Stepanov type pseudo Bloch periodic functions and applications to some evolution equations in Banach spaces

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Abstract

The main purpose of this paper is to investigate some quasi-Bloch periodic functions in Stepanov sense and their applications in abstract spaces. We introduce quasi-Bloch periodic functions such as Stepanov type Bloch periodic functions and Stepanov type pseudo Bloch periodic functions, and establish some properties of these functions including completeness, composition and convolution theorems. We also apply the obtained results to investigate the existence and uniqueness of pseudo Bloch periodic solutions to some semi-linear evolutionary equations in Banach spaces.

Keywords: Stepanov type pseudo Bloch periodic functions, existence and uniqueness, evolution equations.

Mathematics Subject Classification (2010): 34K13, 58D25, 46E40.

1 Introduction

The periodicity is a natural and important phenomenon in the real word, and evolution equations are usually expected to have periodic solutions [1–4]. As it is known, when a periodic function or an anti-periodic function carries different perturbations, it is not necessarily a periodic function or an anti-periodic function, but may have other recurrence, such as pseudo periodicity or pseudo anti-periodicity, which was pointed out in [5] as generalizations of periodicity or anti-periodicity. However the aforementioned functions are usually studied in a bounded continuous space. If the continuity is weaken to the measurability and integrability in the sense of Lebesgue, a new generalized periodic function can be obtained [6, 7]. With the support of these theories in [6, 7], a large number of studies in abstract spaces have emerged, see for instance [8–18]. Particularly, Xia [17] and Alvarez [18] introduced some new concepts, and further generalized pseudo periodic functions and pseudo anti-periodic functions from the perspective of Stepanov boundedness, respectively.

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On the other hand, the Bloch periodic function widely exists in the condensed matter and solid state physics [20, 21], which includes periodic functions and anti-periodic functions. Similar to cases for the usual periodic functions, various quasi-Bloch periodic functions under different perturbations in abstract spaces have been studied [22]. For instance, Hasler and N'Guérékata [23] considered the perturbation that disappears at infinity and initiated the concept of asymptotic Bloch periodic functions. Wei and Chang [25] introduced pseudo Bloch periodic functions. Salah, Miraoui and Khemili [26] further presented measure pseudo S-asymptotically Bloch periodic functions in Banach spaces. However the pseudo Bloch periodic function [25] in Stepanov sense has not been considered yet. Thus we introduce some quasi-Blcoh periodic functions in Banach spaces called Stepanov type Bloch periodic functions and Stepanov type pseudo Bloch periodic functions, and establish the completeness, composition and convolution theorems for such functions. It can be shown that the Stepanov type Bloch periodic function extends the Bloch periodic function (see Remark 3.1) and the Stepanov type pseudo Bloch periodic function generalizes the pseudo Bloch periodic function [25] (see Lemma 3.6). In addition, it is easy to see that Stepanov type pseudo periodic functions [17] and Stepanov type pseudo antiperiodic functions [18] are special cases of Stepanov type pseudo Bloch periodic functions at $k\omega = 2\pi$ and $k\omega = \pi$, respectively. Finally, we investigate the existence and uniqueness of pseudo Bloch periodic solutions to evolution equations with Stepanov force term in Banach spaces.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we introduce notions of Stepanov type Bloch periodic functions and Stepanov type pseudo Bloch periodic functions, and explore some further properties. Section 4 is devoted to applications to some evolution equations in Banach spaces. A conclusion is summarized in Section 5.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ be the set of all real numbers and complex numbers, respectively. Let $(X, \|\cdot\|)$ be a Banach space and $BC(\mathbb{R}, X)$ be the Banach space formed by all bounded continuous functions $f : \mathbb{R} \to X$ with sup-norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. The set $C_0(\mathbb{R}, X)$ consists of all functions $f : \mathbb{R} \to X$ with $\lim_{t\to\infty} \|f(t)\| = 0$. The space $L^p(\mathbb{R}, X)$ denotes the Banach space of p-Bochner integrable functions defined on $\mathbb{R}$ with values in X. The notation $L^p_{loc}(\mathbb{R}, X)$ stands for the set of all measurable functions $f : \mathbb{R} \to X$ such that the restriction of f to every bounded subinterval I of $\mathbb{R}$ is in $L^p(I, X)$. Furthermore, we denote by $BC(\mathbb{R} \times X, X)$ the set of all functions $f : \mathbb{R} \times X \to X$ such that $f(\cdot, x) \in BC(\mathbb{R}, X)$ uniformly for each x in any bounded subset of X, $\mathcal{B}(X)$ the space of all bounded linear operators from X into itself.

The following Definition 2.1 and Lemma 2.1 can be found in [23] for details.

Definition 2.1 For given $\omega, k \in \mathbb{R}$, a function $f \in BC(\mathbb{R}, X)$ is called be Bloch periodic if for all $t \in \mathbb{R}$, $f(t + \omega) = e^{ik\omega} f(t)$. We denote by $BP_{\omega,k}(\mathbb{R}, X)$ the space of all Bloch periodic functions from $\mathbb{R}$ to X.

Lemma 2.1 Let $g \in BP_{\omega,k}(\mathbb{R}, X)$ and $\epsilon > 0$ be given. Then there exist $s_1, ..., s_m \in \mathbb{R}$ such that $\mathbb{R} = \bigcup_m^{i=1}(s_i + C_\epsilon)$, where $C_\epsilon := \{t \in \mathbb{R} : \|g(t) - g(0)\| < \epsilon\}$.

Next, to facilitate the definition of new concepts, we introduce the following spaces.

$$\mathscr{E}(\mathbb{R},X):=\Big\{h\in BC(\mathbb{R},X):\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|h(t)\|dt=0\Big\};$$

$$\mathscr{E}(\mathbb{R}\times X,X):=\Big\{h\in BC(\mathbb{R}\times X,X):\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|h(t,x)\|dt=0 \\ \text{uniformly for } x \text{ in any compact subset of } X\Big\}.$$

Definition 2.2 [25] A function $f\in BC(\mathbb{R},X)$ is called to be a pseudo Bloch periodic function, if there exists $g\in BP_{\omega,k}(\mathbb{R},X)$ and $h\in\mathscr{E}(\mathbb{R},X)$ such that

$$f=g+h.$$

We denote the set of all such functions by $PBP_{\omega,k}(\mathbb{R},X)$.

The following contents are specified in [9, 28].

Definition 2.3 The Bochner transform $f^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, of a function $f:\mathbb{R}\to X$ is defined by $f^b(t,s)=f(t+s)$.

Remark 2.1 (i) A function $\varphi(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$ is the Bochner transform of a certain function f,

$$\varphi(t,s)=f^b(t,s),$$

if and only if

$$\varphi(t+\tau,s-\tau)=\varphi(s,t),$$

for all $t\in\mathbb{R}$, $s\in[0,1]$, and $\tau\in[s-1,s]$.

(ii) Note that if $f=g+h$, then $f^b=g^b+h^b$. Moreover, $(\lambda f)^b=\lambda f^b$ for any $\lambda\in\mathbb{R}$.

Definition 2.4 The Bochner transform $F^b(t,s,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $u\in X$ of a a function $F(t,u)$ on $\mathbb{R}\times X$, with values in X, is defined by

$$F^b(t,s,u)=F(t+s,u),$$

for each $u\in X$.

We always let $p\in[1,\infty)$ throughout this paper.

Definition 2.5 The space $BS^p(\mathbb{R},X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\to X$ such that $f^b\in L^\infty(\mathbb{R},L^p([0,1],X))$. It is a Banach space with the norm

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}.$$

The notation $BS^p(\mathbb{R}\times X, X)$ represents all functions $f:\mathbb{R}\times X\to X$, which is Stepanov bounded uniformly in $x\in X$.

Lemma 2.2 [29] Let $f\in BS^p(\mathbb{R},X)$, then $f^b\in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$ if and only if for every $\epsilon>0$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{M_{T,\epsilon}(f)} dt = 0,$$

where $M_{T,\epsilon}(f)=\Big\{t\in[-T,T]:\Big(\int_t^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\geq\epsilon\Big\}$.

3 Generalized Bloch periodic functions in Stepanov sense

In this section, we introduce two kinds of functions which are Stepanov generalizations of the functions in [25], and explore their properties.

3.1 Stepanov type Bloch periodic functions

Definition 3.1 A function $f\in BS^p(\mathbb{R},X)$ is called Stepanov type Bloch periodic (or S^p-Bloch periodic) if $f^b\in BP_{\omega,k}(\mathbb{R},L^p([0,1],X))$.

In other words, a function $f\in L^p_{loc}(\mathbb{R},X)$ is said to be Stepanov type Bloch periodic if its Bochner transform $f^b:\mathbb{R}\to L^p([0,1],X)$ is Bloch periodic in the sense that for given $\omega,k\in\mathbb{R}$,

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|^p ds\Big)^{\frac{1}{p}}=0.$$

The collection of all such functions will be denoted by $S^pBP_{\omega,k}(\mathbb{R},X)$.

Remark 3.1 It is clear that if $f\in BP_{\omega,k}(\mathbb{R},X)$, then $f\in S^pBP_{\omega,k}(\mathbb{R},X)$ for each $1\leq p<\infty$.

Lemma 3.1 Assume that $1\leq q<p<\infty$ and $f\in S^pBP_{\omega,k}(\mathbb{R},X)$. Then $f\in S^qBP_{\omega,k}(\mathbb{R},X)$.

Proof: Since $\|f\|_{S^q}=\sup_{t\in\mathbb{R}}\big(\int_t^{t+1}\|f(s)\|^q ds\big)^{\frac{1}{q}}\leq\sup_{t\in\mathbb{R}}\big(\int_t^{t+1}\|f(s)\|^p ds\big)^{\frac{1}{p}}=\|f\|_{S^p}$ for $q\in[1,p)$, we have $f\in BS^q(\mathbb{R},X)$. Similarly, by the definition of S^p-Bloch periodic functions, it is easy to see that

$$\Big(\int_t^{t+1}\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|^q ds\Big)^{\frac{1}{q}}\leq\Big(\int_t^{t+1}\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|^p ds\Big)^{\frac{1}{p}},$$

which implies that $f^b\in BP_{\omega,k}(\mathbb{R},L^q([0,1],X))$. The proof is complete. ■

Lemma 3.2 Let $f_1,f_2,f\in S^pBP_{\omega,k}(\mathbb{R},X)$. Then the following holds:

(1) $f_1+f_2\in S^pBP_{\omega,k}(\mathbb{R},X)$, and $cf\in S^pBP_{\omega,k}(\mathbb{R},X)$ for each $c\in\mathbb{C}$.

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(2) The translated $f_a := f(t+a) \in S^pBP_{\omega,k}(\mathbb{R}, X)$ for any $a \in \mathbb{R}$.

(3) The space $(S^pBP_{\omega,k}(\mathbb{R}, X), \|\cdot\|_{S^p})$ is a Banach space.

Proof: (1) By the Minkowski's Lemma, we get

$$\begin{aligned} \|f_1 + f_2\|_{S^p} &= \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f_1(s) + f_2(s)\|^p ds \Big)^{\frac{1}{p}} \\ &\leq \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f_1(s)\|^p ds \Big)^{\frac{1}{p}} + \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f_2(s)\|^p ds \Big)^{\frac{1}{p}} \\ &\leq \|f_1\|_{S^p} + \|f_2\|_{S^p}, \end{aligned}$$

and

$$\begin{aligned} \|cf\|_{S^p} &= \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|cf(s)\|^p ds \Big)^{\frac{1}{p}} \\ &\leq |c| \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p ds \Big)^{\frac{1}{p}} \\ &\leq |c| \|f\|_{S^p}, \end{aligned}$$

which indicates that $f_1 + f_2$, $cf \in BS^p(\mathbb{R}, X)$. Similarly, by Remark 2.1 and Definition 3.1, we have $(f_1 + f_2)^b$, $(cf)^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$.

(2) Obviously, for any $a \in \mathbb{R}$,

$$\|f_a\|_{S^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s+a)\|^p ds \right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}} \left(\int_{t+a}^{t+a+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} = \|f\|_{S^p},$$

and

$$\begin{aligned} &\left(\int_t^{t+1} \|f_a(s+\omega) - e^{ik\omega} f_a(s)\|^p ds \right)^{\frac{1}{p}} \\ &= \left(\int_t^{t+1} \|f(s+a+\omega) - e^{ik\omega} f(s+a)\|^p ds \right)^{\frac{1}{p}} \\ &= \left(\int_{t+a}^{t+a+1} \|f(s+\omega) - e^{ik\omega} f(s)\|^p ds \right)^{\frac{1}{p}} \\ &= \left(\int_{t_1}^{t_1+1} \|f(s+\omega) - e^{ik\omega} f(s)\|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

it is obvious that $f_a \in S^pBP_{\omega,k}(\mathbb{R}, X)$ by $f \in S^pBP_{\omega,k}(\mathbb{R}, X)$. Thus $S^pBP_{\omega,k}(\mathbb{R}, X)$ is translation invariant.

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(3) We can deduce that $S^pBP_{\omega,k}(\mathbb{R},X)$ is a closed subspace of $BS^p(\mathbb{R},X)$. Let $\{f_n\}_n \subset S^pBP_{\omega,k}(\mathbb{R},X)$ be a Cauchy sequence for the norm $\|\cdot\|_{S^p}$ and $f_n \to f$ as $n \to \infty$. Then for any $\epsilon > 0$, there exists a constant $N > 0$ such that

$$\left(\int_t^{t+1} \|f_n(s) - f(s)\|^p ds\right)^{\frac{1}{p}} < \frac{\epsilon}{2},$$

for every $n > N$ and $t \in \mathbb{R}$. Notice that

$$\|f\|_{S^p} \leq \|f - f_n\|_{S^p} + \|f_n\|_{S^p} < \frac{\epsilon}{2} + \|f_n\|_{S^p},$$

then $f \in BS^p(\mathbb{R},X)$. Next, we show that $f^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1],X))$, i.e.,

$$\begin{aligned}
&\left(\int_t^{t+1} \|f(s+\omega) - e^{ik\omega}f(s)\|^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_t^{t+1} \|f(s+\omega) - f_n(s+\omega)\|^p ds\right)^{\frac{1}{p}} + \left(\int_t^{t+1} \|f_n(s+\omega) - e^{ik\omega}f_n(s)\|^p ds\right)^{\frac{1}{p}} \\
&\quad + \left(\int_t^{t+1} \|e^{ik\omega}f_n(s) - e^{ik\omega}f(s)\|^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_t^{t+1} \|f(s+\omega) - f_n(s+\omega)\|^p ds\right)^{\frac{1}{p}} + \left(\int_t^{t+1} \|f_n(s+\omega) - e^{ik\omega}f_n(s)\|^p ds\right)^{\frac{1}{p}} \\
&\quad + \left(\int_t^{t+1} \|f_n(s) - f(s)\|^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_t^{t+1} \|f(s+\omega) - f_n(s+\omega)\|^p ds\right)^{\frac{1}{p}} + \left(\int_t^{t+1} \|f_n(s) - f(s)\|^p ds\right)^{\frac{1}{p}} < \epsilon,
\end{aligned}$$

as $n \to \infty$. Therefore $f \in S^pBP_{\omega,k}(\mathbb{R},X)$, which implies that the space $S^pBP_{\omega,k}(\mathbb{R},X)$ is a closed subspace of $BS^p(\mathbb{R},X)$. Thus $S^pBP_{\omega,k}(\mathbb{R},X)$ is a Banach space with the norm $\|\cdot\|_{S^p}$. ■

Theorem 3.1 Let $f \in BS^p(\mathbb{R} \times X, X)$. Assume that the following conditions hold:

(I) $f(t+\omega, e^{ik\omega}x) = e^{ik\omega}f(t,x)$ a.e. $t \in \mathbb{R}$ and each $x \in X$,

(II) There exists a constant $L > 0$ such that for all x, $y \in X$ and $t \in \mathbb{R}$,

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|.$$

Then for each $\varphi \in S^pBP_{\omega,k}(\mathbb{R},X)$, $f(\cdot,\varphi(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R},X)$.

Proof: For each $\varphi \in S^pBP_{\omega,k}(\mathbb{R},X)$ and all $t \in \mathbb{R}$, we have

$$\left(\int_t^{t+1} \left\|f(s+\omega, \varphi(s+\omega)) - e^{ik\omega}f(s,\varphi(s))\right\|^p ds\right)^{\frac{1}{p}}$$

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$$
\begin{aligned}
&\le \left(\int_t^{t+1} \left\|f(s+\omega,\varphi(s+\omega)) - f(s+\omega, e^{ik\omega}\varphi(s))\right\|^p ds\right)^{\frac{1}{p}} \\
&\quad + \left(\int_t^{t+1} \left\|f(s+\omega, e^{ik\omega}\varphi(s)) - e^{ik\omega} f(s,\varphi(s))\right\|^p ds\right)^{\frac{1}{p}} \\
&\le L\left(\int_t^{t+1} \left\|\varphi(s+\omega) - e^{ik\omega}\varphi(s)\right\|^p ds\right)^{\frac{1}{p}} \\
&\quad + \left(\int_t^{t+1} \left\|f(s+\omega, e^{ik\omega}\varphi(s)) - e^{ik\omega} f(s,\varphi(s))\right\|^p ds\right)^{\frac{1}{p}} \\
&= 0,
\end{aligned}
$$

that is to say that $f(t+\omega,\varphi(t+\omega)) = e^{ik\omega} f(t,\varphi(t))$ a.e. $t\in\mathbb{R}$ and consequently $f(\cdot,\varphi(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R},X)$. ■

From the above proof, we can see that (II) can be simplified to the condition (III).

(III): there exists a constant $L^*>0$ such that for all z_1, $z_2\in BS^p(\mathbb{R},X)$ and $t\in\mathbb{R}$,

$$
\left(\int_t^{t+1} \|f(s,z_1(s)) - f(s,z_2(s))\|^p ds\right)^{\frac{1}{p}} \le L^*\left(\int_t^{t+1} \|z_1(s)-z_2(s)\|^p ds\right)^{\frac{1}{p}}.
$$

Thus, we get the following corollary.

Corollary 3.1 Let $f\in BS^p(\mathbb{R}\times X,X)$. If (I) and (III) hold, then for each $\varphi\in S^pBP_{\omega,k}(\mathbb{R},X)$, $f(\cdot,\varphi(\cdot))\in S^pBP_{\omega,k}(\mathbb{R},X)$.

Theorem 3.2 Let $p\in(1,+\infty)$. Assume that $f\in BS^p(\mathbb{R}\times X,X)$ and verifies (I) and the following condition:

(IV) There exists a function $l(t)\in BS^r(\mathbb{R},\mathbb{R}^+)$ with $r\ge\max\{p,\frac{p}{p-1}\}$ such that for all x, $y\in X$ and $t\in\mathbb{R}$,

$$
\|f(t,x)-f(t,y)\| \le l(t)\|x-y\|.
$$

Then for each $\varphi\in S^pBP_{\omega,k}(\mathbb{R},X)$, there exists $q\in[1,p)$ such that $f(\cdot,\varphi(\cdot))\in S^qBP_{\omega,k}(\mathbb{R},X)$.

Proof: It is easy to get that $f\in BS^q(\mathbb{R}\times X,X)$. Next, since $r\ge p/(p-1)$, we can find a constant $q\in[1,p)$ such that $r=pq/(p-q)$. Let $p'=p/(p-q)$, $q'=p/q$, there are $p'>1$, $q'>1$ and $1/p'+1/q'=1$. Then for each $\varphi\in S^pBP_{\omega,k}(\mathbb{R},X)$, we have

$$
\begin{aligned}
&\left(\int_t^{t+1} \left\|f(s+\omega,\varphi(s+\omega)) - e^{ik\omega} f(s,\varphi(s))\right\|^q ds\right)^{\frac{1}{q}} \\
&\le \left(\int_t^{t+1} \left\|f(s+\omega,\varphi(s+\omega)) - f(s+\omega, e^{ik\omega}\varphi(s))\right\|^q ds\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_t^{t+1}\left\|f(s+\omega,e^{ik\omega}\varphi(s))-e^{ik\omega}f(s,\varphi(s))\right\|^q ds\right)^{\frac{1}{q}}\\
\le&\left(\int_t^{t+1} l^q(s+\omega)\left\|\varphi(s+\omega)-e^{ik\omega}\varphi(s)\right\|^q ds\right)^{\frac{1}{q}}\\
\le&\left(\int_t^{t+1} l^{qp'}(s+\omega)ds\right)^{\frac{1}{qp'}}\left(\int_t^{t+1}\left\|\varphi(s+\omega)-e^{ik\omega}\varphi(t)\right\|^{qq'} ds\right)^{\frac{1}{qq'}}\\
\le&\left(\int_t^{t+1} l^r(s+\omega)ds\right)^{\frac{1}{r}}\left(\int_t^{t+1}\left\|\varphi(s+\omega)-e^{ik\omega}\varphi(t)\right\|^p ds\right)^{\frac{1}{p}}\\
\le&\left(\int_{t+\omega}^{t+\omega+1} l^r(s)ds\right)^{\frac{1}{r}}\left(\int_t^{t+1}\left\|\varphi(s+\omega)-e^{ik\omega}\varphi(t)\right\|^p ds\right)^{\frac{1}{p}}\\
\le&\|l\|_{S^r}\left(\int_t^{t+1}\left\|\varphi(s+\omega)-e^{ik\omega}\varphi(t)\right\|^p ds\right)^{\frac{1}{p}}\\
\le&\, 0,
\end{aligned}
$$

which implies that $f(\cdot,\varphi(\cdot))\in S^qBP_{\omega,k}(\mathbb{R},X)$. ■

Theorem 3.3 Let $f\in BS^p(\mathbb{R}\times X,X)$ satisfying (I) and the following condition:

(V) For each $\epsilon>0$, there exists a constant $\delta>0$ such that for all $t\in\mathbb{R}$ and $z_1,\, z_2\in BS^p(\mathbb{R},X)$ with $\left(\int_t^{t+1}\|z_1(s)-z_2(s)\|^p ds\right)^{\frac{1}{p}}<\delta$,

$$
\left(\int_t^{t+1}\|f(s,z_1(s))-f(s,z_2(s))\|^p ds\right)^{\frac{1}{p}}<\epsilon.
$$

Then for each $\varphi\in S^pBP_{\omega,k}(\mathbb{R},X)$, $f(\cdot,\varphi(\cdot))\in S^pBP_{\omega,k}(\mathbb{R},X)$.

Proof: From the condition (V), we get that for any $\varphi\in S^pBP_{\omega,k}(\mathbb{R},X)$,

$$
\left(\int_t^{t+1}\left\|f(s+\omega,\varphi(s+\omega))-f(s+\omega,e^{ik\omega}\varphi(s))\right\|^p ds\right)^{\frac{1}{p}}<\epsilon
$$

holds for all $t\in\mathbb{R}$. Next, by the Minkowski's inequality, we have

$$
\begin{aligned}
&\left(\int_t^{t+1}\left\|f(s+\omega,\varphi(s+\omega))-e^{ik\omega}f(s,\varphi(s))\right\|^p ds\right)^{\frac{1}{p}}\\
\le&\left(\int_t^{t+1}\left\|f(s+\omega,\varphi(s+\omega))-f(s+\omega,e^{ik\omega}\varphi(s))\right\|^p ds\right)^{\frac{1}{p}}\\
&+\left(\int_t^{t+1}\left\|f(s+\omega,e^{ik\omega}\varphi(s))-e^{ik\omega}f(s,\varphi(s))\right\|^p ds\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\leq \left(\int_t^{t+1} \left\| f(s+\omega, \varphi(s+\omega)) - f(s+\omega, e^{ik\omega}\varphi(s)) \right\|^p ds \right)^{\frac{1}{p}}$$
$$< \epsilon,$$

for $\varphi \in S^pBP_{\omega,k}(\mathbb{R}, X)$. Thus, $f(\cdot, \varphi(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$. ∎

Theorem 3.4 Assume that $f \in BS^p(\mathbb{R} \times X, X)$ satisfying (I) and the following condition:

(VI) For each $\epsilon > 0$, there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}$ and any z_1, $z_2 \in BS^p(\mathbb{R}, X)$ with $\left(\int_t^{t+1} \|z_1(s) - z_2(s)\|^p ds\right)^{\frac{1}{p}} < \delta$,

$$\left(\int_t^{t+1} \|f(s, z_1(s)) - f(s, z_2(s))\|^p ds \right)^{\frac{1}{p}} < \ell(t)\epsilon,$$

where $\ell : \mathbb{R} \to \mathbb{R}^+$ is bounded.

Then for each $\varphi \in S^pBP_{\omega,k}(\mathbb{R}, X)$, $f(\cdot, \varphi(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$.

Proof: Similar to Theorem 3.3, we can easily get that

$$\left(\int_t^{t+1} \left\| f(s+\omega, \varphi(s+\omega)) - e^{ik\omega} f(s, \varphi(s)) \right\|^p ds \right)^{\frac{1}{p}} < \ell(t)\epsilon$$

via the condition (VI). This show that $f(\cdot, \varphi(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$ by the boundedness of ℓ. ∎

Remark 3.2 As can be seen from the proofs in Theorems 3.1, 3.2, 3.3 and 3.4, the condition (I) can be weakened by

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(t+\omega, e^{ik\omega}x) - e^{ik\omega} f(t, x)\|^p ds \right)^{\frac{1}{p}} = 0,$$

uniformly in $x \in X$.

We give the following assumptions:

(H1) For strongly continuous functions $\mathcal{T} : [0, \infty) \to \mathfrak{B}(X)$, there exists $\phi \in L^1(\mathbb{R}^+)$ such that $\|\mathcal{T}(t)\| \leq \phi(t)$ for all $t \in \mathbb{R}$.

Theorem 3.5 Let $p > 1$. Assume that $(\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous family of bounded linear operators satisfying the assumption (H1), where ϕ is nonincreasing. If $f \in S^pBP_{\omega,k}(\mathbb{R}, X)$, then

$$u(t) = \int_{-\infty}^t \mathcal{T}(t-s) f(s) ds \in BP_{\omega,k}(\mathbb{R}, X).$$

Proof: Let
$$u_n(t)=\int_{t-n}^{t-n+1}\mathcal{T}(t-s)f(s)ds.$$
It follows from $f\in L^p_{loc}(\mathbb{R},X)$ that for each $n\in N$ and $t\in\mathbb{R}$,
$$\begin{aligned}\|u_n(t+h)-u_n(t)\| &\le \int_{t-n}^{t-n+1}\phi(t-s)\|f(s+h)-f(s)\|ds\\ &\le \phi(n-1)\left(\int_{t-n}^{t-n+1}\|f(s+h)-f(s)\|^p ds\right)^{\frac{1}{p}}\to 0,\quad as\ h\to 0.\end{aligned}$$
This shows that $u_n(t)$ is continuous. By the Hölder inequality, we have
$$\begin{aligned}\|u_n(t)\| &\le \int_{t-n}^{t-n+1}\|\mathcal{T}(t-s)f(s)\|\,ds\\ &\le \phi(n-1)\left(\int_{t-n}^{t-n+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\\ &\le \phi(n-1)\|f\|_{S^p},\end{aligned}$$
since
$$\begin{aligned}\sum_{n=1}^{\infty}\phi(n-1)\|f\|_{S^p} &\le \left(\phi(0)+\sum_{n=2}^{\infty}\int_{n-2}^{n-1}\phi(t)dt\right)\|f\|_{S^p}\\ &\le \left(\phi(0)+\|\phi\|_{L^1}\right)\|f\|_{S^p}<\infty,\end{aligned}$$
then $\sum_{n=1}^{\infty}u_n(t)$ is uniformly convergent on $\mathbb{R}$. Thus $u(t):=\int_{-\infty}^{t}\mathcal{T}(t-s)f(s)ds=\sum_{n=1}^{\infty}u_n(t)\in BC(\mathbb{R},X)$. In addition,
$$\begin{aligned}\|u(t+\omega)-e^{ik\omega}u(t)\| &= \left\|\int_{-\infty}^{t+\omega}\mathcal{T}(t+\omega-s)f(s)ds-e^{ik\omega}\int_{-\infty}^{t}\mathcal{T}(t-s)f(s)ds\right\|\\ &= \left\|\int_{-\infty}^{t}\mathcal{T}(t-s)\left[f(s+\omega)-e^{ik\omega}f(s)\right]ds\right\|.\end{aligned}$$
Let $Y_n(t)=\int_{t-n}^{t-n+1}\mathcal{T}(t-s)\big[f(s+\omega)-e^{ik\omega}f(s)\big]ds$, we know that
$$\begin{aligned}\|Y_n(t)\| &= \left\|\int_{t-n}^{t-n+1}\mathcal{T}(t-s)\big[f(s+\omega)-e^{ik\omega}f(s)\big]ds\right\|\\ &\le \int_{t-n}^{t-n+1}\big\|\mathcal{T}(t-s)\big\|\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|ds\\ &\le \phi(0)\int_{t-n}^{t-n+1}\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|ds\\ &\le \phi(0)\left(\int_{t-n}^{t-n+1}\big\|f(s+\omega)-e^{ik\omega}f(s)\big\|^p ds\right)^{\frac{1}{p}}\end{aligned}$$

$$\le \phi(0)\left(\int_t^{t+1}\left\|f(s+n+\omega)-e^{ik\omega}f(s+n)\right\|^p ds\right)^{\frac{1}{p}}$$
$$\le 0$$

by Lemma 3.2 and $f \in S^pBP_{\omega,k}(\mathbb{R},X)$. It follows that $\sum_{n=1}^{\infty} Y_n(t)$ is uniform convergent to $\int_{-\infty}^{t}\mathcal{T}(t-s)\big[f(s+\omega)-e^{ik\omega}f(s)\big]ds$ on $\mathbb{R}$ and $\sum_{n=1}^{\infty} Y_n(t)=0$. Thus, $u(t)\in BP_{\omega,k}(\mathbb{R},X)$. ■

3.2 Stepanov type pseudo Bloch periodic functions

Definition 3.2 A function $f \in BS^p(\mathbb{R},X)$ is said to be Stepanov type pseudo Bloch periodic (or S^p-pseudo Bloch periodic) if it can be decomposed as $f = g + h$, where $g^b \in BP_{\omega,k}(\mathbb{R},L^p([0,1],X))$ and $h^b \in \mathscr{E}(\mathbb{R},L^p([0,1],X))$.

In other words, a function $f \in L^p_{loc}(\mathbb{R},X)$ is said to be Stepanov type pseudo Bloch periodic if its Bochner transform $f^b:\mathbb{R}\to L^p([0,1],X)$ is pseudo Bloch periodic in the sense that there exist two functions g, $h:\mathbb{R}\to X$ such that $f=g+h$, where $g^b\in BP_{\omega,k}(\mathbb{R},L^p([0,1],X))$ and $h^b\in\mathscr{E}(\mathbb{R},L^p([0,1],X))$, i.e.,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h(s)\|^p ds\right)^{\frac{1}{p}} dt = 0.$$

The set of all such functions is denoted by $S^pPBP_{\omega,k}(\mathbb{R},X)$.

Remark 3.3 Especially, when h^b in Definition 3.2 belongs to $C_0(\mathbb{R},L^p([0,1],X))$, we can get the concept of Stepanov type asymptotically Bloch periodicity (or S^p-asymptotically Bloch periodicity).

Lemma 3.3 Let $f\in S^pPBP_{\omega,k}(\mathbb{R},X)$ be such that $f=g+h$, where $g^b\in BP_{\omega,k}(\mathbb{R},L^p([0,1],X))$ and $h^b\in\mathscr{E}(\mathbb{R},L^p([0,1],X))$, then

$$\{g(t+\cdot):t\in\mathbb{R}\}\subseteq\overline{\{f(t+\cdot):t\in\mathbb{R}\}} \quad \text{in} \quad L^p([0,1],X).$$

Proof: If the assertion is not true, then there exists a constant $t_0\in\mathbb{R}$ such that $g(t_0+\cdot)\notin\overline{\{f(t+\cdot):t\in\mathbb{R}\}}$. Without losing generality, let $t_0=0$, we can get that there exists $\epsilon>0$, such that

$$\|g(\cdot)-f(t+\cdot)\|_p\ge 2\epsilon, \text{ for all } t\in\mathbb{R}.$$

By Lemma 2.1, we obtain

$$\|h(t+\cdot)\|_p=\|f(t+\cdot)-g(t+\cdot)\|_p\ge\|f(t+\cdot)-g(\cdot)\|_p-\|g(\cdot)-g(t+\cdot)\|_p>\epsilon,$$

for all $t\in D_\epsilon$, where $D_\epsilon:=\{t\in\mathbb{R}:\|g(\cdot)-g(t+\cdot)\|_p<\epsilon\}$, $\mathbb{R}=\bigcup_m^{i=1}(s_i^*+D_\epsilon)$. Hence,

$$\|h(t+\cdot-s_i^*)\|_p>\epsilon,$$

for each $i \in 1,...,m$ and $t \in s_i^* + D_\epsilon$. Now, we define the function Q by

$$Q(t+\cdot) = \sum_{i=1}^{m} \|h(t+\cdot - s_i^*)\|_p.$$

From the above inequality, we can see that

$$Q(t+\cdot) > \epsilon,$$

for all $t \in \mathbb{R}$. On the other hand, by the translation invariance of $\mathscr{E}(\mathbb{R}, L^p([0,1], X))$ which has been proved in Theorem 3.2 of Ref. [11], we conclude that $h^b(t - s_i) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$ for all $i \in 1,...,m$, and thus $Q^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$, which is contradiction to $Q^b(t) > \epsilon$. So, the conclusion is true. ■

Proposition 3.1 The decomposition of a Stepanov type pseudo Bloch periodic function in Definition 3.2 is unique.

Proof: Assume that $f = g_1 + h_1 = g_2 + h_2$ with $g_i^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$ and $h_i^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$ for $i = 1, 2$. Then we have $0 = (g_1 - g_2) + (h_1 - h_2) \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ with $(g_1 - g_2)^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$ and $(h_1 - h_2)^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. From Lemma 3.3, we obtain $(g_1 - g_2)(\mathbb{R} + \cdot) \subseteq 0$. Hence, we have $g_1 = g_2$ and $h_1 = h_2$. ■

Lemma 3.4 Assume that f, f_1, $f_2 \in S^pPBP_{\omega,k}(\mathbb{R}, X)$. Then the following holds:

(1) $f_1 + f_2 \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

(2) $cf \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ for any $c \in \mathbb{C}$.

(3) The translated $f_a \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ for each $a \in \mathbb{R}$.

Proof: Let $f_1 = g_1 + h_1$, $f_2 = g_2 + h_2$, $f = g + h$ where g_1, g_2, $g \in S^pBP_{\omega,k}(\mathbb{R}, X)$, h_1^b, h_2^b, $h^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. It is easy to get that $f_1 + f_2$, $cf \in BS^p(\mathbb{R}, X)$ and obtain that $g_1 + g_2$, $cg \in S^pBP_{\omega,k}(\mathbb{R}, X)$ by Lemma 3.2. Similarly, we have

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|h_1(s) + h_2(s)\|^p ds\right)^{\frac{1}{p}} dt \\
&\leq \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|h_1(s)\|^p ds\right)^{\frac{1}{p}} dt + \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|h_2(s)\|^p ds\right)^{\frac{1}{p}} dt \\
&\to 0 \ \ as\ T \to \infty,
\end{aligned}$$

and

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|ch(s)\|^p ds\right)^{\frac{1}{p}} dt = \frac{|c|}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|h(s)\|^p ds\right)^{\frac{1}{p}} dt \to 0 \ \ as\ T \to \infty,$$

that is $(h_1 + h_2)^b$, $(ch)^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. Therefore, $f_1 + f_2$, $cf \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

(3) It follows from Lemma 3.2 that $g_a \in S^pBP_{\omega,k}(\mathbb{R}, X)$. In addition, $h_a^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$ can be deduced by the proof of Theorem 3.2 in [11]. Thus, the conclusion is obtained. ■

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Lemma 3.5 The space $\left(S^pPBP_{\omega,k}(\mathbb{R},X), \|\cdot\|_{S_p}\right)$ is a Banach space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $S^pPBP_{\omega,k}(\mathbb{R},X)$. We write $f_n = g_n + h_n$ with $g_n^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1],X))$, $h_n^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$. From Lemma 3.3, we have

$$\|(g_{m_1} - g_{m_2})(t+\cdot)\|_p \leq \|(f_{m_1} - f_{m_2})(t+\cdot)\|_p,$$

that is

$$\|g_{m_1} - g_{m_2}\|_{S^p} \leq \|f_{m_1} - f_{m_2}\|_{S^p},$$

therefore $\{g_n\}$ is a Cauchy sequence in the Banach space $(S^pBP_{\omega,k}(\mathbb{R},X), \|\cdot\|_{S^p})$. There exists a function $g \in S^pBP_{\omega,k}(\mathbb{R},X)$, that is $g^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1],X)$, such that $\|g_n - g\|_{S^p} \to 0$ as $n \to \infty$. Based on the above facts, there exists a function $h \in BS^p(\mathbb{R},X)$ such that $\|h_n - h\|_{S^p} \to 0$ as $n \to \infty$.

Next, we prove that $h^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$, i.e.,

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h(s)\|^p ds\right)^{\frac{1}{p}} dt \\
&\leq \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h_n(s) - h(s)\|^p ds\right)^{\frac{1}{p}} dt + \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h_n(s)\|^p ds\right)^{\frac{1}{p}} dt \\
&\leq \|h_n - h\|_{S^p} + \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h_n(s)\|^p ds\right)^{\frac{1}{p}} dt.
\end{aligned}$$

Let $n \to \infty$, we get that $h^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$ with the help of $h_n^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$, and conclude $\lim_{n\to\infty} f_n = g + h \in S^pPBP_{\omega,k}(\mathbb{R},X)$. These arguments imply that the space $S^pPBP_{\omega,k}(\mathbb{R},X)$ is a closed subspace of $BS^p(\mathbb{R},X)$. Therefore, $S^pPBP_{\omega,k}(\mathbb{R},X)$ is a Banach space equipped with the norm $\|\cdot\|_{S^p}$. ■

Lemma 3.6 Assume that $f \in PBP_{\omega,k}(\mathbb{R},X)$, then $f \in S^pPBP_{\omega,k}(\mathbb{R},X)$. Further, if there is a constant q satisfying $1 \leq q < p$, then $f \in S^qPBP_{\omega,k}(\mathbb{R},X)$.

Proof: Let $f = g + h$ where $g \in BP_{\omega,k}(\mathbb{R},X)$ and $h \in \mathscr{E}(\mathbb{R},X)$. First of all, it is obvious that $f \in BS^p(\mathbb{R},X)$. And from Remark 3.1, we have that $g^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1],X))$. To prove this lemma, it suffices to show that $h^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$. For $T > 0$, we have the following inequality

$$\begin{aligned}
\int_{-T}^{T}\left(\int_0^1 \|h(t+s)\|^p ds\right)^{\frac{1}{p}} dt &\leq \int_{-T}^{T}\left(\int_0^1 \sup_{s\in[0,1]}\|h(t+s)\|^p ds\right)^{\frac{1}{p}} dt \\
&\leq \int_{-T}^{T}\left(\sup_{s\in[0,1]}\|h(t+s)\|^p\right)^{\frac{1}{p}} dt.
\end{aligned}$$

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Let $s_0 \in [0,1]$ such that $\sup_{s\in[0,1]} \|h(t+s)\| = \|h(t+s_0)\|$. Using the above inequality, it follows that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \|h(t+s)\|^p ds\right)^{\frac{1}{p}} dt &\le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\sup_{s\in[0,1]} \|h(t+s)\|^p\right)^{\frac{1}{p}} dt \\ &\le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|h(t+s_0)\| dt \\ &= 0\end{aligned}$$

by using the fact that $\mathscr{E}(\mathbb{R}, X)$ is translation invariant. Thus, $f \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

Finally, the proof of $f \in S^qPBP_{\omega,k}(\mathbb{R}, X)$ is similar to that of Lemma 3.1 and the details are omitted here. ■

We also assume that the following condition hold:

(A1) $g \in BS^p(\mathbb{R}\times X, X)$ and $g(t+\omega, e^{ik\omega}x) = e^{ik\omega}g(t,x)$ a.e. $t \in \mathbb{R}$ and each $x \in X$.

Theorem 3.6 Let $f = g + h \in BS^p(\mathbb{R}\times X, X)$, where g satisfies (A1) and $h^b \in \mathscr{E}(\mathbb{R}\times X, L^p([0,1], X))$. Assume that $u = \alpha + \beta \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ with $\mathfrak{I} := \overline{\alpha([-T,T])}$ compact. If the following condition holds:

(C1) There exists constants $L_1 > 0$ and $L_2 > 0$ such that for all $x, y \in X$ and $t \in \mathbb{R}$,

$$\|f(t,x) - f(t,y)\| \le L_1\|x-y\|, \quad \|g(t,x) - g(t,y)\| \le L_2\|x-y\|.$$

Then $f(\cdot, u(\cdot)) \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

Proof: The function f can be re-written as

$$\begin{aligned}f(t,u(t)) &= g(t,\alpha(t)) + f(t,u(t)) - g(t,\alpha(t)) \\ &= g(t,\alpha(t)) + f(t,u(t)) - f(t,\alpha(t)) + h(t,\alpha(t)).\end{aligned}$$

Set

$$F(t) := g(t,\alpha(t)),\ G(t) := f(t,u(t)) - f(t,\alpha(t)),\ H(t) := h(t,\alpha(t)).$$

It is easy to see that $F^b(\cdot) = g^b(\cdot, u^b(\cdot)) : \mathbb{R} \to L^p([0,1], X)$, where $g^b(t, u^b(t)) = g(t+s, u(t+s))$, $t \in \mathbb{R}$, $s \in [0,1]$. Similarly,

$$G^b(t) = f^b(t, u^b(t)) - f^b(t, \alpha^b(t)),\ H^b(t) = h^b(t, \alpha^b(t)).$$

To complete the proof, it is enough to show $F(t) \in S^pBP_{\omega,k}(\mathbb{R}, X)$, $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$ and $H^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. According to (A1) and Theorem 3.1, we have $g(t,\alpha(t)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$, that is $F^b(t) = g^b(t, \alpha^b(t)) \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$.

Next, we will prove that $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. In fact, $G(t) \in BS^p(\mathbb{R}, X)$ and

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \|G(s)\|^p ds\right)^{\frac{1}{p}} dt$$

$$= \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|f(s,u(s))-f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} dt$$
$$\leq \frac{L_1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}} dt.$$

Since $\beta^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$, $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$ is true. In the end, we will show that $H^b(t) = h^b(t,\alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$. From $h^b \in \mathscr{E}(\mathbb{R}\times X, L^p([0,1],X))$, it is obvious that for any $\epsilon > 0$, there exists $T_0 > 0$ such that

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,x)\|^p ds\right)^{\frac{1}{p}} dt < \epsilon, \tag{3.1}$$

for each $T > T_0$ and $x \in X$. Secondly, due to the compactness of set $\mathfrak{I}$, one can find finite open balls $\mathfrak{B}_k$ $(k = 1,2,\cdots,m)$ with center $x_k \in \mathfrak{I}$ and radius $\delta < \epsilon$ such that $\mathfrak{I} \subseteq \bigcup_{k=1}^m \mathfrak{B}_k$. Set $\mathcal{O}_k = \{t \in [-T,T] : \alpha(t) \in \mathfrak{B}_k\}$, then $[-T,T] = \bigcup_{k=1}^m \mathcal{O}_k$. Let

$$\mathfrak{E}_1 = \mathcal{O}_1,\ \mathfrak{E}_k = \mathcal{O}_k \setminus \bigcup_{j=1}^{k-1} \mathcal{O}_j,\ 2 \leq k \leq m,$$

then $\mathfrak{E}_i \cap \mathfrak{E}_j = \emptyset$ if $i \neq j$, $1 \leq i,j \leq m$. Define a function $\bar{x} : [-T,T] \to X$ by $\bar{x}(t) = x_k$ for $t \in \mathfrak{E}_k$, $k = 1,2,...m$, then we have

$$\|\alpha(t) - \bar{x}(t)\| < \delta,\ t \in [-T,T]. \tag{3.2}$$

It follows from the Minkowski inequality that

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} dt\\
\leq &\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\alpha(s))-h(s,\bar{x}(s))\|^p\, ds\right)^{\frac{1}{p}} dt + \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\bar{x}(s))\|^p\, ds\right)^{\frac{1}{p}} dt\\
:= &I_1(T) + I_2(T),
\end{aligned} \tag{3.3}$$

where

$$I_1(T) = \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\alpha(s))-h(s,\bar{x}(s))\|^p\, ds\right)^{\frac{1}{p}} dt, \tag{3.4}$$

$$I_2(T) = \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\bar{x}(s))\|^p\, ds\right)^{\frac{1}{p}} dt. \tag{3.5}$$

Finally, we estimate I_1 and I_2, respectively. It follows from the inequality (3.2) that

$$I_1(T) = \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\alpha(s))-h(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}} dt$$

$$
\begin{aligned}
\leq & \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|f(s,\alpha(s))-f(s,\bar{x}(s))\|^{p}ds\right)^{\frac{1}{p}}dt \\
& + \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|g(s,\alpha(s))-g(s,\bar{x}(s))\|^{p}ds\right)^{\frac{1}{p}}dt \\
\leq & \frac{L_1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|\alpha(s)-\bar{x}(s)\|^{p}ds\right)^{\frac{1}{p}}dt \\
& + \frac{L_2}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|\alpha(s)-\bar{x}(s)\|^{p}ds\right)^{\frac{1}{p}}dt \\
\leq & L_1\cdot\delta+L_2\cdot\delta \\
< & (L_1+L_2)\epsilon,
\end{aligned}
\tag{3.6}
$$

for all $T > T_0$. And by (3.1), we get

$$
\begin{aligned}
I_2(T) = & \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\bar{x}(s))\|^{p}\,ds\right)^{\frac{1}{p}}dt \\
= & \frac{1}{2T}\int_{-T}^{T}\left[\sum_{k=1}^{m}\left(\int_{(t,t+1)\bigcap\mathfrak{E}_k}\|h(s,x_k)\|^{p}ds\right)\right]^{\frac{1}{p}}dt \\
\leq & m^{\frac{1}{p}}\frac{1}{2T}\int_{-T}^{T}\sum_{k=1}^{m}\left(\int_{(t,t+1)\bigcap\mathfrak{E}_k}\|h(s,x_k)\|^{p}ds\right)^{\frac{1}{p}}dt \\
\leq & m^{\frac{1}{p}}\sum_{k=1}^{m}\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,x_k)\|^{p}ds\right)^{\frac{1}{p}}dt \\
< & m^{1+\frac{1}{p}}\cdot\epsilon.
\end{aligned}
\tag{3.7}
$$

Therefore,

$$
\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\|h(s,\alpha(s))\|^{p}ds\right)^{\frac{1}{p}}dt < (L_1+L_2)\epsilon + m^{1+\frac{1}{p}}\cdot\epsilon,
$$

which means that $H^b(t) = h^b(t,\alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$. The conclusion is proved. ■

From the above proof, we get that when f and g satisfy condition (III) with f and g instead of F, respectively, Theorem 3.6 is still true.

Theorem 3.7 Let $p > 1$. Assume that $f = g + h \in BS^p(\mathbb{R}\times X, X)$ with g satisfying (A1) and $h^b \in \mathscr{E}(\mathbb{R}\times X, L^p([0,1],X))$. Suppose further that $u = \alpha + \beta \in S^pPBP_{\omega,k}(\mathbb{R},X)$ with $\mathfrak{I}$ compact. If the following condition holds:

(C2) There exists two functions $l_f(t)$ and $l_g(t) \in BS^r(\mathbb{R},\mathbb{R}^+)$ with $r \geq \max\{p, \frac{p}{p-1}\}$ such that for all x, $y \in X$ and $t \in \mathbb{R}$,

$$
\|f(t,x)-f(t,y)\| \leq l_f(t)\|x-y\|, \quad \|g(t,x)-g(t,y)\| \leq l_g(t)\|x-y\|.
$$

Then there exists $q \in [1,p)$ such that $f(\cdot,u(\cdot)) \in S^qPBP_{\omega,k}(\mathbb{R},X)$.

Proof: The proof follows a similar procedure in Theorem 3.6. Do the same decomposition of f, we can get that $g(t,\alpha(t)) \in S^pBP_{\omega,k}(\mathbb{R},X) \subset S^qBP_{\omega,k}(\mathbb{R},X)$ by Lemma 3.1 and Theorem 3.2, i.e. $F^b(t) = g^b(t,\alpha^b(t)) \in BP_{\omega,k}(\mathbb{R},L^q([0,1],X))$. Next, we show that $G^b(t) \in \mathscr{E}(\mathbb{R},L^q([0,1],X))$. It is obvious that $G^b(t) \in BS^p(\mathbb{R},X)$. We have by the Hölder inequality that

$$
\begin{aligned}
\left(\int_t^{t+1}\|G(s)\|^q ds\right)^{\frac{1}{q}} &= \left(\int_t^{t+1}\|f(s,u(s))-f(s,\alpha(s))\|^q ds\right)^{\frac{1}{q}}\\
&\le \left(\int_t^{t+1} l_f^q(s)\|\beta(s)\|^q ds\right)^{\frac{1}{q}}\\
&\le \left(\int_t^{t+1} l_f^{qp'}(s)ds\right)^{\frac{1}{qp'}}\left(\int_t^{t+1}\|\beta(s)\|^{qq'}ds\right)^{\frac{1}{qq'}}\\
&\le \left(\int_t^{t+1} l_f^{r}(s)ds\right)^{\frac{1}{r}}\left(\int_t^{t+1}\|\beta(s)\|^{p}ds\right)^{\frac{1}{p}}\\
&\le \|l_f\|_{S^r}\left(\int_t^{t+1}\|\beta(s)\|^{p}ds\right)^{\frac{1}{p}},
\end{aligned}
$$

where the parameters p' and q' here come from the proof of Theorem 3.2, then

$$
\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|G(s)\|^q ds\right)^{\frac{1}{q}}dt \le \frac{\|l_f\|_{S^r}}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}dt \to 0,
$$

as $T\to\infty$ by $\beta^b \in \mathscr{E}(\mathbb{R},L^p([0,1],X))$. Therefore, $G^b(t) \in \mathscr{E}(\mathbb{R},L^q([0,1],X))$. Finally, we prove that $H^b(t) = h^b(t,\alpha^b(t)) \in \mathscr{E}(\mathbb{R},L^q([0,1],X))$. As can be seen from the proof of Theorem 3.6, we just need to re-estimate $I_1(T)$ here. $l_f \in BS^p(\mathbb{R},\mathbb{R}^+)$ and $l_g \in BS^p(\mathbb{R},\mathbb{R}^+)$ can be obtained by $r \ge p$. According to this conclusion and the condition (C2), it is further deduced that for all $T > T_0$,

$$
\begin{aligned}
I_1(T) =& \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h(s,\alpha(s))-h(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}}dt\\
\le& \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|f(s,\alpha(s))-f(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}}dt\\
&+ \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|g(s,\alpha(s))-g(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}}dt\\
\le& \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}(l_f(s)\|\alpha(s)-\bar{x}(s)\|)^p ds\right)^{\frac{1}{p}}dt\\
&+ \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}(l_g(s)\|\alpha(s)-\bar{x}(s)\|)^p ds\right)^{\frac{1}{p}}dt
\end{aligned}
$$

$$\leq \frac{\delta}{2T}\int_{-T}^{T}\left(\int_t^{t+1}(l_f(s))^p ds\right)^{\frac{1}{p}} dt + \frac{\delta}{2T}\int_{-T}^{T}\left(\int_t^{t+1}(l_g(s))^p ds\right)^{\frac{1}{p}} dt$$
$$\leq (\|l_f\|_{S^p} + \|l_g\|_{S^p})\delta < (\|l_f\|_{S^p} + \|l_g\|_{S^p})\epsilon.$$

Thus, combining the above estimate with (3.7), we get

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^q ds\right)^{\frac{1}{q}} dt$$
$$\leq \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} dt$$
$$\leq I_1(T) + I_2(T) < (\|l_f\|_{S^p} + \|l_g\|_{S^p})\epsilon + m^{1+\frac{1}{p}} \cdot \epsilon,$$

that is $H^b(t) = h^b(t, \alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^q([0,1], X))$. The proof is complete. ■

Theorem 3.8 Let $f = g + h \in BS^p(\mathbb{R}\times X, X)$, where g satisfies the condition (A1) and $h^b \in \mathscr{E}(\mathbb{R}\times X, L^p([0,1], X))$. Assume that $u = \alpha + \beta \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ with $\mathfrak{J}$ compact. If the following condition holds:

(C3) For each $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$\left(\int_t^{t+1}\|f(s,z_1(s)) - f(s,z_2(s))\|^p\,ds\right)^{\frac{1}{p}} < \epsilon, \quad \left(\int_t^{t+1}\|g(s,z_1(s)) - g(s,z_2(s))\|^p\,ds\right)^{\frac{1}{p}} < \epsilon,$$

for all $t \in \mathbb{R}$, and any $z_1,\, z_2 \in BS^p(\mathbb{R}, X)$ with $\left(\int_t^{t+1}\|z_1(s) - z_2(s)\|^p\,ds\right)^{\frac{1}{p}} < \delta$.

Then $f(\cdot, u(\cdot)) \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

Proof: Do the same decomposition for f as in Theorem 3.6. By Theorem 3.3 and condition (C3), we can get that $F^b(t) \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$. Next, we prove that $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. In fact, for each $t \in \mathbb{R}$, $\left(\int_t^{t+1}\|\beta(s)\|^p\,ds\right)^{\frac{1}{p}} = \left(\int_t^{t+1}\|u(s) - \alpha(s)\|^p\,ds\right)^{\frac{1}{p}} < \delta$, $s \in [t, t+1]$ implies that for all $t \in \mathbb{R}$,

$$\left(\int_t^{t+1}\|G(s)\|^p ds\right)^{\frac{1}{p}} = \left(\int_t^{t+1}\|f(s,u(s)) - f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} < \epsilon.$$

So we get $M_{T,\epsilon}(G) = M_{T,\epsilon}(f(\cdot,u(\cdot)) - f(\cdot,\alpha(\cdot))) \subseteq M_{T,\delta}(\beta)$. It follows form Lemma 2.2 that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{M_{T,\delta}(\beta)} dt = 0.$$

Thus

$$\lim_{T\to\infty}\frac{1}{2T}\int_{M_{T,\epsilon}(G)} dt = 0,$$

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which show that $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$.

Next, we deduce that $H^b(t) = h^b(t, \alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. Similar to the proof in Theorem 3.6, from the compactness of set $\mathfrak{I}$ and $h^b \in \mathscr{E}(\mathbb{R} \times X, L^p([0,1], X))$, we can obtain (3.2) and (3.7). Here the radius δ is given in condition (C3). It is obvious that (3.2) implies that $\left(\int_t^{t+1} \|\alpha(s) - \bar{x}(s)\|^p \, ds\right)^{\frac{1}{p}} < \delta$, so

$$
\begin{aligned}
I_1(T) \leq & \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|f(s, \alpha(s)) - f(s, \bar{x}(s))\|^p ds \right)^{\frac{1}{p}} dt \\
& + \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|g(s, \alpha(s)) - g(s, \bar{x}(s))\|^p ds \right)^{\frac{1}{p}} dt \\
< & \epsilon \frac{1}{2T} \int_{-T}^{T} dt + \epsilon \frac{1}{2T} \int_{-T}^{T} dt \\
< & 2\epsilon.
\end{aligned}
$$

Therefore, for all $T > T_0$,

$$
\frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|h(s, \alpha(s))\|^p ds \right)^{\frac{1}{p}} dt \leq I_1(T) + I_2(T) < 2\epsilon + m^{1+\frac{1}{p}} \cdot \epsilon,
$$

which implies that $H^b(t) = h^b(t, \alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$. Thus the conclusion is proved. ■

Theorem 3.9 Let $f = g + h \in BS^p(\mathbb{R} \times X, X)$ with g satisfying (A1) and $h^b \in \mathscr{E}(\mathbb{R} \times X, L^p([0,1], X))$. Suppose that $u = \alpha + \beta \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ with $\mathfrak{I}$ compact. If the following condition holds:

(C4) For each $\epsilon > 0$, there exists a constant $\delta > 0$, such that for all $t \in \mathbb{R}$ and any z_1, $z_2 \in BS^p(\mathbb{R}, X)$ with $\left(\int_t^{t+1} \|z_1(s) - z_2(s)\|^p \, ds\right)^{\frac{1}{p}} < \delta$,

$$
\left(\int_t^{t+1} \|f(s, z_1(s)) - f(s, z_2(s))\|^p \, ds \right)^{\frac{1}{p}} < \ell_1(t)\epsilon,
$$

$$
\left(\int_t^{t+1} \|g(s, z_1(s)) - g(s, z_2(s))\|^p \, ds \right)^{\frac{1}{p}} < \ell_2(t)\epsilon,
$$

where $\ell_1 : \mathbb{R} \to \mathbb{R}^+$ satisfies $\frac{1}{2T}\int_{-T}^{T} \ell_1(t) dt < \infty$ and $\ell_2 : \mathbb{R} \to \mathbb{R}^+$ is bounded.

Then $f(\cdot, u(\cdot)) \in S^pPBP_{\omega,k}(\mathbb{R}, X)$.

Proof: Decompose the function $f(\cdot, u(\cdot))$ similar to that in Theorem 3.6, we can get that $F^b(t) \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$ by Theorem 3.4 and (C4). Next, we prove that $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$.

It is known from the condition (C4) that for each $t \in \mathbb{R}$, $\left(\int_t^{t+1} \|u(s)-\alpha(s)\|^p\, ds\right)^{\frac{1}{p}} < \delta$, $s \in [t,t+1]$ implies that

$$\left(\int_t^{t+1} \|G(s)\|^p ds\right)^{\frac{1}{p}} = \left(\int_t^{t+1} \|f(s,u(s)) - f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} < \ell_1(t)\epsilon.$$

So $M_{T,\ell_1(t)\epsilon}(G) = M_{T,\ell_1(t)\epsilon}(f(\cdot,u(\cdot)) - f(\cdot,\alpha(\cdot))) \subseteq M_{T,\delta}(\beta)$ is hold. Combine Lemma 2.2 and $\beta^b \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$, we have that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{M_{T,\ell_1(t)\epsilon}(G)} dt = 0. \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} \|G(s)\|^p ds\right)^{\frac{1}{p}} dt \\
=&\frac{1}{2T}\int_{M_{T,\ell_1(t)\epsilon}(G)} \left(\int_t^{t+1} \|G(s)\|^p ds\right)^{\frac{1}{p}} dt + \frac{1}{2T}\int_{[-T,T]\setminus M_{T,\ell_1(t)\epsilon}(G)} \left(\int_t^{t+1} \|G(s)\|^p ds\right)^{\frac{1}{p}} dt \\
<&\frac{\|G\|_{S^p}}{2T}\int_{M_{T,\ell_1(t)\epsilon}(G)} dt + \frac{\epsilon}{2T}\int_{[-T,T]\setminus M_{T,\ell_1(t)\epsilon}(G)} \ell_1(t) dt \\
<&\frac{\|G\|_{S^p}}{2T}\int_{M_{T,\ell_1(t)\epsilon}(G)} dt + \frac{\epsilon}{2T}\int_{-T}^{T} \ell_1(t) dt,
\end{aligned}$$

which show that $G^b(t) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$ by the condition (C4) and the equation (3.8). In the end, we use techniques in Theorem 3.6 to show that $H^b(t) = h^b(t,\alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$. For $\alpha(t)$ and $\bar{x}(t)$ satisfying (3.2), $\left(\int_t^{t+1} \|\alpha(s)-\bar{x}(s)\|^p\, ds\right)^{\frac{1}{p}} < \delta$ is also true. Let the radius δ be given in (C4), then we get that

$$\begin{aligned}
I_1(T) \le&\frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} \|f(s,\alpha(s)) - f(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}} dt \\
&+ \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} \|g(s,\alpha(s)) - g(s,\bar{x}(s))\|^p ds\right)^{\frac{1}{p}} dt \\
\le&\frac{\epsilon}{2T}\int_{-T}^{T} (\ell_1(t)+\ell_2(t))\, dt.
\end{aligned}$$

It follows from (3.7) and (3.3) that for all $T > T_0$,

$$\frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} \|h(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} dt \le I_1(T) + I_2(T) < \frac{\epsilon}{2T}\int_{-T}^{T} (\ell_1(t)+\ell_2(t))\, dt + m^{1+\frac{1}{p}} \cdot \epsilon.$$

Therefore, $H^b(t) = h^b(t,\alpha^b(t)) \in \mathscr{E}(\mathbb{R}, L^p([0,1],X))$ is true under the condition (C4). This completes the proof. ■

Remark 3.4 It is easy to get that the condition (A1) in Theorems 3.6, 3.7, 3.8 and 3.9 can be replaced by $g \in BS^p(\mathbb{R} \times X, X)$ and

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \big\| g(t+\omega, e^{ik\omega}x) - e^{ik\omega} g(t,x) \big\|^p ds \Big)^{\frac{1}{p}} = 0,$$

uniformly in $x \in X$.

Theorem 3.10 Let $(\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous family of bounded linear operators satisfying the assumption (H1), where ϕ is nonincreasing. If $f \in S^pPBP_{\omega,k}(\mathbb{R}, X)$, then

$$u(t) = \int_{-\infty}^t \mathcal{T}(t-s) f(s) ds \in PBP_{\omega,k}(\mathbb{R}, X).$$

Proof: Let $f(t) = g(t) + h(t) \in S^pPBP_{\omega,k}(\mathbb{R}, X)$ with $g^b \in BP_{\omega,k}(\mathbb{R}, L^p([0,1], X))$ and $h^b \in \mathscr{E}(\mathbb{R}, L^p([0,1], X))$, then

$$u(t) = \int_{-\infty}^t \mathcal{T}(t-s) g(s) ds + \int_{-\infty}^t \mathcal{T}(t-s) h(s) ds := K(t) + E(t).$$

From Theorem 3.5, we can get that $K(t) \in BP_{\omega,k}(\mathbb{R}, X)$. $E(t) \in BC(\mathbb{R}, X)$ can be conducted similarly. Next, we show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T \|E(t)\| dt = 0.$$

Consider the integrals

$$E_n(t) = \int_{t-n}^{t-n+1} \mathcal{T}(t-s) h(s) ds,$$

it follows that

$$\begin{aligned} \|E_n(t)\| &\leq \phi(n-1) \left(\int_{t-n}^{t-n+1} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq \phi(0) \left(\int_{t-n}^{t-n+1} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq \phi(0) \left(\int_{t}^{t+1} \|h(s-n)\|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

then

$$\frac{1}{2T} \int_{-T}^T \|E_n(t)\| dt \leq \frac{\phi(0)}{2T} \int_{-T}^T \left(\int_t^{t+1} \|h(s-n)\|^p ds \right)^{\frac{1}{p}} dt \to 0, \quad as\ T \to \infty,$$

by the translation invariance of $\mathscr{E}(\mathbb{R}, L^p([0,1], X))$. Hence, $E_n \in \mathcal{E}(\mathbb{R}, X)$, their uniform limit $E \in \mathcal{E}(\mathbb{R}, X)$. The proof is complete. ∎

4 Bloch periodic solutions

In this section, we mainly investigate the existence and uniqueness of (pseudo) Bloch periodic mild solutions to a semi-linear evolution equation and a fractional integro-differential equation with Stepanov type force term respectively.

We first consider the existence and uniqueness of Bloch periodic mild solutions to the following well-known semi-linear evolution equation

$$u'(t) = Au(t) + f(t, u(t)),\ t \in \mathbb{R}, \tag{4.1}$$

where the nonlinearity $f \in BS^p(\mathbb{R} \times X, X)$ is a given function with suitable properties and $A : D(A) \subset X \mapsto X$ is a densely closed linear operator which generates an exponentially stable C_0-semigroup $\{\mathfrak{T}(t)\}_{t\geq 0}$, i.e. there exist constants M, $\sigma > 0$ such that

$$\|\mathfrak{T}(t)\| \leq Me^{-\sigma t}, \quad \text{for each } t \geq 0.$$

Definition 4.1 A function $u : \mathbb{R} \to X$ is called a mild solution of (4.1), if it verifies

$$u(t) = \int_{-\infty}^{t} \mathfrak{T}(t-s) f(s, u(s)) ds,\ t \in \mathbb{R}.$$

Theorem 4.1 Assume that $f \in BS^p(\mathbb{R} \times X, X)$ and satisfies (I) and (II) in Theorem 3.1 with $\frac{ML}{\sigma} < 1$. Then the equation (4.1) has a unique mild solution in $BP_{\omega,k}(\mathbb{R}, X)$.

Proof: Define the operator $\mathcal{P} : BP_{\omega,k}(\mathbb{R}, X) \to BP_{\omega,k}(\mathbb{R}, X)$ by

$$(\mathcal{P}u)(t) = \int_{-\infty}^{t} \mathfrak{T}(t-s) f(s, u(s)) ds. \tag{4.2}$$

If $u \in BP_{\omega,k}(\mathbb{R}, X) \subset S^pBP_{\omega,k}(\mathbb{R}, X)$, from Theorem 3.1, it is not difficult to see that $f(\cdot, u(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$. And by Theorem 3.5, we get that $\mathcal{P}u \in BP_{\omega,k}(\mathbb{R}, X)$, so $\mathcal{P}$ is well defined.

For any u, $v \in BP_{\omega,k}(\mathbb{R}, X)$

$$\begin{aligned}
\|(\mathcal{P}u)(t) - (\mathcal{P}v)(t)\| &\leq \int_{-\infty}^{t} \|\mathfrak{T}(t-s)\left(f(s, u(s)) - f(s, v(s))\right)\| ds \\
&\leq ML \int_{-\infty}^{t} e^{-\sigma(t-s)} \|u(s) - v(s)\| ds \\
&\leq ML \int_{0}^{\infty} e^{-\sigma s} \|u(t-s) - v(t-s)\| ds \\
&\leq \frac{ML}{\sigma} \|u - v\|
\end{aligned}$$

by the Banach contraction mapping principle, $\mathcal{P}$ has a unique fixed point in $BP_{\omega,k}(\mathbb{R}, X)$, which is the unique mild solution to the equation (4.1). ■

Theorem 4.2 Let $f \in BS^p(\mathbb{R} \times X, X)$ satisfy (I) in Theorem 3.1 and (IV) in Theorem 3.2. Then the equation (4.1) has a unique mild solution in $BP_{\omega,k}(\mathbb{R}, X)$ provided that $\|l\|_{S^r} \leq \frac{1-e^{-\sigma}}{M}\left(\frac{r_0\sigma}{1-e^{-r_0\sigma}}\right)^{\frac{1}{r_0}}$, where $\frac{1}{r_0} + \frac{1}{r} = 1$.

Proof: Define the operator $\mathcal{P}$ as in (4.2). If $u \in BP_{\omega,k}(\mathbb{R}, X) \subset S^pBP_{\omega,k}(\mathbb{R}, X)$, from Theorem 3.2, it is not difficult to see that $f(\cdot, u(\cdot)) \in S^pBP_{\omega,k}(\mathbb{R}, X)$. And by Theorem 3.5, we get that $\mathcal{P}u \in BP_{\omega,k}(\mathbb{R}, X)$, so $\mathcal{P}$ is well defined.

For any u, $v \in BP_{\omega,k}(\mathbb{R}, X)$, we have

$$
\begin{aligned}
\|(\mathcal{P}u)(t) - (\mathcal{P}v)(t)\| &\leq \int_{-\infty}^{t} \|\mathfrak{T}(t-s)\,(f(s,u(s)) - f(s,v(s))\|\,ds \\
&\leq M \int_{-\infty}^{t} e^{-\sigma(t-s)} l(s)\,\|u(s) - v(s)\|\,ds \\
&\leq M \sum_{k=1}^{\infty} \left(\int_{t-k}^{t-k+1} e^{-r_0\sigma(t-s)} ds\right)^{\frac{1}{r_0}} \|l\|_{S^r}\|u - v\| \\
&\leq M \sqrt[r_0]{\frac{e^{r_0\sigma} - 1}{r_0\sigma}} \sum_{k=1}^{\infty} e^{-\sigma k}\|l\|_{S^r}\|u - v\| \\
&\leq M \sqrt[r_0]{\frac{1 - e^{-r_0\sigma}}{r_0\sigma}} \sum_{k=0}^{\infty} e^{-\sigma k}\|l\|_{S^r}\|u - v\| \\
&\leq \frac{M}{1 - e^{-\sigma}} \left(\frac{1 - e^{-r_0\sigma}}{r_0\sigma}\right)^{\frac{1}{r_0}} \|l\|_{S^r}\|u - v\|,
\end{aligned}
$$

where $\frac{1}{r_0} + \frac{1}{r} = 1$. By the Banach contraction mapping principle, $\mathcal{P}$ has a unique fixed point in $BP_{\omega,k}(\mathbb{R}, X)$, which was the unique mild solution to the equation (4.1). ∎

Next we investigate the existence and uniqueness of pseudo Bloch periodic mild solutions to the following fractional integro-differential equation, which was initially studied in [32]

$$
D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t, u(t)), \tag{4.3}
$$

where $f \in BS^p(\mathbb{R} \times X, X)$ satisfies some additional conditions, A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X, $a \in L^1_{loc}(\mathbb{R}_+)$, $\alpha > 0$ and the fractional derivative is understood in the sense of Weyl.

Given a function $f : \mathbb{R} \to X$, the Weyl fractional integral of order $\alpha > 0$ is defined by $D^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s)ds,\ t \in \mathbb{R}$, when this integral is convergent. The Weyl fractional derivative $D^\alpha f$ of order $\alpha > 0$ is defined by $D^\alpha f(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} f(t),\ t \in \mathbb{R}$, where $n = [\alpha] + 1$. It is known that $D^\alpha D^{-\alpha} g = g$ for any $\alpha > 0$, and $D^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. We can see more details in [30].

Definition 4.2 [32] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X, and $\alpha > 0$. Given $a \in L^1_{loc}(\mathbb{R}_+)$, we say that A is the generator of an α-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0, \infty) \to \mathcal{B}(X)$ such that $\left\{ \frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : Re\lambda > \omega \right\} \subset \rho(A)$, the resolvent set of A, and for all $x \in X$,

$$(\lambda^\alpha - (1 + \hat{a}(\lambda))A)^{-1} x = \frac{1}{1 + \hat{a}(\lambda)} \big(\frac{\lambda^\alpha}{1 + \hat{a}(\lambda)} - A \big)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad Re\lambda > \omega,$$

where $\hat{a}$ denotes the Laplace transform of a. In this case, $\{S_\alpha(t)\}_{t \geq 0}$ is called the α-resolvent family generated by A.

Definition 4.3 A function $u : \mathbb{R} \to X$ is said to be a mild solution of (4.3) if

$$u(t) = \int_{-\infty}^{t} S_\alpha(t - s) f(s, u(s)) ds, \quad t \in \mathbb{R},$$

where $\{S_\alpha(t)\}_{t \geq 0}$ is the α-resolvent family generated by A, whenever it exists.

Theorem 4.3 Assume that $f = g + h \in BS^p(\mathbb{R} \times X, X)$ where g satisfies the condition (A1) and $h^b \in \mathscr{E}(\mathbb{R} \times X, L^p([0, 1], X))$. If (C1) in Theorem 3.6 and the following condition hold:

(H2) the operator A generates an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ such that $\|S_\alpha(t)\| \leq \phi_\alpha(t)$ for all $t \in \mathbb{R}$, where $\phi_\alpha(\cdot) \in L^1(\mathbb{R}^+)$ is nonincreasing.

then the equation (4.3) has a unique mild solution in $PBP_{\omega,k}(\mathbb{R}, X)$ provided that $L_1 \|\phi_\alpha\|_1 < 1$.

Proof: Define the operator $\mathcal{F} : PBP_{\omega,k}(\mathbb{R}, X) \to PBP_{\omega,k}(\mathbb{R}, X)$ by

$$(\mathcal{F}u)(t) = \int_{-\infty}^{t} S_\alpha(t - s) f(s, u(s)) ds, \ t \in \mathbb{R}. \tag{4.4}$$

Let $u = \alpha + \beta \in PBP_{\omega,k}(\mathbb{R}, X)$, where $\alpha \in BP_{\omega,k}(\mathbb{R}, X)$ and $h \in \mathscr{E}(\mathbb{R}, X)$. By Definition 2.1, α is continuous, then $\alpha([-T, T])$ is compact. So, $\mathfrak{I}$ is a compact set. In addition, $u \in PBP_{\omega,k}(\mathbb{R}, X) \subset S^p PBP_{\omega,k}(\mathbb{R}, X)$, from Theorem 3.6, it is not difficult to see that $f(\cdot, u(\cdot)) \in S^p PBP_{\omega,k}(\mathbb{R}, X)$. By Theorem 3.10, $\mathcal{F}u \in PBP_{\omega,k}(\mathbb{R}, X)$, so $\mathcal{F}$ is well defined.

Now for u, $v \in PBP_{\omega,k}(\mathbb{R}, X)$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &\leq \int_{-\infty}^{t} \|S_\alpha(t - s) \left(f(s, u(s)) - f(s, v(s)) \right\| ds \\
&\leq L_1 \int_{-\infty}^{t} \phi_\alpha(t - s) \|u(s) - v(s)\| ds \\
&\leq L_1 \int_0^\infty \phi_\alpha(s) \|u(t - s) - v(t - s)\| ds \\
&\leq L_1 \|u - v\| \int_0^\infty \phi_\alpha(s) ds \\
&\leq L_1 \|\phi_\alpha\|_1 \|u - v\|.
\end{aligned}$$

This prove that $\mathcal{F}$ is a contraction, so by the Banach fixed point theorem there exists a unique $u \in PBP_{\omega,k}(\mathbb{R}, X)$ such that $\mathcal{F}u = u$. ■

Theorem 4.4 Let $f = g + h \in BS^p(\mathbb{R} \times X, X)$, where g satisfies the condition (A1) and $h^b \in \mathscr{E}(\mathbb{R} \times X, L^p([0,1], X))$. Assume that conditions (C2) in Theorem 3.7 and (H2) in Theorem 4.3 hold. Further suppose that $\phi_0 := \sum_{n=0}^{\infty} \phi_\alpha(n) < \infty$, then equation (4.3) has a unique mild solution in $PBP_{\omega,k}(\mathbb{R}, X)$ whenever $\phi_0 < \frac{1}{\|l_f\|_{S^r}}$.

Proof: Define the operator $\mathcal{F}$ as in (4.4). It is followed by Theorem 3.7 and 3.10 that $\mathcal{F}$ is well defined. Let u, $v \in PBP_{\omega,k}(\mathbb{R}, X)$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &\leq \int_{-\infty}^{t} \|S_\alpha(t-s)\left(f(s,u(s)) - f(s,v(s))\right)\| ds \\
&\leq \int_{-\infty}^{t} \phi_\alpha(t-s) l_f(s) \|u(s) - v(s)\| ds \\
&\leq \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} \phi_\alpha(t-s) l_f(s) ds \|u - v\| \\
&\leq \sum_{k=1}^{\infty} \left(\int_{t-k}^{t-k+1} \phi_\alpha^{r_0}(t-s) ds\right)^{\frac{1}{r_0}} \|l_f\|_{S^r} \|u - v\| \\
&\leq \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} \phi_\alpha^{r_0}(s) ds\right)^{\frac{1}{r_0}} \|l_f\|_{S^r} \|u - v\| \\
&\leq \left(\phi_\alpha(0) + \sum_{k=1}^{\infty} \phi_\alpha(k)\right) \|l_f\|_{S^r} \|u - v\| \\
&\leq \left(\sum_{k=0}^{\infty} \phi_\alpha(k)\right) \|l_f\|_{S^r} \|u - v\| \\
&\leq \phi_0 \|l_f\|_{S^r} \|u - v\|,
\end{aligned}$$

where $\frac{1}{r_0} + \frac{1}{r} = 1$, which shows that $\mathcal{F}$ is a contraction, so by the Banach fixed point theorem there exists a unique $u \in PBP_{\omega,k}(\mathbb{R}, X)$ such that $\mathcal{F}u = u$. ∎

Theorem 4.5 Let $p > 1$ and $f = g + h \in BS^p(\mathbb{R} \times X, X)$, where g satisfies the condition (A1) and $h^b \in \mathscr{E}(\mathbb{R} \times X, L^p([0,1], X))$. Assume that (C2) in Theorem 3.7 and the following condition (H3) are satisfied:

(H3) The operator A generates an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$, and there exist $C > 0$, $\xi > 0$ such that $\|S_\alpha(t)\| \leq Ce^{-\xi t}$ for all $t \in \mathbb{R}$.

Then equation (4.3) has a unique mild solution in $PBP_{\omega,k}(\mathbb{R}, X)$.

Proof: $\mathcal{F}$ is well defined as in (4.4). Let u, $v \in PBP_{\omega,k}(\mathbb{R}, X)$, we have

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \leq \int_{-\infty}^{t} \|S_\alpha(t-s)\left(f(s,u(s)) - f(s,v(s))\right)\| ds$$

$$\begin{aligned}&\le \int_{-\infty}^{t} Ce^{-\xi(t-s)} l_f(s)\,\|u(s)-v(s)\|\,ds\\&\le C\|u-v\|\int_{-\infty}^{t} l_f(s)ds.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|\mathcal{F}^2u-\mathcal{F}^2v\right\| &\le C\int_{-\infty}^{t} l_f(s)\,\|(\mathcal{F}u)(s)-(\mathcal{F}v)(s)\|\,ds\\&\le C^2\|u-v\|\int_{-\infty}^{t} l_f(s)\int_{-\infty}^{s} l_f(\theta)d\theta ds\\&\le \frac{C^2}{2!}\|u-v\|\left(\int_{-\infty}^{t} l_f(s)ds\right)^2.\end{aligned}$$

By induction, we have

$$\|\mathcal{F}^nu-\mathcal{F}^nv\| \le \frac{C^n}{n!}\|u-v\|\left(\int_{-\infty}^{t} l_f(s)ds\right)^n \le \frac{(C\|l_f\|_1)^n}{n!}\|u-v\|.$$

For sufficiently large n, we have $\frac{(C\|l_f\|_1)^n}{n!}<1$. Thus $\mathcal{F}$ has a unique fixed point in $PBP_{\omega,k}(\mathbb{R},X)$ by the Banach contraction mapping principle. ■

Theorem 4.6 Assume that $f=g+h\in BS^p(\mathbb{R}\times X,X)$, where g satisfies (A1) and $h^b\in\mathscr{E}(\mathbb{R}\times X,L^p([0,1],X))$, and conditions (C2) in Theorem 3.7 and (H3) in Theorem 4.5 hold. If the integral $\int_{-\infty}^{t} l_f(s)ds$ exists for all $t\in\mathbb{R}$, then equation (4.3) has a unique mild solution in $PBP_{\omega,k}(\mathbb{R},X)$.

Proof: Define an equivalent norm on $PBP_{\omega,k}(\mathbb{R},X)$ as $\|f\|_h=\sup_{t\in\mathbb{R}}\{e^{-h\gamma(t)}\|f\|_\infty\}$, where $h>C$ and $\gamma(t)=\int_{-\infty}^{t} l_f(s)ds$. The operator $\mathcal{F}$ has the same definition as before. Let u, $v\in PBP_{\omega,k}(\mathbb{R},X)$, we have

$$\begin{aligned}\|\mathcal{F}u-\mathcal{F}v\|_\infty &\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\|S_\alpha(t-s)\left(f(s,u(s))-f(s,v(s))\right)\|\,ds\\&\le C\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta(t-s)} l_f(s)\,\|u(s)-v(s)\|\,ds\\&\le C\|u-v\|_h\int_{-\infty}^{t} e^{-\xi(t-s)} l_f(s)e^{h\gamma(s)}ds\\&\le C\|u-v\|_h\int_{-\infty}^{t} l_f(s)e^{h\gamma(s)}ds\\&\le C\|u-v\|_h\int_{-\infty}^{t} \gamma'(s)e^{h\gamma(s)}ds\\&\le \frac{C}{h}\|u-v\|_h e^{h\gamma(t)},\end{aligned}$$

therefore,

$$\|\mathcal{F}u-\mathcal{F}v\|_h \le \frac{C}{h}\|u-v\|_h,$$

we get that $\mathcal{F}$ is a contraction mapping by $h>C$. So equation (4.3) has a unique mild solution in $PBP_{\omega,k}(\mathbb{R},X)$. ∎

Example 4.1 We put $A=-\varrho I$ in $X=\mathbb{R}$, $a(t)=\frac{\varrho}{4}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\varrho>0$, $0<\alpha<1$, and $f(t,u)=g(t,u)+h(t,u)\in BS^p(\mathbb{R}\times X,X)$ where $g(t,u)=\gamma(t)\varsigma(u)$, $h(t,u)=\frac{1}{1+t^2}\cos u$. Assume that γ and ς are bounded (not necessarily continuous) functions and satisfy $\gamma(t+\omega)=\gamma(t)$, $\varsigma(e^{ik\omega}u)=e^{ik\omega}\varsigma(u)$ and $\|\varsigma(u)-\varsigma(v)\|\le l_\varsigma\|u-v\|$ with $l_\varsigma>0$. Then, we have that $g\in BS^p(\mathbb{R}\times X,X)$ and

$$g(t+\omega,e^{ik\omega}u)=\gamma(t+\omega)\varsigma(e^{ik\omega}u)=e^{ik\omega}\gamma(t)\varsigma(u)=e^{ik\omega}g(t,u),$$

$$\|g(t,u)-g(t,v)\|=\|\gamma(t)\varsigma(u)-\gamma(t)\varsigma(v)\|\le l_\varsigma\|\gamma\|\|u-v\|.$$

On the other hand, by Proposition 3.6, $h^b\in\mathscr{E}(\mathbb{R}\times X,L^p([0,1],X))$ is easy to get because $h\in\mathscr{E}(\mathbb{R}\times X,X)$. Thus we have that

$$\|f(t,u)-f(t,v)\|\le(\|\gamma\|l_\varsigma+1)\,\|u-v\|.$$

Now equation (4.3) takes the form

$$D^\alpha u(t)=-\varrho u(t)-\frac{\varrho^2}{4}\int_{-\infty}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+f(t,u(t)),\quad t\in\mathbb{R}. \tag{4.5}$$

From Example 4.17 in [32], we get that A generates an α-resolvent family such that its Laplace transform satisfying

$$\hat{S}_\alpha(\lambda)=\frac{\lambda^\alpha}{(\lambda^\alpha+2/\varrho)^2}=\frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha+2/\varrho)}\cdot\frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha+2/\varrho)},$$

and

$$S_\alpha(t)=(\kappa*\kappa)(t),\quad t>0,$$

with $\kappa(t)=t^{\frac{\alpha}{2}-1}E_{\alpha,\frac{\alpha}{2}}(-\frac{\varrho}{2}t^\alpha)$, and where $E_{\alpha,\frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function. The Mittag-Leffler function [31] is defined as follows:

$$E_{\alpha,\beta}(z):=\sum\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_{Ha}e^\eta\frac{\eta^{\alpha-\beta}}{\eta^\alpha-z}ds,\qquad \alpha,\beta>0,\ z\in\mathbb{Z},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\eta|\le|z|^{1/\alpha}$ counterclockwise. Then, by Theorem 4.3, equation (4.5) has a unique mild solution $u(\cdot)\in PBP_{\omega,k}(\mathbb{R},X)$ provided $\|S_\alpha\|\le\frac{1}{(\|\gamma\|l_\varsigma+1)}$.

5 Conclusions

In this paper, we have introduced notions of the Stepanov type Bloch periodic function and the Stepanov type pseudo Bloch periodic function, and shown some basic properties on the completeness, the composition theorems and the convolution theorem of such functions. In addition, we have applied some theorems including composition and convolution theorems to investigate the existence and uniqueness of (pseudo) Bloch periodic mild solutions to a semilinear evolution equation and a fractional integro-differential equation with Stepanov type force term. Recently, Salah, Miraoui and Khemili [26] revisited the pseudo S-asymptotically Bloch periodic function [19] via the measure ergodic function. We can further discuss the Stepanov type pseudo Bloch periodic function in Definition 3.2 via the measure ergodic function as does in [26].

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