beta} = \vec{a}_{\beta,\alpha}.\vec{a}^\delta$.

Let $\underline{\underline{\underline{\underline{E}}}}= (E^{\alpha\beta\lambda\mu})_{\alpha\beta\lambda\mu}$ be the elasticity tensor and $\underline{\underline{G}}= (G^{\alpha\beta})_{\alpha\beta}$ be the metric tensor which we assume to satisfy the usual symmetries and to be bounded and uniformly strictly positive. For an homogeneous isotropic material with Young modulus $E > 0$ and Poisson ratio $\nu \in (0,\frac{1}{2})$, we have

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}),\ \ G^{\alpha\beta} = \frac{E}{2(1+\nu)}a^{\alpha\beta}.$$

Let $\vec{u} \in H^1(\omega,R^3)$ be a midsurface displacement field and $\vec{\theta} \in H^1(\omega,R^3)$ be a rotation of the normal vector given in covariant components by $\vec{u} = u_i\vec{a}^i$ and $\vec{\theta} = \theta_\alpha\vec{a}^\alpha$. The linearized changes of the curvature tensor $\underline{\underline{\Upsilon}} = (\Upsilon_{\alpha\beta})_{\alpha,\beta}$, the transverse shear tensor $\underline{\Phi} = (\Phi_\alpha)_\alpha$ and of the membrane tensor $\underline{\underline{\Lambda}} = (\Lambda_{\alpha\beta})_{\alpha,\beta}$ read in covariant components:

$$\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta}) = \tfrac{1}{2}(\theta_{\alpha/\beta} + \theta_{\beta/\alpha} - b^\gamma_\alpha(u_{\gamma/\beta} - b_{\gamma\beta}u_3) - b^\gamma_\beta(u_{\gamma/\alpha} - b_{\gamma\alpha}u_3)),$$

$$\Phi_\alpha(\vec{u},\vec{\theta}) = u_{3,\alpha} + b^\gamma_\alpha u_\gamma + \theta_\alpha,$$

$$\Lambda_{\alpha\beta}(\vec{u}) = \tfrac{1}{2}(u_{\alpha/\beta} + u_{\beta/\alpha}) - b_{\alpha\beta}u_3,$$

where $v_{\alpha/\beta} = v_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}v_\gamma$ and $b^\gamma_\beta = a^{\alpha\gamma}b_{\gamma\beta}$.

Let ε be the shell thickness. The reference shell is defined by:

$$C = \left\{ \begin{array}{c} M \in \mathbb{R}^3,\ \overrightarrow{OM} = \vec{\psi}(x_1,x_2) + x_3\vec{a}_3(x_1,x_2), \\ (x_1,x_2) \in \overline{\omega},\ \frac{-\varepsilon}{2} \le x_3 \le \frac{\varepsilon}{2} \end{array} \right\}.$$

We suppose the shell clamped on a part $\Gamma \subset \partial\omega$ and set

$$\begin{aligned} H^1_\Gamma &= \{v \in H^1(\omega), v_{/\Gamma} = 0\}, \\ V &= \{(\vec{v},\vec{\delta}) = (v_i\vec{a}^i, \delta_\alpha\vec{a}^\alpha),\ v_i \in H^1_\Gamma,\ \delta_\alpha \in H^1_\Gamma\}. \end{aligned} \tag{1}$$

Let $A_b(.,.)$, $A_m(.,.)$, $A_s(.,.)$ be respectively the bilinear forms associated to the bending energy, membrane energy and shear energy and given by:

$$\begin{aligned} A_b(\vec{u},\vec{\theta};\vec{v},\vec{\delta}) &= \int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta})\Upsilon_{\lambda\mu}(\vec{v},\vec{\delta})\sqrt{a}dx \\ A_m(\vec{u},\vec{\theta};\vec{v},\vec{\delta}) &= \int_\omega E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx \\ A_s(\vec{u},\vec{\theta};\vec{v},\vec{\delta}) &= \int_\omega G^{\alpha\beta}\Phi_\alpha(\vec{u},\vec{\theta})\Phi_\beta(\vec{v},\vec{\delta})\sqrt{a}dx. \end{aligned}$$

The total energy $\mathbb{E}$ of the deformed shell is given by:

$$\mathbb{E}(\vec{v},\vec{\delta}) = \frac{\varepsilon^3}{2}A_b(\vec{v},\vec{\delta};\vec{v},\vec{\delta}) + \frac{\varepsilon}{2}A_m(\vec{v},\vec{\delta};\vec{v},\vec{\delta}) + \frac{\varepsilon}{2}A_s(\vec{v},\vec{\delta};\vec{v},\vec{\delta}) - \tilde{L}(\vec{v}), \tag{2}$$

where $\tilde{L}(\vec{v}) = \int_\omega \vec{F}.\vec{v}\sqrt{a}dx$ and $\vec{F}$ is the external force applied to the thin elastic linear shell. Let G be the inextensional displacement space defined by:

$$G = \{(v_i\vec{a}^i, \delta_\alpha\vec{a}^\alpha) \in V \text{ such that } \Lambda_{\alpha\sigma}(\vec{v}) = \Phi_\alpha(\vec{v},\vec{\delta}) = 0\}.$$

The space G also called subspace of admissible pure bending displacements corresponds to zero membrane and shear energies. The shell is called flexural or non inhibited or bending dominated for suitable non zero loading and boundary conditions and if the inextensional displacements subspace G contains non zero displacements. Since the limit problem is posed over G and to get a well posed flexural shell problem, the external applied forces are scaled in the form by $\vec{F} = \varepsilon^3\vec{f}$ and $\vec{f} \in L^2(\omega,\mathbb{R}^3)$ is independent of ε [6, 14].

3. The Naghdi model for a bending dominated shell with obstacle

Assuming that the deformed shell remains in a given half space $\mathbb{H} = \{y \in \mathbb{R}^3; oy.p \geq 0\}$, where p is a given non-zero vector in $\mathbb{R}^3$ [10]. Then, the unknown displacement field $u_i(x)\vec{a}^i(x) + x_3\theta_\alpha(x)\vec{a}^\alpha(x)$ of a point of the deformed shell C is determined such that the energy $\mathbb{E}(\vec{v},\vec{\delta})$ is minimized over a strict subset U of V given by:

$$U = \{(\vec{u},\vec{\theta}) \in V; (\vec{\psi}(x) + u_i(x)\vec{a}^i(x) + x_3\theta_\alpha\vec{a}^\alpha).p \geq 0 \ \forall x \in \omega,\ \forall x_3 \in [-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]\}. \tag{3}$$

The constrained minimization problem will be the following:

$$\mathscr{P}_{min}\left\{\begin{array}{c} \text{Find } (\vec{u},\vec{\theta}) \in U \\ \mathbb{E}(\vec{u},\vec{\theta}) = \inf_{(\vec{v},\vec{\delta})\in U}\mathbb{E}(\vec{v},\vec{\delta}) \end{array}\right. \tag{4}$$

As a consequence of the projection theorem, the problem $(\mathscr{P}_{min})$ is equivalent to the following problem $(\mathscr{P}_{\mathscr{U}})$ of variational inequalities [9]:

$$(\mathscr{P}_{\mathscr{U}})\left\{\begin{array}{c} \text{Find } (\vec{u},\vec{\theta}) \in U \\ \frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta})\Upsilon_{\lambda\mu}(\vec{v}-\vec{u},\vec{\delta}-\vec{\theta})\sqrt{a}dx \\ +\varepsilon^{-2}\int_\omega E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u})\Lambda_{\lambda\mu}(\vec{v}-\vec{u})\sqrt{a}dx+ \\ \varepsilon^{-2}\int_\omega G^{\alpha\beta}\Phi_\alpha(\vec{u},\vec{\theta})\Phi_\beta(\vec{v}-\vec{u},\vec{\delta}-\vec{\theta})\sqrt{a}dx \geq l(\vec{v}-\vec{u}) \\ \forall\ (\vec{v},\vec{\delta}) \in U \end{array}\right. \tag{5}$$

where $l(\vec{v}) = \int_\omega \vec{f}.\vec{v}$.

Theorem 3.1. *The minimisation problem (4) is well-posed.*

Proof. Since $\Gamma \neq \emptyset$ and due to the Korn's inequality, we deduce the ellipticity on V of the symmetric bilinear form $A_b(.,.) + \varepsilon^{-2}(A_m(.,.) + A_s(.,.))$ associated to the problem (5)[3]. Moreover, U is a non-empty, closed and convex subspace of V. Then the problem (4) has a unique solution [9]. □

Let G_U be the non-empty, closed and convex subspace of G such that the deformed shell remains in $\mathbb{H}$ [10]:

$$G_U = \{(v_i\vec{a}^i, \delta_\alpha\vec{a}^\alpha) \in G; (\vec{\psi}(x) + v_i(x)\vec{a}^i + x_3\delta_\alpha(x)\vec{a}^\alpha(x)).p \geq 0 \;\forall x \in \omega\}. \tag{6}$$

The limit problem of $(\mathscr{P}_{\mathscr{U}})$ when ε vanishes is formulated in G_U and we have the following result:

Theorem 3.2. *The solution of the scaled flexural bidimensional problem $(\mathscr{P}_{\mathscr{U}})$ tends to the solution of the limit flexural problem posed over the convex G_U and given by:*

$$\left\{\begin{array}{c} \text{Find } \; (\vec{u},\vec{\theta}) \in G_U \\ \frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta})\Upsilon_{\lambda\mu}(\vec{v}-\vec{u},\vec{\delta}-\vec{\theta})\sqrt{a}dx \geq l(\vec{v}-\vec{u})\sqrt{a}dx \\ \\ \forall\;(\vec{v},\vec{\delta}) \in G_U \end{array}\right. \tag{7}$$

Proof. The proof is similar to those of theorem 6 in [10] for a bending dominated Koiter shell subject to a confinement condition.
We note that the bilinear form associated to the variational problem (7) is V-coercive, the linear form $l(.)$ is continuous over V and that G_U is a convex closed subset of V, then the problem (7) has a unique pure bending solution for which the deformed shell remains on one side of the plane $\mathbb{H}$.
We deduce from (5) that $\forall \varepsilon > 0$ and $\forall\;(\vec{v},\vec{\delta}) \in U$, we have:

$$\begin{array}{c} \frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\sqrt{a}dx + \varepsilon^{-2}(\int_\omega E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\Lambda_{\lambda\mu}(\vec{u}^\varepsilon)\sqrt{a}dx \\ \\ +\int_\omega G^{\alpha\beta}\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Phi_\beta(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\sqrt{a}dx) \leq \frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{v},\vec{\delta})\sqrt{a}dx \\ \\ +\varepsilon^{-2}(\int_\omega E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx + \int_\omega G^{\alpha\beta}\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Phi_\beta(\vec{v},\vec{\delta})\sqrt{a}dx) \\ \\ -l(\vec{v}-\vec{u}^\varepsilon) \end{array} \tag{8}$$

Using the ellipticity of the bilinear form $A_b(.,.) + \varepsilon^{-2}(A_m(.,.) + A_s(.,.))$ and the continuity of the linear form $l(.)$, there exist constants $C_c, C_b > 0$ such that:

$$\begin{array}{c} C_c^{-1}\|(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\|_V^2 \leq C_b(\|(\vec{v},\vec{\delta})\|_V\|(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\|_V + 2\varepsilon^{-2}\|(\vec{v},\vec{\delta})\|_V\|(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\|_V \\ +\|(\vec{v},\vec{\delta})\|_V + \|(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\|_V). \end{array} \tag{9}$$

Then by taking $(\vec{v},\vec{\delta}) = (0,0)$, we get $\|(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\|_V \leq C_cC_b$.
Moreover, from the uniform positive definiteness of the ellipticity tensor, we get:

$$\Sigma_{\alpha,\beta}\|\varepsilon^{-1}\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\|_{L^2(\omega)}^2 + \Sigma_\alpha\|\varepsilon^{-1}\Phi_\alpha(\vec{u}^\varepsilon)\|_{L^2(\omega)}^2) \leq C. \tag{10}$$

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We deduce from the boundedness of the solution $(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)$ in V that there exists a subsequence also denoted $(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)$ such that $(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\rightharpoonup(\vec{u}^*,\vec{\theta}^*)$ in V. Then:

$$\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\rightharpoonup\Lambda_{\alpha\beta}(\vec{u}^*),\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\rightharpoonup\Phi_\alpha(\vec{u}^*,\vec{\theta}^*) \tag{11}$$

By similar arguments, there exist subsequences $\varepsilon^{-1}\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)$ and $\varepsilon^{-1}\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)$ that converge weakly respectively to $\Lambda_{\alpha\beta}$ and Φ_α. Then $\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)$ and $\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)$ converge to zero in $L^2(\omega)$. The uniqueness of the limit imply that $\Lambda_{\alpha\beta}(\vec{u}^*)=0$ and $\Phi_\alpha(\vec{u}^*,\vec{\theta}^*)=0$. Then $(\vec{u}^*,\vec{\theta}^*)\in G_U$.

Given $(\vec{v},\vec{\delta})\in G_U$, we have:

$$\begin{aligned}\int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\sqrt{a}dx &\le \int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{v},\vec{\delta})\sqrt{a}dx\\ &+\varepsilon^{-2}\Big(\int_\omega E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx+\int_\omega G^{\alpha\beta}\Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Phi_\beta(\vec{v},\vec{\delta})\sqrt{a}dx\Big)\\ &-l(\vec{v}-\vec{u}^\varepsilon)=\frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{v},\vec{\delta})\sqrt{a}dx-l(\vec{v}-\vec{u}^\varepsilon),\end{aligned} \tag{12}$$

and

$$\begin{aligned}&2\int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u}^*,\vec{\theta}^*)\sqrt{a}dx-\int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^*,\vec{\theta}^*)\Upsilon_{\lambda\mu}(\vec{u}^*,\vec{\theta}^*)\sqrt{a}dx\\ &\le \int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\sqrt{a}dx.\end{aligned} \tag{13}$$

Since ε tends to zero, we obtain:

$$\frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u}^*,\vec{\theta}^*)\Upsilon_{\lambda\mu}(\vec{v}-\vec{u}^*,\vec{\delta}-\vec{\theta}^*)\sqrt{a}dx\ge l(\vec{v}-\vec{u}^*),\ \forall(\vec{v},\vec{\delta})\in G_U.$$

Therefore, $(\vec{u}^*,\vec{\theta}^*)=(\vec{u},\vec{\theta})$ is the unique solution of (7).

The weak convergence $(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\rightharpoonup(\vec{u}^*,\vec{\theta}^*)$ in V imply the following strong convergence in $L^2(\omega)$:

$$\Lambda_{\alpha\beta}(\vec{u}^\varepsilon)\to\Lambda_{\alpha\beta}(\vec{u}^*),\ \Phi_\alpha(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\to\Phi_\alpha(\vec{u}^*,\vec{\theta}^*). \tag{14}$$

The usual based Sobolev space V norm is given by

$$\|(\vec{v},\vec{\delta})\|_V=\Big(\sum_{i=1}^{3}\|v_i\|_{H^1(\omega)}+\sum_{\alpha=1}^{2}\|\delta_\alpha\|_{H^1(\omega)}\Big)^{\frac{1}{2}} \tag{15}$$

Since $(\vec{u},\vec{\theta})\in G_U$ and from (12),(13) we have:

$$\begin{aligned}\|(\vec{u}^\varepsilon-\vec{u},\vec{\theta}^\varepsilon-\vec{\theta})\|_V &\le \int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon-\vec{u},\vec{\theta}^\varepsilon-\vec{\theta})\Upsilon_{\lambda\mu}(\vec{u}^\varepsilon-\vec{u},\vec{\theta}^\varepsilon-\vec{\theta})\sqrt{a}dx\\ &\le \frac{1}{12}\int_\omega E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u},\vec{\theta})\sqrt{a}dx-l(\vec{u}-\vec{u}^\varepsilon)\\ &-2\int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\Upsilon_{\lambda\mu}(\vec{u},\vec{\theta})\sqrt{a}dx+\int_\omega \frac{E^{\alpha\beta\lambda\mu}}{12}\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta})\Upsilon_{\lambda\mu}(\vec{u},\vec{\theta})\sqrt{a}dx\end{aligned} \tag{16}$$

Then, we deduce that $(\vec{u}^\varepsilon,\vec{\theta}^\varepsilon)\to(\vec{u},\vec{\theta})$ in V. □

The limit solution of the problem (5) deduced by the asymptotic analysis is an inextensional displacement such that the deformed bending dominated Naghdi shell remains in the same half-space $\mathbb{H}$. A degradation of the approximation occurs for small thickness when using the standard finite element methods for a bending dominated shell [1, 6, 14].
This phenomenon is called the numerical locking. It occurs when the standard finite element techniques fail to approximate the inextensional displacement continuous space, and the discrete one is usualy reduced to zero. Mixed formulations are proposed to avoid locking [1, 4, 7]. Numerical studies of the mixed numerical scheme proposed in [1] show a good properties of convergence as predicted [8, 11, 12].

4. A mixed formulation for a flexural Naghdi shell with obstacle

Using the same technique of partial selective integration of membrane and shear energies proposed in [1], we introduce the following auxiliary variables which represent the membrane and shear stress aside a multiplicator factor:

$$\underline{\underline{\lambda}} = (\lambda^{\alpha\beta})_{\alpha\beta},\ \lambda^{\alpha\beta} = (\frac{1}{\varepsilon^2} - c_o)E^{\alpha\beta\lambda\mu}\Lambda_{\lambda\mu}(\vec{u}), \tag{17}$$

$$\underline{\chi} = (\chi^{\alpha})_{\alpha},\ \chi^{\alpha} = (\frac{1}{\varepsilon^2} - c_o)G^{\alpha\beta}\Phi_{\beta}(\vec{u},\vec{\theta}), \tag{18}$$

where $c_0 \in]0, \varepsilon^{-2}[$.
Consider the set $W = \{(\underline{\underline{\varphi}}, \underline{\eta}) /\ \varphi^{\alpha\beta}, \eta^{\alpha} \in L^2(\omega)\}$ and the following bilinear forms:

$$\begin{aligned} A(\vec{u},\vec{\theta};\vec{v},\vec{\delta}) = \int_{\omega} \frac{1}{12} E^{\alpha\beta\lambda\mu}\Upsilon_{\alpha\beta}(\vec{u},\vec{\theta})\Upsilon_{\lambda\mu}(\vec{v},\vec{\delta})\sqrt{a}dx + \\ c_0 \int_{\omega} G^{\alpha\beta}\Phi_{\alpha}(\vec{u},\vec{\theta})\Phi_{\beta}(\vec{v},\vec{\delta})\sqrt{a}dx + c_0\int_{\omega} E^{\alpha\beta\lambda\mu}\Lambda_{\alpha\beta}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx, \end{aligned} \tag{19}$$

$$B(\vec{v},\vec{\delta};\underline{\underline{\xi}},\underline{\mu}) = \int_{\omega}\Phi_{\alpha}(\vec{v},\vec{\delta})\mu^{\alpha}\sqrt{a}dx + \int_{\omega}\Lambda_{\alpha\beta}(\vec{v})\xi^{\alpha\beta}\sqrt{a}dx, \tag{20}$$

$$C(\underline{\underline{\lambda}},\underline{\chi};\underline{\underline{\xi}},\underline{\mu}) = \int_{\omega}(G^{-1})_{\alpha\beta}\chi^{\beta}\mu^{\alpha}\sqrt{a}dx + \int_{\omega}(E^{-1})_{\alpha\beta\lambda\mu}\lambda^{\lambda\mu}\xi^{\alpha\beta}\sqrt{a}dx. \tag{21}$$

For $(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \in U \times W$, the total energy is given by:

$$\mathbb{E}(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) = \varepsilon^3(\frac{1}{2}A(\vec{u},\vec{\theta};\vec{u},\vec{\theta}) + \frac{\varepsilon^2}{1-c_0\varepsilon^2}C(\underline{\underline{\lambda}},\underline{\chi};\underline{\underline{\lambda}},\underline{\chi}) - l(\vec{u})). \tag{22}$$

An extension of the classical inf-sup condition is done by assuming that there exists a constant $\sigma > 0$ such that [15]:

$$|||\underline{\underline{\lambda}},\underline{\chi}||| \geq \sigma \|\underline{\underline{\lambda}},\underline{\chi}\|_W \tag{23}$$

where $\|.\|_W$ is the standard L^2 product norm and $|||.|||$ is the semi norm:

$$|||\underline{\underline{\lambda}},\underline{\chi}||| = \sup_{(\vec{v},\vec{\delta})\in U\setminus\{0_V\}} \frac{B(\vec{v},\vec{\delta};\underline{\underline{\lambda}},\underline{\chi})}{\|\vec{v},\vec{\delta}\|_V} \quad \forall\ (\underline{\underline{\lambda}},\underline{\chi}) \in W.$$

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Theorem 4.1. *The constrained minimisation problem*

$$(24)\qquad \begin{cases} \text{Find } (\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \in U\times W \text{ such that} \\ \mathbb{E}(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) = \inf_{(\vec{v},\vec{\delta};\underline{\underline{\xi}},\underline{\mu})\in U\times W} \mathbb{E}(\vec{v},\vec{\delta};\underline{\underline{\xi}},\underline{\mu}) \end{cases}$$

has a unique solution and is equivalent to the following problem:

$$(25)\qquad \begin{cases} \text{seek pairs}(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \in U\times W \text{ such that} \\ A(\vec{u},\vec{\theta};\vec{v}-\vec{u},\vec{\delta}-\vec{\theta}) + B(\vec{v}-\vec{u},\vec{\delta}-\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \geq l(\vec{v}-\vec{u}) \ \forall\ (\vec{v},\vec{\delta})\in U, \\ B(\vec{u},\vec{\theta};\underline{\underline{\xi}},\underline{\mu}) - \frac{\varepsilon^2}{1-c_0\varepsilon^2}C(\underline{\underline{\lambda}},\underline{\chi};\underline{\underline{\xi}},\underline{\mu}) = 0 \ \ \forall\ (\underline{\underline{\xi}},\underline{\mu})\in W. \end{cases}$$

Proof. We define the bilinear form $\widetilde{\widetilde{A}}$:

$$\widetilde{\widetilde{A}}(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi};\vec{v},\vec{\delta};\underline{\underline{\xi}},\underline{\mu}) = A(\vec{u},\vec{\theta};\vec{v},\vec{\delta}) + B(\vec{v},\vec{\delta};\underline{\underline{\lambda}},\underline{\chi})$$

$$-B(\vec{u},\vec{\theta};\underline{\underline{\xi}},\underline{\mu}) + \frac{\varepsilon^2}{1-c_0\varepsilon^2}C(\underline{\underline{\lambda}},\underline{\chi};\underline{\underline{\xi}},\underline{\mu}).$$

The bilinear forms A, B, C are continuous respectively on $V\times V$, $V\times W$ and $W\times W$. We note also that A is $V-$elliptic and C is $W-$elliptic, so the bilinear form $\widetilde{\widetilde{A}}$ is $V\times W$ elliptic. Moreover, $U\times W$ is a non empty closed and convex subspace of $V\times W$. Then the problem (25) has a unique solution. □

Remark 4.2. *If $(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \in U\times W$ is a solution of (25) then $(\vec{u},\vec{\theta})$ is a solution of (5). Moreover there exists $(\underline{\underline{\lambda}},\underline{\chi})\in W$ such that $(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi})\in U\times W$ is a solution of (25) whenever $(\vec{u},\vec{\theta})$ is a solution of (5).*

Besides, the solution of (25) is bounded and we have the following result:

Theorem 4.3. *There exists a constant $C>0$ such that the solution of (25) verify:*

$$(26)\qquad \|\vec{u},\vec{\theta}\|_V + \|\underline{\underline{\lambda}},\underline{\chi}\|_W \leq C\|l\|_{V'}$$

Proof. By replacing $(\vec{v},\vec{\delta})$ by zero in the first line of (25), we get:

$$(27)\qquad A(\vec{u},\vec{\theta};\vec{u},\vec{\theta}) + B(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \leq l(\vec{u})$$

or $B(\vec{u},\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) = \frac{\varepsilon^2}{1-c_0\varepsilon^2}C(\underline{\underline{\lambda}},\underline{\chi};\underline{\underline{\lambda}},\underline{\chi}) \geq 0$ then by using the coercivity of $A(.,)$, $C(.,)$ and the continuity of $B(.,)$ we get:

$$(28)\qquad \begin{aligned} C_1\|\vec{u},\vec{\theta}\|_V &\leq \|l\|_{V'} \\ C_2\|\underline{\underline{\lambda}},\underline{\chi}\|_W &\leq (\varepsilon^{-2}-c_0)\|B\|\|\vec{u},\vec{\theta}\|_V \end{aligned}$$

It follows that (26) is verified with $C = C_1^{-1} + (C_1C_2)^{-1}(\varepsilon^{-2} - c_0)\|B\|$.
Once more, we have:

$$B(\vec{v}-\vec{u},\vec{\delta}-\vec{\theta};\underline{\underline{\lambda}},\underline{\chi}) \leq (\|A\|\|\vec{u},\vec{\theta}\|_V + \|l\|_{V'})\|\vec{v}-\vec{u},\vec{\delta}-\vec{\theta}\|_V \tag{29}$$

Since (23) is verified, then:

$$\beta\|\underline{\underline{\lambda}},\underline{\chi}\|_W \leq \|A\|\|\vec{u},\vec{\theta}\|_V + \|l\|_{V'} \tag{30}$$

Then, (26) is verified with $C = \beta^{-1}(\|A\|C_1^{-1}+1)+C_1^{-1}$ independent of the small parameter ε. □

5. Conclusion

The equilibrium state of a flexural elastic thin Naghdi shell with obstacle solves a constrained minimization problem over a convex space. In this work, we use the asymptotic analysis to show the bending dominated behavior of a bending dominated Naghdi shell with obstacle therefore numerical locking is expected. The standard finite element methods are unable to approximate well a non inhibited shell problem for low thickness. Then, we propose a locking free mixed variational formulation to approximate a non trivial inextensional displacements for a non inhibited Naghdi shell with obstacle. The theoretical well-posedness of the proposed mixed scheme is established. Numerical investigations should justify the robustness of the proposed mixed problem in front of the numerical locking.

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