On the fourth degree classical forms

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Abstract:

Classical forms correspond to classical orthogonal polynomials: sequences of orthogonal polynomials whose derivatives also constitutes an orthogonal sequence. In the present paper, our aim is to extend to the four degree the works of Beghdadi et al. [2] and Ben Salah [4]. Precisely, we determine all the classical forms which are of fourth degree. Moreover, we show that Hermite, Laguerre and Bessel forms are not of fourth degree. Only Jacobi forms satisfying some conditions possess this property.

Keywords: Classical forms, fourth degree forms, orthogonal polynomials.

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Contents

1 Introduction

Orthogonal polynomials (OPs) have been a subject of research in the last 150 years. The orthogonality considered here is related to a regular form not only to an inner product. For more details, we refer the reader to the monograph [7], the articles [2, 11] and the references therein. A natural generalization of the classical character is the semi-classical one introduced by J. A. Shohat [20] in 1939. Since 1985, this theory has been developed from algebraic and distributional aspects by P. Maroni, and it has been extensively studied by P. Maroni and coworkers in last decades [1, 2, 3, 8, 11, 13, 15, 16, 17].

A form (linear functional) u is called regular if there exists a sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$, $\deg P_n = n$ which is orthogonal with respect to u. Such a form is said to be of second degree if there are polynomials B, C and D suth that the Stieltjes function $S(u)(z) := -\sum_{n\geq 0} \frac{\langle u, x^n\rangle}{z^{n+1}}$ satisfies a quadratic equation

$$B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0.$$

The most famous second degree forms are Tchebychev forms of the first and the second kinds and more generally, Bernstein-Szegö forms and their generalizations [9, 10, 18, 19]. In [2], the authors determine

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all the classical forms which are of second degree. They show that Hermite, Laguerre and Bessel forms are not of second degree. Only Jacobi forms which satisfy a certain condition possess this property.

In [4, 5], third-degree regular forms TDRF have been introduced. These forms are characterized by the fact that their formal Stieltjes function $S(u)$ satisfies a cubic equation

$$A(z)S^3(u)(z)+B(z)S^2(u)(z)+C(z)S(u)(z)+D(z)=0,$$

where A,B,C and D are polynomials. Some examples of TDRF arising from $2-$orthogonality are studied (see [8, 13]). In [4], the author has determined all classical forms which are TDRF. Precisely, he proved that only Jacobi forms $\mathcal{J}(k+\frac{q}{3},l-\frac{q}{3})$, where l,k are integers provided $k+l\geq-1$ and $q\in\{1,2\}$, possess this property.

Motivated by the works mentioned above, in this paper, our aim is to extend to the four degree the works of Beghdadi et al. [2] and Ben Salah [4]. Precisely, we determine all the classical forms which are of fourth degree. This paper is organized as follows. In the second section, we mention the preliminaries and notations used in the sequel. In the third section, we define and present some properties of fourth degree forms. In the fourth section, we show that Hermite, Laguerre and Bessel forms are not fourth degree forms. Finally, we determine all the Jacobi forms which are fourth degree forms.

2 Preliminaries and notations

Let us recall some results related to the algebraic theory of orthogonal polynomials and their applications to semi-classical forms. For more details we refer to [2, 4, 11, 14].
Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u,f\rangle$ the effect of $u\in\mathcal{P}'$ on $f\in\mathcal{P}$.
Let $S(u)(z)$ be the formal Stieltjes function of the form u, defined by

$$S(u)(z):=-\sum_{n\geq0}\frac{(u)_n}{z^{n+1}}, \tag{2.1}$$

where $(u)_n:=\langle u,x^n\rangle$, $n\geq0$, are the moments of u.
The left product of a form by a polynomial is defined by

$$\langle fu,p\rangle:=\langle u,fp\rangle\,,\text{ for each polynomial } p. \tag{2.2}$$

The right product of a form by a polynomial is defined by

$$(uf)(x):=\left\langle u,\frac{xf(x)-\xi f(\xi)}{x-\xi}\right\rangle=\sum_{m=0}^{n}\left(\sum_{\nu=m}^{n}a_\nu(u)_{\nu-m}\right)x^m, \tag{2.3}$$

where $f(x)=\sum_{\nu=0}^{n}a_\nu x^\nu$. Next, it is possible to define a multiplicative product (Cauchy product) of two forms through

$$\langle uv,p\rangle:=\langle u,vp\rangle\,,\quad u,v\in\mathcal{P}',p\in\mathcal{P}. \tag{2.4}$$

From the definition of the formal Stieltjes function of a form, we have [11]

$$S(uv)(z)=-zS(u)(z)S(v)(z). \tag{2.5}$$

$$S(fu)(z)=f(z)S(u)(z)+(u\theta_0 f)(z), \tag{2.6}$$

where f is a polynomial and

$$(\theta_c f)(x)=\frac{f(x)-f(c)}{x-c}. \tag{2.7}$$

The derivative of a form u denoted by u', is defined by

$$\langle u',f\rangle:=-\langle u,f'\rangle, \tag{2.8}$$

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for each polynomial f, f' is the derivative of f.
The form u is called regular if there exists a polynomial sequence $\{P_n\}_{n\geq 0}$, such that

$$\langle u, P_nP_m\rangle = k_n\delta_{n,m}, \quad n,m\geq 0, \quad k_n\neq 0, \quad n\geq 0.$$

Then $\deg P_n = n$.
The sequence $\{P_n\}_{n\geq 0}$, satisfies the linear recurrence relation of order two

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x-\beta_0, \\ &P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n\geq 0. \end{aligned} \tag{2.9}$$

The regularity of u means that we must have $\gamma_{n+1}\neq 0, n\geq 0$.
Let us recall some results about the semi-classical character [11, 14].
A form u is called semi-classical if it is regular and there exist two polynomials ϕ and ψ, ϕ monic, $\deg\phi = t\geq 0$, $\deg\psi = p\geq 1$ such that

$$(\phi u)' + \psi u = 0. \tag{2.10}$$

Or equivalently, the formal Stieltjes function of u satisfies a first order no homogeneous linear differential equation

$$A_0(z)S^{'}(u)(z) = C_0(z)S(u)(z) + D_0(z), \tag{2.11}$$

where

$$A_0 = \phi, \quad C_0 = -\phi' - \psi, \quad D_0 = -(u\theta_0\phi)' - (u\theta_0\psi).$$

Moreover, if u is semi-classical satisfying (2.10), the class s of u is defined by

$$s := \min(\max(\deg\phi - 2, \deg\psi - 1))\geq 0,$$

where the minimum is taken over all pairs (ϕ,ψ) satisfying (2.10). In this case, $\{P_n\}_{n\geq 0}$ is said to be semi-classical of class s. In particular, when $s = 0$ the form u is classical (Hermite, Laguerre, Bessel and Jacobi).

Definition 2.1 *A regular form u is called a second degree form if there exist two polynomials B and C such that [2, 12]*

$$B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0,$$

where D depends on B, C and u:

$$D(z) = -(u^2\theta_0^2B)(z) + (u\theta_0C)(z).$$

Theorem 2.1 *[2] Among the classical forms, only the Jacobi forms $\mathcal{J}(k-\frac{1}{2}, l-\frac{1}{2})$ are second degree forms, provided $k+l\geq 0$, $k,l\in\mathbb{Z}$.*

Definition 2.2 *[4] The form u is called a third degree regular form (TDRF) if it is regular and there exist three polynomials A monic, B and C such that*

$$A(z)S^3(u)(z) + B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0, \tag{2.12}$$

where D depends on A, B, C and u.

The form u is a TDRF if and only if the following conditions are fulfilled

$$\begin{aligned} A(x)u^3 - xB(x)u^2 + x^2C(x)u &= 0, \\ \langle u^3, \theta_0^2A\rangle - \langle u^2, \theta_0B\rangle + \langle u, C\rangle &= 0, \\ \langle u^3, \theta_0A\rangle - \langle u^2, B\rangle + \langle u, xC(x)\rangle &= 0. \end{aligned}$$

Then

$$D(z) = (u^3\theta_0^3A)(z) - (u^2\theta_0^2B)(z) + (u\theta_0C)(z).$$

When a form u is a TDRF, and it is not a second degree form ($A(x)$ doesn't vanish), we shall call u a strict third degree regular form STDRF.

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Theorem 2.2 *[4] Among the classical forms, only the Jacobi forms* $\mathcal{J}(k+\frac{q}{3}, l-\frac{q}{3})$ *are STDRF, provided* $k+l \geq -1, \quad k, l \in \mathbb{Z},\ q \in \{1,2\}$.

Let us consider the shifted form $w := (h_{a^{-1}} \circ \tau_{-b})\, u$, where for each polynomial p

$$\langle \tau_b u, p\rangle := -\langle u, \tau_{-b} p\rangle = \langle u, p(x+b)\rangle,\ b \in \mathbb{C},$$

$$\langle h_a u, p\rangle := \langle u, h_a p\rangle = \langle u, p(ax)\rangle,\ a \in \mathbb{C}^*.$$

A shift leaves the equation (2.12) invariant.

Proposition 2.1 *[4] Let u be a regular form and $w := (h_{a^{-1}} \circ \tau_{-b})\, u$. The form u is TDRF if and only if w is TDRF. Moreover, w fulfils*

$$\tilde{A}(z)S^3(w)(z) + \tilde{B}(z)S^2(w)(z) + \tilde{C}(z)S(w)(z) + \tilde{D}(z) = 0,$$

with

$$\tilde{A}(z) = a^{-t}A(az+b), \quad \tilde{B}(z) = a^{1-t}B(az+b),$$
$$\tilde{C}(z) = a^{2-t}C(az+b), \quad \tilde{D}(z) = a^{3-t}D(az+b),$$

where $t = \deg A$.

3 Fourth degree forms

3.1 Definition and properties of fourth degree forms

Definition 3.1 *The form u is called a fourth degree form if it is regular and there exists four polynomials $A_i(z)$, $i = 1, ..., 4$, such that*

$$A_4(z)S^4(u)(z) + A_3(z)S^3(u)(z) + A_2(z)S^2(u)(z) + A_1(z)S(u)(z) + A_0(z) = 0, \tag{3.1}$$

where A_0 depends of A_i, for $i = 1, \ldots, 4$ and u.

When the form u is regular and is not a $i-1$ degree form for $i = 3, 4$, we call u a strict fourth degree regular form SFDRF.

Proposition 3.1 *Let u be a FDRF satisfying (3.1). Then the shifted form $\widehat{u} = (h_{a^{-1}} \circ \tau_{-b})u$ fulfils:*

$$\sum_{i=0}^{4} \widehat{A}_i(z)S^i(\widehat{u})(z) = 0, \tag{3.2}$$

with

$$\widehat{A}_i(z) = a^{4-i-t}A_i(az+b),\ t = degA_4. \tag{3.3}$$

Proof: The formal Stieltjes function of $\widehat{u}$ satisfies [6]

$$S(\widehat{u})(z) = aS(u)(az+b).$$

Therefore, using (3.1) we deduce (3.2) and (3.3). ∎

Proposition 3.2 *Let u be a semi-classical form satisfying (2.11) with polynomials A, C and D, and*

$$A(z) = \prod_{i=1}^{m}(z-a_i)^{k_i}, \tag{3.4}$$

where $a_1, \ldots, a_m$ are complex numbers and $k_1, \ldots, k_m$ are positive integers such that $k_1 + \ldots + k_m = t = \deg A$. If u is a SFDRF, necessarily, we have the following conditions:

(a) The rational fraction $\frac{C}{A}$ has only simple poles and if $\alpha_1, \ldots, \alpha_m$ are the residues of $\frac{C}{A}$, then

$$\frac{C}{A} = \sum_{i=1}^{m} \frac{\alpha_i}{z-a_i}. \tag{3.5}$$

(b) $4\alpha_1, \ldots, 4\alpha_m \in \mathbb{Z}$.

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Proof: Let u be a SFDRF, then u satisfies

$$\sum_{i=0}^{4} A_i(z)S^i(u)(z) = 0,$$

dividing by A_4, we obtain

$$\sum_{i=0}^{4} p_i(z)S^i(u)(z) = 0, \quad \text{where} \quad p_i = \frac{A_i}{A_4}.$$

Deriving this equation, multiplying by A and using (2.11), we get

$$\sum_{i=0}^{4} \left(p_i' AS^i(u) + ip_iCS^i(u) + ip_iDS^{i-1}(u)\right) = 0.$$

Which is equivalent to

$$4CS^4(u) + 4DS^3 + \sum_{i=0}^{3} \left(p_i' AS^i(u) + ip_iCS^i(u) + ip_iDS^{i-1}(u)\right) = 0.$$

However, we have from (3.1)

$$S^4(u) = -\sum_{i=0}^{3} p_i S^i(u).$$

We obtain

$$p_i' A + (i-4)p_iC + (i+1)Dp_{i+1} = 0, \text{ for } 0 \leq i \leq 3. \tag{3.6}$$

Using recurrence, we get, for $2 \leq k \leq 3$,

$$p_{4-k} = \frac{(3)\dots(4-k+1)}{k!4^{k-1}p_3^k} \quad \text{and} \quad k\alpha_1, \dots, k\alpha_m \notin \mathbb{Z}.$$

From (3.6), we have the following equations:

$$p_0' A - 4p_0C + p_1D = 0, \tag{3.7}$$

$$p_1' A + (1-4)p_1C + 2p_2D = 0. \tag{3.8}$$

After some calculations, we obtain

$$4C\left(p_0 - \frac{1}{4^4}p_3^4\right) = A\left(p_0' - \frac{1}{4^3}p_3^3\right).$$

Since $\Delta := p_0 - \frac{1}{4^4}p_3^4 \neq 0$, we have

$$\frac{C}{A} = \frac{\Delta'}{4\Delta}. \tag{3.9}$$

This gives that,

$$\frac{C}{A} = \sum_{i=1}^{m} \frac{\alpha_i}{z-a_i}.$$

Using the two last equations, we obtain

$$\Delta = \mu \prod_{j=1}^{m} (z-a_j)^{4\alpha_j},$$

where $\mu \in \mathbb{C}$. As Δ is a rational fraction, we obtain $4\alpha_1, \dots, 4\alpha_m \in \mathbb{Z}$. ∎

Remarks 3.1 *1. From relations (3.6), we have*

$$A\frac{-p_3'}{4} = C\frac{-p_3}{4} + D.$$

Using this equation and the relation (3.9), we have, for u a semi-classical fourth degree form,

$$S(u)(z) = \frac{-A_3(z)}{4A_4(z)} + \lambda\prod_{j=1}^{m}(z-a_j)^{\alpha_j}, \lambda \in \mathbb{C}. \tag{3.10}$$

2. Let $\nu := \mathcal{J}(\frac{q}{4}, -1-\frac{q}{4})$ be the Jacobi forms of parameters $\frac{q}{4}$ and $-1-\frac{q}{4}$. It is well known that ν is a classical form and satisfy [11]

$$A_0(z)S^{'}(\nu)(z) = C_0(z)S(\nu)(z) + D_0(z), \tag{3.11}$$

where

$$A_0(z) = z^2 - 1, \quad C_0(z) = -z - 1 - \frac{2q}{4}; \quad D_0(z) = 0.$$

If we divide (3.11) by $S(\nu)(z)$ and after a simple calculation, we obtain

$$\frac{S'(\nu)(z)}{S(\nu)(z)} = \frac{-1-\frac{q}{4}}{z-1} + \frac{\frac{q}{4}}{z+1}.$$

Then, by integrating we have

$$S(\nu)(z) = \frac{(z+1)^{\frac{q}{4}}}{(z-1)^{\frac{q+4}{4}}} = \left(\frac{(z+1)^q}{(z-1)^{q+4}}\right)^{\frac{1}{4}}.$$

Consequently, we obtain

$$(z-1)^{q+4}S^4(\nu)(z) - (z+1)^q = 0.$$

Then, ν is a SFDRF and this is how we were able to construct and define a first example of such forms.

Proposition 3.3 *If u is a semi-classical form satisfying (2.11) with (3.4) and u is also a fourth degree form, then for the residues $\alpha_1, \ldots, \alpha_m$ of the rational fraction $\frac{C}{A}$, the following statements hold:*

- *(a) $\exists j_0 \in \mathbb{Z}$, $1 \leq j_0 \leq m$ such that the corresponding residue $\alpha_{j_0} \notin \mathbb{Z}$.*
- *(b) $\alpha_1 + \ldots + \alpha_m \in \mathbb{Z}$.*

Proof:

- (a) The regularity of u implies that $S(u)$ cannot be a rational fraction, therefore, using (3.10) there exist $1 \leq j_0 \leq m$ such that $\alpha_{j_0} \notin \mathbb{Z}$.
- (b) Put $A_3(z) = b_r z^r + \ldots + b_0$ and $A_4(z) = z^k + \ldots + h_0$, where $b_r, \ldots, b_0$, $h_{k-1}, \ldots, h_0 \in \mathbb{C}$, the relation (3.10) becomes

$$-\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}} = -\frac{1}{4}\frac{b_r z^r + \ldots + b_0}{z^k + \ldots + h_0} + \lambda\prod_{j=1}^{m}(z-a_j)^{\alpha_j}. \tag{3.12}$$

Then,

$$-\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}} = -\frac{z^{r-k}}{4}\frac{\sum_{j=0}^{r} b_{r-j}z^{-j}}{1+\sum_{j=1}^{k} h_{k-j}z^{-j}} + \lambda z^{\alpha_1+\ldots+\alpha_m}\prod_{j=1}^{m}(1-\frac{a_j}{z})^{\alpha_j}. \tag{3.13}$$

Since $\frac{\sum_{j=0}^{r} b_{r-j}z^{-j}}{1+\sum_{j=1}^{k} h_{k-j}z^{-j}}$ is expanded as a formal power series on $\frac{1}{z}$, and $\prod_{j=1}^{m}(1-\frac{a_j}{z})^{\alpha_j}$ is also a formal power series on $\frac{1}{z}$. Thus, taking into account (3.13), one has $\alpha_1 + \ldots + \alpha_m$ is integer.

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■

Proposition 3.4 *Let u be a semi-classical form satisfying (2.11) with (3.4). The form u is a fourth degree form if and only if the differential equation*

$$Ay' = Cy + D, \tag{3.14}$$

has a rational fraction solution with

1. *the rational fraction $\frac{C}{A}$ has only simple poles,*
2. $\alpha_1 + \ldots + \alpha_m \in \mathbb{Z}$,
3. $4\alpha_1, \ldots, 4\alpha_m \in \mathbb{Z}$,
4. $\exists j_0 \in \mathbb{N},\ 1 \leq j_0 \leq m$ *such that the corresponding residue* $\alpha_{j_0} \notin \mathbb{Z}$, *where* $\alpha_1, \ldots, \alpha_m$ *are the residues of the rational fraction* $\frac{C}{A}$.

Proof: From the last proposition we prove that the condition is necessary.
The condition is also sufficient. Indeed, let the rational fraction $F = \frac{P}{Q}$ be the solution of equation (3.14). Then

$$S(u) - F = \lambda \prod_{i=1}^{m} (z - a_i)^{\alpha_i},$$

and

$$\sum_{k=0}^{4} C_k^4 Q^k S^k(u)(-P)^{4-k} - Q^4 \lambda^4 \prod_{i=1}^{m} (z - a_i)^{4\alpha_i} = 0. \tag{3.15}$$

Writing

$$\prod_{i=1}^{m} (z - a_i)^{4\alpha_i} = \frac{X}{Y},$$

and replacing this last equation in (3.15), we have u is a fourth degree form. ■

3.2 Fourth degree classical forms

An orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called classical, if the sequence of derivatives, $\{Q_n\}_{n\geq 0}$ ($Q_n = \frac{1}{n+1} P'_{n+1}$, $n \geq 0$), is also orthogonal. Its associated form is also classical. The classical forms are Hermite, Laguerre, Bessel and Jacobi forms. They satisfy

$$(\phi u)' + \psi u = 0,$$

where ϕ and ψ are polynomials, $\deg \phi \leq 2$ and $\deg \psi = 1$. For the regularity, if $\deg \phi = 2$ and $\psi = a_1 x + a_0$, we must have $a_1 \notin \mathbb{N}^*$.

Proposition 3.5 *The Hermite, Laguerre and Bessel forms are not SFDRF.*

Proof:

- **Hermite case**: The formal Stieltjes function of Hermite form denoted by H fulfils [11]

  $$S'(H)(z) = -2zS(H)(z) - 2.$$

  Therefore $\dfrac{C}{A} = -2z$. According to Proposition 3.4, we conclude that H is not SFDRF.

- **Laguerre case**: The formal Stieltjes function of Laguerre form denoted by $L(\alpha)$ fulfils [11]

  $$zS'(L)(z) = -(z-1)S(L)(z) - 1.$$

  Therefore $\dfrac{C}{A} = -1 + \dfrac{\alpha}{z}$. According to Proposition 3.4, we conclude that L is not SFDRF.

- **Bessel case**: The formal Stieltjes function of Bessel form denoted by $B(\alpha)$ fulfills [11]
$$z^2S'(B)(z) = ((2\alpha-1)z+2)S(B)(z) + 2\alpha - 1.$$
Therefore $\dfrac{C}{A} = \dfrac{2}{z^2} + \dfrac{2\alpha-2}{z}$. According to Proposition 3.4, we conclude that B is not SFDRF.

∎

Theorem 3.1 *Among the classical forms, only the Jacobi forms $\mathcal{J}(k+\frac{q}{4}, l-\frac{q}{4})$ are SFDRF, provided that $k+l \geq -1, \quad k,l \in \mathbb{Z}$ and the greatest common divisor of q and 4 is 1.*

For the proof of this theorem, we need the following lemmas.

Lemma 3.1 *If the Jacobi form $\mathcal{J}(\alpha,\beta)$ is SFDRF, necessarily we must have $\alpha = k+\frac{q}{4}$, $\beta = l-\frac{q}{4}$ provided $k+l \geq -1, \quad k,l \in \mathbb{Z}$.*

Proof: The Jacobi form $\mathcal{J}(\alpha,\beta)$ is a classical form and satisfies [11]
$$A(z)S'(\mathcal{J})(z) = C(z)S(\mathcal{J})(z) + D(z)$$
where
$$\begin{aligned} A(z) &= z^2-1,\\ C(z) &= (\alpha+\beta)z+\beta-\alpha,\\ D(z) &= \alpha+\beta+1. \end{aligned}$$
After a simple calculation, we obtain
$$\frac{C(z)}{A(z)} = \frac{\beta}{z-1} + \frac{\alpha}{z+1}.$$
Since the rational fraction $\dfrac{C}{A}$ has simple poles with residues α , β and using the condition 1 in Proposition 3.4, we infer that 4α and 4β are integers numbers. Therefore, we have:

- (i) If $4\alpha = 4k$ and $4\beta = 4k_1$, where k and k_1 are integers, then we have $\alpha = k$ and $\beta = k_1$ which is contradictory with Proposition 3.4.
- (ii) If $4\alpha = 4k$ and $4\beta = 4k_1+q$, where k and k_1 are integers and $q \in \{1,2,3\}$, then we have $\alpha = k$ and $\beta = k_1 + \dfrac{q}{4}$ which is contradictory with condition 1 in Proposition 3.4.
- (iii) Analogously, if $4\alpha = 4k+q$ and $4\beta = 4k_1$ where k and k_1 are integers and $q \in \{1,2,3\}$.
- (iv) If $4\alpha = 4k+q$ and $4\beta = 4k_1+q$ where k and k_1 are integers and $q \in \{1,2,3\}$, then $\alpha+\beta$ is not integer due to Proposition 3.4.
- (v) If $4\alpha = 4k+q$ and $4\beta = 4k_1+q'$, where k and k_1 are integers, $q \in \{1,2,3\}$ and $q' = 4-q$. So $\alpha = k+\frac{q}{4}$ and $\beta = l-\frac{q}{4}$ with $k+l \geq -1, \quad k,l \in \mathbb{Z}$.

∎

Lemma 3.2 *Let u and v be two regular forms satisfying the following relation*
$$M(x)u = N(x)v, \tag{3.16}$$
where M and N are polynomials. If u is a SFDRF satisfying (3.1), then v is also a SFDRF and satisfies
$$\sum_{k=0}^{4} \widetilde{A_k}(z)S^k(v)(z) = 0, \tag{3.17}$$
with
$$\begin{cases} \widetilde{A}_4(z) = A_4(z)N^4(z),\\ \widetilde{A}_3(z) = N^3(z)\left[A_3(z)M(z) + 4A_4(z)\left((v\theta_0N)(z) - (u\theta_0M)(z)\right)\right],\\ \vdots\\ \widetilde{A}_0(z) = M^4(z)A_0(z) + A_1M(z)\left((v\theta_0N)(z) - (u\theta_0M)(z)\right)^3 + \ldots + A_4(z)\left((v\theta_0N)(z) - (u\theta_0M)(z)\right)^4. \end{cases}$$

Proof: From formula (2.6), we have

$$S(Nv)(z) = N(z)S(v)(z) + (v\theta_0 N)(z),$$
$$S(Mu)(z) = M(z)S(u)(z) + (u\theta_0 M)(z).$$

This gives

$$M(z)S(u)(z) = N(z)S(u)(z) + (v\theta_0 N)(z) - (u\theta_0 M)(z).$$

∎

Lemma 3.3 *The Jacobi form $\mathcal{J}(\alpha,\beta)$ is SFDRF if and only if $\mathcal{J}(\beta,\alpha)$ is SFDRF.*

Proof: The desired result is obtained by using Proposition 3.1 and the fact that $h_{-1}\mathcal{J}(\alpha,\beta) = \mathcal{J}(\beta,\alpha)$.
∎

Lemma 3.4 *We have, for $\nu = \mathcal{J}(\frac{q}{4}, -\frac{q}{4}-1)$,*

1. *The form $u = \mathcal{J}(k+\frac{q}{4}, l-\frac{q}{4})$, $k,l \in \mathbb{N}$ satisfying $au = (x-1)^{l+1}(x+1)^k\nu$ where $a = \langle \nu, (x-1)^{l+1}(x+1)^k\rangle$.*
2. *The form $u = \mathcal{J}(-k+\frac{q}{4}, l-\frac{q}{4})$, $k \in \mathbb{N}^*$, $l \in \mathbb{N}$ satisfying: $a(x+1)^k u = (x-1)^{l+1}\,\nu$ where $a = \dfrac{\langle \nu, (x-1)^{l+1}\rangle}{\langle u, (x+1)^k\rangle}$.*
3. *The form $u = \mathcal{J}(k+\frac{q}{4}, -l-\frac{q}{4})$, $k,l \in \mathbb{N}$ satisfying $a(x-1)^{l-1}u = (x+1)^k\nu$ where $a = \dfrac{\langle \nu, (x+1)^k\rangle}{\langle u, (x-1)^{l-1}\rangle}$.*

Proof:

1. The form $\nu = \mathcal{J}(\frac{q}{4}, -\frac{q}{4}-1)$ verifies the following equation:
$$((x^2-1)\nu)' + (-x + \frac{2q}{4} + 1)\nu = 0.$$
However, we have $(X\phi u)' + (X\psi - X'\phi)u = 0$.
Taking $X(x) = (x-1)^{l-1}(x+1)^k$, we obtain
$$((x^2-1)(X\nu))' + (-(k+l+2)x + k - l + \frac{2q}{4})X\nu = 0.$$
2. The form $u = \mathcal{J}(-k+\frac{q}{4}, l-\frac{q}{4})$ verifies the following equation:
$$((x^2-1)u)' + (-(l-k+2)x - k - l + \frac{2q}{4})u = 0.$$
Then the form $(x+1)^k u$ satisfies $X(x) = (x+1)^k$. Then we obtain
$$((x^2-1)(x+1)^k u)' + (-(l+2)x - l + \frac{2q}{4})(x+1)^k u = 0.$$
This gives that the form $(x+1)^k u$ is the form $\mathcal{J}(\frac{q}{4}, l-\frac{q}{4})$ up to a factor. In the same manner, we have that the form $(x-1)^{l+1}\nu$ is the form $\mathcal{J}(\frac{q}{4}, l-\frac{q}{4})$ up to a factor.
3. The form $u = \mathcal{J}(k+\frac{q}{4}, -l-\frac{q}{4})$ verifies
$$((x^2-1)u)' + (-(k-l+2)x + l + k + \frac{2q}{4})u = 0.$$
Then the form $(x-1)^{l-1}u$ verifies the following equation
$$((x^2-1)(x-1)^{l-1}u)' + (-(k+1)x + 1 + k + \frac{2q}{4})((x-1)^{l-1}u) = 0.$$
This gives that the form $(x-1)^{l-1}u$ is the form $\mathcal{J}(k+\frac{q}{4}, -1-\frac{q}{4})$. In the same manner, we have that the form $(x+1)^k\nu$ is the form $\mathcal{J}(k+\frac{q}{4}, -1-\frac{q}{4})$ up to a factor.

∎

3.2.1 Proof of Theorem 3.1

1. If $l \geq 1$, we have two cases:

   (a) If $k \geq 0$, from Lemma 3.4 , we have

   $$au = (x-1)^{l+1}(x+1)^k \nu.$$

   Using the fact that the form ν is SFDRF and the Lemma 3.2, we conclude that u is also SFDRF.

   (b) If $k \leq -1$, in this case we have $u = \mathcal{J}(-(-k)+\frac{q}{4}, l-\frac{q}{4})$, where $(-k) \geq 1$, $l \in \mathbb{N}$ and $l+k \geq -1$. From Lemma 3.4 , we have

   $$a(x+1)^{-k}u = (x-1)^{l+1}\nu.$$

   This gives the form u is SFDRF.

2. If $l \leq 0$, we have two cases:

   (a) If $k \geq 0$, in this case, we have $u = \mathcal{J}(k+\frac{q}{4}, -(-l)-\frac{q}{4})$, where $(-l) \geq 0$, $k \geq 0$ and $l+k \geq -1$. From Lemma 3.4 , we have

   $$a(x-1)^{-l-1}u = (x+1)^k \nu.$$

   This gives the form u is SFDRF.

   (b) If $k = -1$, in this case, we have $u = \mathcal{J}(-1+\frac{q}{4}, l-\frac{q}{4})$, since $l+k \geq -1$, we obtain $l = 0$. Therefore, $u = \mathcal{J}(-1+\frac{q}{4}, -\frac{q}{4})$, This gives the form u is SFDRF.

Theorem 3.2 *Let u be a classical form. We have u is a fourth degree form if and only if $\exists P, Q \in \mathcal{P}$ such that*

$$Pu = Q\nu.$$

Proof: We have $u = \mathcal{J}(k+\frac{q}{4}, l-\frac{q}{4})$, with $l+k \geq -1$, $q \in \{1,2,3\}$ and $\Delta(q,4) = 1$. From the remark 3.1, we have

$$S(u)(z) = \frac{-A_3(z)}{4A_4(z)} + \lambda(z+1)^{k+\frac{q}{4}}(z-1)^{l-\frac{q}{4}}, \ ,\lambda \in \mathbb{C}.$$

Or, we have the form ν is a fourth degree form satisfying

$$S(\nu)(z) = (z+1)^{\frac{q}{4}}(z-1)^{-1-\frac{q}{4}}.$$

Therefore,

$$S(u)(z) = \frac{-A_3(z)}{4A_4(z)} + \lambda(z+1)^k(z-1)^{l+1}S(\nu)(z).$$

This gives,

$$Pu = Q\nu, \quad P, Q \in \mathcal{P}.$$

4 Conclusion

What we can notice in the results founded in this work and in [2, 4] is that the Stieltjes functions of an important class of certain Jacobi forms verify algebraic equations of the second, third and fourth degrees. Solving these equations generally provides some information on the behavior of their moments. Faced with the difficulties encountered in solving algebraic equations of high degrees, our essential objective is to try, in the future, to study and extend this work to forms of n^{th} degree.

■

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The author declare that they have no conflicts of interest.

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