A construction of the real numbers based on Weierstrass's approach

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Abstract

In this paper, we put forth a construction of real numbers rooted in the Weierstraß approach. We depart from the original method by replacing the aliquot parts with rational numbers and employing epsilontics in lieu of comparisons relying on component parts.

1 Introduction

Weierstraß was highly interested in the soundness of calculus in general. In this context, he gave a definition of the real numbers before Méray [27], Heine [19], Dedekind [10], Cantor [6] or Tannery [37] (let us also acknowledge more recent approaches in [4, 9, 29, 36, 21], for instance). The reader interested in the history of the construction of the real numbers can consult [13, 14, 15, 25, 3, 32, 28, 42], for example. It is important to emphasize that Weierstraß's perspective on this subject was an integral part of his course, which he never published. Our understanding of his viewpoint has solely been conveyed through the notes of his students over many versions of the course (primarily Dantscher, Hettner, Hurwitz, Kossak, Pincherle and Thieme [12]), showing slight variations from one version to another.

Mathematicians and historians of mathematics have expressed measured concerns, underscoring reservations about these written notes [17, 16, 12, 14, 3, 30]. However, there is no consensus on this matter, with others considering the theory to be perfectly rigorous [11, 39]. It is possible that this discrepancy stems from errors in interpreting the concepts presented in diverse courseworks, composed by different authors. Without taking a definitive stance, it is worth noting that the existence of divergent opinions highlights the potential difficulty in assimilating these concepts directly from the original sources. Given

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Weierstraß's decision not to publish his work, Heine expressed concern about the absence of a comprehensive source where the developed statements could be located within their proper context (..."so dass es keine Stelle giebt, an welcher man die Sätze im Zusammenhänge entwickelt findet.") [19]. This observation is one of the factors leading Heine to propose his own construction of the real numbers (which is similar to Cantor's view) [19]. That being said, although not the most popular, the construction of real numbers provided by Weierstraß is considered by some to be the most intuitive [8, 39, 33].

The constructions of Weierstraß and Cantor exhibit shared structural elements. In the realm of the real numbers they seek to define, Cantor delves into converging sequences, while Weierstraß focuses on absolutely converging series. Clearly, with the concept of real numbers yet to be defined, attempting a direct recourse to the notion of convergence becomes an exercise in futility, as it inevitably leads us into a vicious circle. Therefore, Cantor examines sequences of rational numbers, whereas Weierstraß considers multisets of positive unit fractions. It is worth emphasing that Weierstraß eschews the application of epsilon-based language, a technique often attributed to him [31], in his approach.

The goal of this work is twofold. Firstly, we endeavor to provide a modern exposition and adaptation of the Weierstraß approach, which suffers from a lack of popularity and, to our knowledge, is not featured in any undergraduate book, unlike those of Cantor and Dedekind, for instance. In doing so, we aim to demonstrate that there is no well-founded reason for this relative lack of popularity. Secondly, we strive to illuminate the parallels between the methodologies of Cantor and Weierstraß in defining real numbers.

To achieve this, we base our adaptation of Weierstraß's method on multisets of rational numbers. As mentioned earlier, the challenge stems from the inability to directly invoke the notion of convergence. Instead, following Weierstraß's approach, we deal with multisets of rational numbers whose partial sums of absolute values are bounded. We demonstrate here the classical properties of real numbers, delineating them as a complete totally ordered field. This characterisation implies the equivalence of Cantor's and Weierstraß's constructions. This approach also highlights that Weierstraß's perspective represents a direct continuation of Stevin's work [34, 35], as it generalizes the construction of real numbers from the decimal numeral system [38], as detailed in Section 6.

This paper is organized as follows. We commence by providing a concise summary of Weierstraß's treatment of real numbers. Next, we introduce the concept of aggregate and their addition, enabling us to define the set of real numbers. Subsequently, we present illustrative examples. We then endow the set of real numbers with a total-order relation and verify that we have formed an ordered group. Moving forward, we establish the Cauchy completeness of these numbers. Then, we offer a brief introduction to the multiplication and division of two real numbers within this context. Finally, we explain that the constructions of Cantor and Weierstraß can be seen as equivalent and show that our method comes down to Weierstraß's.

While heavily inspired by Weierstraß's approach, the proposed definition of real numbers presented here diverges in certain aspects. We delineate some

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of these distinctions in the text. Notably, we utilize rational numbers without relying on aliquot parts (unit fractions). Additionally, we eschew Weierstraß's notion of component, which allowed for the definition of inequality of aggregates, in favor of employing epsilon-based language. This shift replaces the original algorithmic thinking with a more conceptual approach [20, 24].

2 The concept of number for Weierstraß

In this section, we succinctly outline Weierstraß's method for introducing real numbers in his lectures "Einleitung in die Theorie der analytischen Funktionen". For a more comprehensive understanding, readers are encouraged to consult references such as [12].

In his renowned course, penned by several of its students (see [23, 41, 12, 40], for example), Weierstraß formulates the concept of numbers in a broad sense, commencing with the familiar natural and integer numbers. He introduces the idea that numbers of some kind share a common underlying property. According to Weierstraß, more complex numbers can be viewed as aggregates[1] made of numbers obtained from different units. For instance, a rational number is composed of an integer and a decimal part, which can, in turn, be decomposed into further units. He elucidates this concept using the analogy of a price, consisting of multiple currency units (unit, superunit, and subunit).

He introduces an aliquot part ("Theil der Einheit"), which corresponds to the inverse of a natural number. A numerical value ("Zahlgröße") is a finite multiset of aliquot parts along with a natural number; in Weierstraß's terminology, it is an aggregate consisting of a finite number of the unity and aliquot parts. Transformations can be applied to a multiset without changing the numerical value. For example, $(1/2)$ can be transformed into $(1/4, 1/4)$, since $1/2 = 1/4 + 1/4$. In order to be able to consider several expressions corresponding to a unique number, he clearly introduces the notion of equivalence[2] for finite multisets based on these transformations, denoted by the equal sign. Of course, he does not explicitly use the term "equivalence relation", since this concept was not well defined at that time. A positive (rational) number can be seen as a numerical value that is a finite multiset. For example, the aggregate $(1/2, 1/4)$ defines the number three fourths; also $(1/4, 1/4, 1/4)$ is equal to $(1/2, 1/4)$ (in the sense that they are equivalent). Weierstraß studies the properties of these numbers and introduces the operations addition and multiplication.

To extend beyond the concept of rational numbers, it becomes necessary to consider multisets that are not finite. Weierstraß refers to such numbers as "numerical values containing infinitely many elements". He observes that these numerical values are not necessarily bounded, prompting the need for a boundedness criterion: a numerical value is called a finite quantity if its finite

[1] The term has gained popularity through Dugac [12, 13], although the word "Aggregat" is explicitly used in diverse manuscripts by Weierstraß's students [41, 40].

[2] Let us point put that some authors do believe that Weierstraß's approach is more subtle than that [33].

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sums are bounded by a (rational) number. These numbers are examined using set-theoretic language. Without delving into the specifics (see [39] for details), a number is made of components, which are finite multisets. Under the equivalence relation where the transformations described above are the generators, two numbers a and b are deemed equal if each component of a is found in b and vice versa. Furthermore, b is strictly larger than a if every component of a is a component of b, but the reverse is not true. The sum of a and b is obtained through the "juxtaposed union" (equivalent to the disjoint union, corresponding to the union of multisets, see Section 4) of the aggregates defining a and b. This set-based approach could have potentially influenced Cantor [12].

To solidify the properties of these numbers, the implicit connection between the elements of an aggregate and the series whose terms are these elements is taken into consideration. At this juncture, Weierstraß clarifies that the numbers developed thus far are insufficient for subtraction. Consequently, he meticulously and comprehensively constructs the negative numbers. Additionally, he defines multiplication and derives division from it. The ultimate outcome of all these constructions is the set of real numbers, a set on which the operations of addition, subtraction, multiplication, and division (by non-zero numbers) are possible and adhere to the commutative, associative, and distributive laws.

In this manner, we can conceive of real numbers as specific (potentially) infinite sums of aliquot parts and their negations, even though Weierstraß did not unequivocally employ the language of series [39].

3 Definition of an aggregate

As Weierstraß in his course, we will suppose that the concepts of natural and integer numbers are known [41]. However, we will also suppose that the rational numbers have been defined, in order to avoid the use of the notion of aliquot part. By doing so, we directly consider negative numbers from the outset. Let us remark that the properties of aliquot parts are not that different from those of nonnegative rational numbers: as mentioned in [39], a rational number is a number that is equivalent to a finite aggregate. An advantage of this approach is that we do not need to define rational numbers starting from real numbers [39]. Moreover, it is easier to compare this construction with those of Dedekind and Cantor, as they both rely on the notion of rational number.

As remarked by Weierstraß in his course, every new definition of a number relies on some property that does not hold with the previous one. For example, the integers together with addition and multiplication define a ring and the rational numbers give rise to a field. The real numbers allow one to work with a complete field, that is a field where any Cauchy sequence converges.

As usual, we will denote the set of natural numbers (without 0) by $\mathbb{N}$, the set of natural numbers with 0 by $\mathbb{N}_0$, the set of integer numbers by $\mathbb{Z}$, and the set of rational numbers by $\mathbb{Q}$.

Definition 3.1. An aggregate is given by a non empty set Λ and, for any $\lambda \in \Lambda$, by a rational number x_λ. This aggregate will be denoted by $(x_\lambda)_{\lambda\in\Lambda}$ or more simply by x. The set of aggregates will be denoted by $\mathcal{A}$.

Definition 3.2. An aggregate $(x_\lambda)_{\lambda\in\Lambda}$ is nonnegative (resp. nonpositive) if $x_\lambda \geq 0$ (resp. $x_\lambda \leq 0$) for any $\lambda \in \Lambda$.

If the set Λ related to an aggregate $(x_\lambda)_{\lambda\in\Lambda}$ is finite, this aggregate represents a rational number through the sum $\sum_{\lambda\in\Lambda} x_\lambda$. If Λ is not finite, quoting Weierstraß (in German), an aggregate is “not necessarily associated to a finite value”. This becomes apparent when considering $\Lambda = \mathbb{N}$ and $x_\lambda = 1/\lambda$ for any $\lambda \in \Lambda$, as this aggregate represents the diverging harmonic series.

Definition 3.3. An aggregate $(x_\lambda)_{\lambda\in\Lambda}$ has a finite value if the set

$$\{\sum_{j\in J} |x_j| : J \subset \Lambda,\ J \text{ finite}\} \tag{1}$$

is bounded (in $\mathbb{Q}$). The set of aggregates with finite value will be denoted by $\mathbb{A}$.

Of course, $\mathbb{A}$ is a subset of $\mathcal{A}$. If $(x_\lambda)_{\lambda\in\Lambda} \in \mathcal{A}$ is such that Λ is finite, then $(x_\lambda)_{\lambda\in\Lambda}$ trivially belongs to $\mathbb{A}$. For the infinite case, we need further considerations.

We did not impose on the set Λ appearing in Definition 3.1 to be countable. Indeed, this imposition is not necessary.

Proposition 3.4. If $(x_\lambda)_{\lambda\in\Lambda}$ belongs to $\mathbb{A}$, then the set of the elements λ of Λ for which one has $x_\lambda \neq 0$ is necessarily countable.

Proof. Given a rational number $\epsilon > 0$, let us show that the set

$$A_\epsilon = \{\lambda \in \Lambda : |x_\lambda| > \epsilon\}$$

is finite. If it is not the case, let m be a rational number that is an upper bound of the set (1) and such that J^ϵ is a finite part of Λ for which

$$m - \frac{\epsilon}{2} < \sum_{j\in J^\epsilon} |x_j|.$$

Since A_ϵ is not finite, there exists $j_0 \in A_\epsilon \setminus J^\epsilon$. By setting $J = J^\epsilon \cup \{j_0\}$, we get

$$\sum_{j\in J} |x_j| = \sum_{j\in J^\epsilon} |x_j| + |x_{j_0}| > m,$$

which is absurd, since m is an upper bound. As the set of the elements λ of Λ that satisfies $x_\lambda \neq 0$ can be written as

$$\bigcup_{k\in\mathbb{N}} \{\lambda \in \Lambda : |x_\lambda| > \frac{1}{k}\} = \bigcup_{k\in\mathbb{N}} A_{1/k},$$

we can conclude, since any countable union of finite sets is necessarily countable. □

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Let us remark that, for any $(x_\lambda)_{\lambda\in\Lambda} \in \mathcal{A}$, we can always suppose that Λ is infinite by eventually setting $x_\lambda = 0$ for infinitely many indices λ. Therefore, from what we have seen, by choosing a bijection from J to $\mathbb{N}$ or $\mathbb{N}_0$, we can always suppose that we have $\Lambda = \mathbb{N}$ or $\Lambda = \mathbb{N}_0$.

At times, we may represent an element $(x_\lambda)_{\lambda\in\Lambda}$ of the set $\mathbb{A}$ simply as

$$\sum_{\lambda\in\Lambda} x_\lambda, \tag{2}$$

when there is no possibility of confusion. Since we have not yet defined the real numbers, this expression should be interpreted symbolically unless it represents a rational number.

Proposition 3.5. For any rational number r, there exists $(x_\lambda)_{\lambda\in\Lambda} \in \mathbb{A}$ such that $\sum_{\lambda\in\Lambda} x_\lambda = r$.

Proof. It suffices to choose $x_1 = r$ and $x_j = 0$ for $j \geq 2$ ($j \in \mathbb{N}$) to get $\sum_{j=1}^{\infty} x_j = r$. □

4 Sum of aggregates

Let us introduce addition and subtraction on $\mathbb{A}$. Let us remark that they could be defined on $\mathcal{A}$.

If Λ_1 and Λ_2 are two non empty sets, let us write

$$\Lambda_1 \sqcup \Lambda_2 = (\Lambda_1 \times \{1\}) \cup (\Lambda_2 \times \{2\})$$

to denote the disjoint union of Λ_1 and Λ_2 [5]. If Λ_1 and Λ_2 are countable, so is $\Lambda_1 \sqcup \Lambda_2$. This operation defines the “juxtaposed union” of Weierstraß, where no element of Λ_1 is in Λ_2 and conversely. The sum of two aggregates x and y is simply the aggregate obtained by setting side by side the elements of x and y.

Definition 4.1. If $(x_j)_{\lambda\in\Lambda_1}$ and $(y_j)_{\lambda\in\Lambda_2}$ are two elements of $\mathbb{A}$, their sum $(z_{\lambda'})_{\lambda'\in\Lambda_1\sqcup\Lambda_2}$ is defined as follows: $z_{\lambda'} = x_\lambda$ if $\lambda' = (\lambda, 1)$ and $z_{\lambda'} = y_\lambda$ if $\lambda' = (\lambda, 2)$. We will naturally write

$$(z_{\lambda'})_{\lambda'\in\Lambda_1\sqcup\Lambda_2} = (x_\lambda)_{\lambda\in\Lambda_1} + (y_\lambda)_{\lambda\in\Lambda_2}.$$

Proposition 4.2. The sum of two elements of $\mathbb{A}$ is again an element of $\mathbb{A}$.

Proof. Let $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ be two elements of $\mathbb{A}$ and $(z_\lambda)_{\lambda\in\Lambda_1\sqcup\Lambda_2}$ be their sum. If J is a finite subset of $\Lambda_1 \sqcup \Lambda_2$, then there exist a finite subset J_1 of Λ_1 and a finite subset J_2 of Λ_2 such that $J = J_1 \sqcup J_2$. We thus have

$$\sum_{j\in J} |z_j| = \sum_{j\in J_1} |x_j| + \sum_{j\in J_2} |y_j|,$$

which is sufficient to conclude. □

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The operation addition is obviously commutative.

Definition 4.3. The additive inverse of an element $(x_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{A}$ is the element $(y_\lambda)_{\lambda\in\Lambda}$ defined by $y_\lambda = -x_\lambda$ for any $\lambda \in \Lambda$. We will denote by $-(x_\lambda)_{\lambda\in\Lambda}$ the additive inverse of $(x_\lambda)_{\lambda\in\Lambda}$. The difference of two elements $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ of $\mathbb{A}$ is defined by

$$(x_\lambda)_{\lambda\in\Lambda_1} - (y_\lambda)_{\lambda\in\Lambda_2} = (x_\lambda)_{\lambda\in\Lambda_1} + (-(y_\lambda)_{\lambda\in\Lambda_2}).$$

Proposition 4.4. If $(x_\lambda)_{\lambda\in\Lambda_1}$ represents the rational number r_1 and $(y_\lambda)_{\lambda\in\Lambda_2}$ represents the rational number r_2, then $(x_\lambda)_{\lambda\in\Lambda_1} + (y_\lambda)_{\lambda\in\Lambda_2}$ (resp. $(x_\lambda)_{\lambda\in\Lambda_1} - (y_\lambda)_{\lambda\in\Lambda_2}$) represents the rational number $r_1 + r_2$ (resp. $r_1 - r_2$).

Proof. This results from the elementary properties of the absolute convergence of series on the set of rational numbers (as a metric space). □

Weierstraß initially only considers positive numbers and later addresses the subtraction of a larger aggregate from a lesser one.

Remark 4.5. In his lectures, for a larger than b, Weierstraß defines $a - b$ as the number such that, when added to b, results in a. He explicitly constructs a number c such that $b + c = a$ (see the proof of Proposition 7.4).

5 The set of real numbers

We can employ the notion of Cauchy sequence for the aggregates. Recall that a series of rational numbers $\sum_{j\in\mathbb{N}} x_j$ satisfies the Cauchy convergence criterion if and only if, for any rational number $\epsilon > 0$, there exists an integer j_0 such that, for any indices p and q satisfying $q \geq p \geq j_0$, the sum $|\sum_{j=p}^{q} x_j|$ is bounded by ϵ. In the context of unconditionally convergent series, an analog of this result can be stated as follow:

Proposition 5.1. If $(x_\lambda)_{\lambda\in\Lambda}$ belongs to $\mathbb{A}$, then, given a rational number $\epsilon > 0$, there exists a finite subset J^ϵ of Λ such that for any finite subset J of Λ that is disjoint from J^ϵ, we have $\sum_{j\in J} |x_j| < \epsilon$.

Proof. If this is not the case, we can find a rational number $\epsilon > 0$ such that, for any finite subset J^ϵ of Λ, there exists a finite subset J of Λ that is disjoint from J^ϵ and such that $\sum_{j\in J} |x_j| \geq \epsilon$.

Then, let us choose a rational number m that is an upper bound of the set (1) in Definition 3.3 such that there exists a finite subset J^ϵ of Λ for which

$$m - \frac{\epsilon}{2} < \sum_{j\in J^\epsilon} |x_j|.$$

Now, let J be a finite subset of Λ that is disjoint from J^ϵ and such that we have $\sum_{j\in J} |x_j| \geq \epsilon$. We then get

$$\sum_{j\in J^\epsilon\cup J} |x_j| > m,$$

which is absurd, by definition of m. □

To be able to introduce the set of real numbers, we need to make two elements of $\mathbb{A}$ that will represent the same real number indistinguishable. For instance, we have

$$1 = \sum_{j=1}^{\infty} \frac{9}{10^j} = \sum_{j=1}^{\infty} \frac{1}{2^j},$$

in $\mathbb{Q}$, implying that $(9/10^j)_{j\in\mathbb{N}}$ and $(1/2^j)_{j\in\mathbb{N}}$ must be equivalent. As we employ epsilontics, we slightly modify Weierstraß's approach (provided in Section 2): two elements of $\mathbb{A}$ will be deemed equivalent if their finite differences are sufficiently small when summing over sufficiently many indices. More rigorously, we have the following definition.

Definition 5.2. If $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ are two elements of $\mathbb{A}$, we will write $(x_\lambda)_{\lambda\in\Lambda_1} \sim (y_\lambda)_{\lambda\in\Lambda_2}$ to signify that, for any rational number $\epsilon > 0$, there exist a finite subset J_1^ϵ of Λ_1 and a finite subset J_2^ϵ of Λ_2 such that, for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively, we have

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j| < \epsilon.$$

Proposition 5.3. The relation $\sim$ introduced in the Definition 5.2 is an equivalence relation.

Proof. Let $x = (x_\lambda)_{\lambda\in\Lambda_1}$ be an element of $\mathbb{A}$. Given a rational number $\epsilon > 0$, let J^ϵ be a finite subset of Λ_1 such that for any finite subset J of Λ_1 that is disjoint from J^ϵ, we have $\sum_{j\in J} |x_j| < \epsilon/2$. If J_1 and J_2 are finite subsets of Λ containing J^ϵ, we have

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_2} x_j| \leq \sum_{j\in J_1 \Delta J_2} |x_j| < \epsilon,$$

where the symbol Δ denotes the symmetric difference.

If $y = (y_\lambda)_{\lambda\in\Lambda_2}$ is an element of $\mathbb{A}$ such that $x \sim y$, then we trivially have $y \sim x$, by definition of the absolute value on $\mathbb{Q}$.

Finally, let $(z_\lambda)_{\lambda\in\Lambda_3}$ be a third element of $\mathbb{A}$ and suppose that we have $x \sim y$ and $y \sim z$. Given a rational number $\epsilon > 0$, there exist finite subsets J_1^ϵ, J_2^ϵ and J_3^ϵ of Λ_1, Λ_2 and Λ_3 respectively such that

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j| < \frac{\epsilon}{2} \quad \text{and} \quad |\sum_{j\in J_2} y_j - \sum_{j\in J_3} z_j| < \frac{\epsilon}{2},$$

for any finite sets J_1, J_2 and J_3 containing J_1^ϵ, J_2^ϵ and J_3^ϵ respectively. We thus get

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_3} z_j| \leq |\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j| + |\sum_{j\in J_2} y_j - \sum_{j\in J_3} z_j| < \epsilon,$$

which implies $x \sim z$. □

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Definition 5.4. The set of real numbers $\mathbb{R}$ is the quotient set of $\mathbb{A}$ by the equivalence relation $\sim$.

As customary, we will often identify an element $[x] = \{x' \in \mathbb{A} : x \sim x'\}$ of $\mathbb{R}$ with an element of the equivalence class $x' \in [x]$, that is, we will not make the distinction between $[x]$ and a representative of $[x]$.

Remark 5.5. In his lectures, Weierstraß employs comparison via component parts to define equality. As the set of real numbers, viewed as a complete ordered field, is unique (up to an isomorphism) [28], both approaches are fundamentally equivalent. This equivalence can also be demonstrated by noting that the difference between two real numbers is essentially the same in both approaches (see proof of Proposition 7.4).

Let us consider the case of the rational numbers in $\mathbb{R}$.

Proposition 5.6. If $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ are two elements of $\mathbb{A}$ such that $\sum_{\lambda\in\Lambda_1} x_\lambda$ and $\sum_{\lambda\in\Lambda_2} y_\lambda$ both represent a rational number r, then we have $(x_\lambda)_{\lambda\in\Lambda_1} \sim (y_\lambda)_{\lambda\in\Lambda_2}$.

Conversely, if we have $(x_\lambda)_{\lambda\in\Lambda_1} \sim (y_\lambda)_{\lambda\in\Lambda_2}$ and if $\sum_{\lambda\in\Lambda_1} x_\lambda$ represents a rational number r, then $\sum_{\lambda\in\Lambda_2} y_\lambda$ also represents r.

Proof. If these two elements of $\mathbb{A}$ represent the rational number r, then, given a rational number $\epsilon > 0$, there exist finite subsets J_1^ϵ and J_2^ϵ of Λ_1 and Λ_2 respectively such that

$$|\sum_{j\in J_1} x_j - r| < \frac{\epsilon}{2} \quad \text{and} \quad |\sum_{j\in J_2} y_j - r| < \frac{\epsilon}{2},$$

for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively. We thus can write

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j| \leq |\sum_{j\in J_1} x_j - r| + |r - \sum_{j\in J_2} y_j| < \epsilon,$$

for such sets.

Now, if we have $(x_\lambda)_{\lambda\in\Lambda_1} \sim (y_\lambda)_{\lambda\in\Lambda_2}$, given a rational number $\epsilon > 0$, let J_1^ϵ and J_2^ϵ be finite subsets of Λ_1 and Λ_2 respectively such that

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j| < \frac{\epsilon}{2},$$

for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively. There also exists a finite subset J_1^r of Λ_1 such that

$$|\sum_{j\in J_1} x_j - r| < \frac{\epsilon}{2},$$

for any finite set J_1 containing J_1^r. Therefore, we have

$$|\sum_{j\in J_2} y_j - r| < |\sum_{j\in J_2} y_j - \sum_{j\in J_1^\epsilon\cup J_1^r} x_j| + |\sum_{j\in J_1^\epsilon\cup J_1^r} x_j - r| < \epsilon,$$

for any finite set J_2 containing J_2^ϵ. $\square$

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Given a rational number r, by choosing $x_r = (x_j^{(r)})_{j\in\mathbb{N}}$ in $\mathbb{A}$ such that $x_1^{(r)} = r$ and $x_j^{(r)} = 0$ for $j \geq 2$, one can symbolically write $\mathbb{Q} \subset \mathbb{R}$ thanks to the canonical injection[3]

$$\iota : \mathbb{Q} \to \mathbb{R} \quad r \mapsto x_r.$$

Let us show that the cardinality of $\mathbb{R}$ is strictly greater than that of $\mathbb{Q}$.

Lemma 5.7. For a given natural number b where $b > 1$, the set

$$E_b = \{(x_j b^{-j})_{j\in\mathbb{N}} : x_j \in \{0, \dots, b-1\} \, \forall j \in \mathbb{N}\}$$

of $\mathbb{A}$ is not countable. Additionally, $\mathbb{R}$ is not countable.

Proof. Consider the subset E_b^* of E_b that excludes the sequences $(x_j b^{-j})_{\in\mathbb{N}}$ of E_b where there exists $j_0 \in \mathbb{N}$ such that $x_j = b-1$ for $j \geq j_0$. The set E_b^* comprises what is commonly referred to as the proper representations in base b of numbers from the interval $[0,1)$. Notably, this set is uncountable in $\mathbb{A}$. Given $A \subset \mathbb{N}$, let $(x_j^{(A)})_{j\in\mathbb{N}}$ denote the aggregate defined such that $x_j^{(A)} = 1$ if $j \in A$ and $x_j^{(A)} = 0$ otherwise. If we define the set

$$E_{\mathbb{N}} = \{(x_j)_{j\in\mathbb{N}} : x_j \in \{0,1\}\},$$

the application

$$\phi : \wp(\mathbb{N}) \to E_{\mathbb{N}} \quad A \mapsto (x_j^{(A)})_{j\in\mathbb{N}},$$

where $\wp(\mathbb{N})$ represents the power set of $\mathbb{N}$, is evidently bijective. This means that the sets $\wp(\mathbb{N})$ and $E_{\mathbb{N}}$ are equinumerous. Furthermore, the set of aggregates $(x_j)_{j\in\mathbb{N}}$ of $E_{\mathbb{N}}$ for which there exists $j_0 \in \mathbb{N}$ such that $j \geq j_0$ implies $x_j = 1$ is countable. Therefore, the sets E_2, E_2^*, $E_{\mathbb{N}}$ and $\wp(\mathbb{N})$ are equinumerous. Consequently, E_b is not countable for any $b > 1$. Alternatively, this observation can be established by adapting Cantor's original proof for E_{10}.

The uncountability holds true in the quotient $\mathbb{R}$ of $\mathbb{A}$ as well: if $x = (x_j b^{-j})_{j\in\mathbb{N}}$ and $x = (y_j b^{-j})_{j\in\mathbb{N}}$ are two distinct elements of E_b^*, these aggregates do not satisfy $x \sim y$. This arises from the fact that

$$\sum_{j=n}^{n'} \frac{b-1}{b^j} < b^{1-n},$$

for any $n, n' \in \mathbb{N}$, with $n \leq n'$ (see [28], for example). Hence, if for $j \in \{1, \cdots n-1\}$, it holds that $x_j = y_j$ and $x_n < y_n$, then the inequality

$$\sum_{j\in J_2} \frac{y_j}{b^j} - \sum_{j\in J_1} \frac{x_j}{b^j} > \epsilon$$

is satisfied[4] for some rational number $\epsilon > 0$, whenever J_1 and J_2 are finite sets that encompass the integers from 1 to n. □

[3] obviously, we should write $r \mapsto [x_r]$.

[4] It is crucial to emphasize that this implication holds true only when working with the proper representations. Otherwise, it would be impossible to satisfy $\epsilon > 0$ in the scenario where $y_n - x_n = 1$, $x_j = b-1$ and $y_j = 0$ for $j > n$.

Remark 5.8. The previous proof establishes that, within the framework of base-b representations, two distinct proper representations correspond to two distinct real numbers.

As a consequence, ι is not a surjection; that is, we cannot have the equivalence $\mathbb{Q} = \mathbb{R}$. In essence, the capacity to represent numbers with aggregates extends beyond rational numbers. In the following, we will treat the rational number r as an element of $\mathbb{R}$, considering r, x_r and $[x_r]$ as synonymous in $\mathbb{R}$.

Remark 5.9. Weierstraß, in his course, merely notes the existence of elements in $\mathbb{A}$ that do not represent any rational number. As an illustration, he considers $x_j = 1/j!$ for $j \in \mathbb{N}_0$, which represents e.

6 Illustrative examples

In this section, our aim is to clarify the nature of $\mathbb{R}$, which we have not yet established as isomorphic to the conventional real numbers. While the set $\mathbb{R}$ has been introduced in Definition 5.4, let $\mathbf{R}$ denote the ordered field of conventional real numbers, as defined by Dedekind or Cantor, for instance [28].

The Riemann series theorem imparts that absolutely convergent series on $\mathbf{R}$ are unconditionally convergent. In simpler terms, for $(x_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{A}$ and any bijection σ from $\mathbb{N}$ to Λ, the series

$$\sum_{j=0}^{\infty} x_{\sigma(j)} \tag{3}$$

is absolutely convergent (and therefore convergent) in $\mathbf{R}$ and the limit does not depend on the bijection σ. Consequently, if Λ is not finite, an element $(x_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{A}$ defines the general term of an absolutely convergent series and can thus be associated to the real number (3). In this context, i.e. with the conventional real numbers and the Riemann series theorem, the notation (2) is meaningful. For instance, by defining

$$x_j = \frac{(-1)^j}{4^j}\left(\frac{2}{4j+1} + \frac{2}{4j+2} + \frac{2}{4j+3}\right)$$

and considering $J = \mathbb{N}_0$, the sequence $(x_j)_{j\in J}$ serves as a representation of the number $\pi \in \mathbf{R}$ in the sense that

$$\pi = \sum_{j=0}^{\infty} x_j,$$

in $\mathbf{R}$ [1].

Any aggregate within $\mathbb{A}$ serves as a representation of a real number of $\mathbf{R}$ as above. On the other hand, we will demonstrate that every real number in $\mathbf{R}$ is represented by an aggregate within $\mathbb{A}$, as every real number has a

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decimal representation (see Remark 7.6). Moreover, any real number can be represented in multiple ways. For instance, consider $b, b' \in \mathbb{N}$, with $b, b' > 1$; for any number $x \in [0,1) \subset \mathbf{R}$, there exist sequences $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ such that $x_j \in \{0,\ldots,b-1\}$, $y_j \in \{0,\ldots,b'-1\}$, and

$$x = \sum_{j=1}^{\infty} \frac{x_j}{b^j} = \sum_{j=1}^{\infty} \frac{y_j}{b'^j},$$

in $\mathbf{R}$. Consequently, we have $(x_j b^{-j})_{j\in\mathbb{N}} \sim (y_j b'^{-j})_{j\in\mathbb{N}}$, so that in $\mathbb{R}$, $(x_j b^{-j})_{j\in\mathbb{N}}$ and $(y_j b'^{-j})_{j\in\mathbb{N}}$ represent the number $x \in \mathbf{R}$.

However, since our goal is to present a standalone construction of real numbers, akin to that of Weierstraß, we cannot assume the prior definition of $\mathbf{R}$. This limitation prevents us from directly employing the arguments of this section. Such considerations gain significance in the realm of isomorphisms, as elucidated in Section 11. Cantor credits Weierstraß with being the first to avoid the logical error of defining a real number by relying on a limit [7].

7 Ordering the set of real numbers

Let us equip our set of real numbers $\mathbb{R}$ with a total order. Once more, given our avoidance of Weierstraß's notions of equivalence and inequality of aggregates[5], we will utilize partial sums of rationals and epsilontics to craft our definition.

Definition 7.1. If $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ are two elements of $\mathbb{A}$, $(x_\lambda)_{\lambda\in\Lambda_1}$ is strictly larger than $(y_\lambda)_{\lambda\in\Lambda_2}$, which will be denoted by $(x_\lambda)_{\lambda\in\Lambda_1} > (y_\lambda)_{\lambda\in\Lambda_2}$, if there exist a rational number $\epsilon > 0$, a finite subset J_1^ϵ of Λ_1 and a finite subset J_2^ϵ of Λ_2 such that for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively, we have

$$\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j > \epsilon.$$

From there, $(x_\lambda)_{\lambda\in\Lambda_1}$ is larger or equal to $(y_\lambda)_{\lambda\in\Lambda_2}$, which will be denoted by $(x_\lambda)_{\lambda\in\Lambda_1} \geq (y_\lambda)_{\lambda\in\Lambda_2}$, if either $(x_\lambda)_{\lambda\in\Lambda_1} > (y_\lambda)_{\lambda\in\Lambda_2}$, or $(x_\lambda)_{\lambda\in\Lambda_1} \sim (y_\lambda)_{\lambda\in\Lambda_2}$.

Of course, $x \leq y$ (resp. $x < y$) means $y \geq x$ (resp. $y > x$). The order relation $\leq$ given above defines a total order on the set of real numbers. Let us first show that it is well defined on this set.

Proposition 7.2. Let x and y be two elements of $\mathbb{A}$ such that $x > y$; if x' and y' are two elements of $\mathbb{A}$ such that $x \sim x'$ and $y \sim y'$, then we have $x' > y'$.

Proof. Let us write $x = (x_\lambda)_{\lambda\in\Lambda_1}$, $y = (y_\lambda)_{\lambda\in\Lambda_2}$, $x' = (x'_\lambda)_{\lambda\in\Lambda_3}$ and $y' = (y'_\lambda)_{\lambda\in\Lambda_4}$. Let $\epsilon > 0$ be a rational number, J_1^ϵ be a finite subset of Λ_1 and J_2^ϵ be a finite subset of Λ_2 such that

$$\sum_{j\in J_1} x_j - \sum_{j\in J_2} y_j > 2\epsilon,$$

[5] These are based on what we earlier called components.

for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively. Let us also consider K_1^ϵ, K_2^ϵ, K_3^ϵ and K_4^ϵ, which are finite subsets of Λ_1, Λ_2, Λ_3 and Λ_4 respectively such that

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_3} x'_j| < \frac{\epsilon}{2} \quad \text{and} \quad |\sum_{j\in J_2} y_j - \sum_{j\in J_4} y'_j| < \frac{\epsilon}{2},$$

for any finite sets J_1, J_2, J_3 and J_4 containing K_1^ϵ, K_2^ϵ, K_3^ϵ and K_4^ϵ respectively.

By setting $E_1^\epsilon = J_1^\epsilon \cup K_1^\epsilon$, $E_2^\epsilon = J_2^\epsilon \cup K_2^\epsilon$, $E_3^\epsilon = J_3^\epsilon \cup K_3^\epsilon$ and $E_4^\epsilon = J_4^\epsilon \cup K_4^\epsilon$, we get

$$\begin{aligned}\sum_{j\in J_3} x'_j - \sum_{j\in J_4} y'_j &= \sum_{j\in E_1^\epsilon} x_j - \sum_{j\in E_2^\epsilon} y_j + \sum_{j\in J_3} x'_j - \sum_{j\in E_1^\epsilon} x_j + \sum_{j\in E_2^\epsilon} y_j - \sum_{j\in J_4} y'_j \\ &> \epsilon,\end{aligned}$$

for any finite sets J_3 and J_4 containing E_3^ϵ and E_4^ϵ respectively. □

Proposition 7.3. The relation $\leq$ defines a total order on $\mathbb{R}$.

Proof. Let $(x_\lambda)_{\lambda\in\Lambda_1}$ and $(y_\lambda)_{\lambda\in\Lambda_2}$ be two elements of $\mathbb{A}$. We cannot have both $(x_\lambda)_{\lambda\in\Lambda_1} > (y_\lambda)_{\lambda\in\Lambda_2}$ and $(y_\lambda)_{\lambda\in\Lambda_2} > (x_\lambda)_{\lambda\in\Lambda_1}$, by definition.

Moreover, if we have neither $(x_\lambda)_{\lambda\in\Lambda_1} > (y_\lambda)_{\lambda\in\Lambda_2}$, nor $(y_\lambda)_{\lambda\in\Lambda_2} > (x_\lambda)_{\lambda\in\Lambda_1}$, then we necessarily have $x \sim y$. □

The following result is of significant importance. We essentially give here the original proof from [41].

Proposition 7.4. If $x \in \mathbb{R}$ is such that $x > 0$, then there exists an element $(y_j)_{j\in\mathbb{N}}$ of $\mathbb{A}$ that is a representative of x (in the sense that $(y_j)_{j\in\mathbb{N}} \sim x$) such that $y_j > 0$ for any $j \in \mathbb{N}$.

Proof. If $x = (x_\lambda)_{\lambda\in\Lambda}$ is such that $x > 0$, as we have $x < n$ for $n \in \mathbb{N}$ sufficiently large, let $n_0 \in \mathbb{N}_0$ be the largest integer such that $x \geq n_0$ and set $y_1 = n_0$. If $x = y_1$, we are done. Otherwise, since we have $0 < x - y_1 < 1$, let n_1 be the smallest natural number such that $x - y_1 \geq 1/n_1$ and set $y_2 = 1/n_1$. Such a number n_1 exists by definition of the inequality $x > y_1$ (given a rational number $\epsilon > 0$, we have $1/n_1 < \epsilon$ for n_1 sufficiently large). If $x = y_1 + y_2$, the construction is complete; otherwise, we proceed to construct n_2 and y_3 in a similar manner. Specifically, assume that n_l and $y_{l+1} = 1/n_l$ have been defined such that

$$\frac{1}{n_l} \leq x - \sum_{k=1}^{l} y_k < \frac{1}{n_l - 1}, \tag{4}$$

for all $l < j$. If $x = \sum_{k=1}^j y_k$, the task is accomplished. Otherwise, let n_j be the smallest natural number such that $x - \sum_{k=1}^j y_k \geq 1/n_j$ and set $y_{j+1} = 1/n_j$. If $x = \sum_{k=1}^j y_k$ for some j, we are finished. If this equality is not satisfied for any j, we can construct a strictly decreasing sequence $(y_j)_{j\in\mathbb{N}}$ such that (4) holds for all l, so that $(y_j)_{j\in\mathbb{N}} \sim x$. □

As a direct consequence of Proposition 7.4, the nonnegative real numbers, following Definition 3.2, are the numbers x that satisfy $x \geq 0$.

Corollary 7.5. A real number x is nonnegative (resp. nonpositive) if and only if $x \geq 0$ (resp. $x \leq 0$).

Remark 7.6. A slight modification to the previous proof demonstrates that any number belonging to $\mathbb{R}$ contains within its equivalence class a representation in base $b \in \mathbb{N}$ ($b > 1$). We simply need to replace the sequence $(y_j)_{j\in\mathbb{N}}$ in the construction with

$$y_j = \frac{a_{j-1}}{b^{j-1}} \quad (j \geq 2),$$

where $a_j \in \{0, \ldots, b-1\}$ is obtained in the same manner (using the greedy algorithm):

$$\frac{a_l}{b^l} \leq x - \sum_{k=1}^{l} y_k < \frac{a_l + 1}{b^l}.$$

8 The group of real numbers

Let us show that the operations related to the sum of two elements of $\mathbb{A}$ are compatible with the equivalence relation defining the real numbers.

Proposition 8.1. If x, x', y and y' are elements of $\mathbb{A}$ such that $x \sim x'$ and $y \sim y'$, then we have $x + y \sim x' + y'$.

Proof. Let us write $x = (x_\lambda)_{\lambda\in\Lambda_1}$, $y = (y_\lambda)_{\lambda\in\Lambda_2}$, $x' = (x'_\lambda)_{\lambda\in\lambda_3}$ and $y' = (y'_\lambda)_{\lambda\in\Lambda_4}$. Given an irrational number $\epsilon > 0$, there exist finite subsets J_1^ϵ, J_2^ϵ, J_3^ϵ and J_4^ϵ of Λ_1, Λ_2, Λ_3 and Λ_4 respectively such that

$$|\sum_{j\in J_1} x_j - \sum_{j\in J_3} x'_j| < \frac{\epsilon}{2} \quad \text{and} \quad |\sum_{j\in J_2} y_j - \sum_{j\in J_4} y'_j| < \frac{\epsilon}{2},$$

for any finite sets J_1, J_2, J_3 and J_4 containing J_1^ϵ, J_2^ϵ, J_3^ϵ and J_4^ϵ respectively. We thus have

$$\begin{aligned}|\sum_{j\in J_1\sqcup J_2} (x+y)_j - \sum_{j\in J_3\sqcup J_4} (x'+y')_j| &= |\sum_{j\in J_1} x_j + \sum_{j\in J_2} y_j - \sum_{j\in J_3} x'_j - \sum_{j\in J_4} y'_j| \\ &< \epsilon,\end{aligned}$$

for these sets. □

Proposition 8.2. If x and x' are two elements of $\mathbb{A}$ such that $x \sim x'$, then we have $-x \sim -x'$. As a consequence, if y and y' are also two elements of $\mathbb{A}$ such that $y \sim y'$, then we have $x - y \sim x' - y'$.

Proof. This is a direct consequence of the properties of the absolute value on $\mathbb{Q}$. □

The set of real numbers equipped with the operation of addition and the order relation defined above is a linearly ordered group.

Proposition 8.3. If x and y are two elements of $\mathbb{A}$ such that $y > x$, then we have $y + z > x + z$, for any $z \in \mathbb{A}$.

Proof. Let $(x_\lambda)_{\lambda\in\Lambda_1}$, $(y_\lambda)_{\lambda\in\Lambda_2}$ and $(z_\lambda)_{\lambda\in\Lambda_3}$ be three elements of $\mathbb{A}$ such that $(y_\lambda)_{\lambda\in\Lambda_2} > (x_\lambda)_{\lambda\in\Lambda_1}$. There exist a rational number $\epsilon > 0$ and two finite subsets J_1^ϵ and J_2^ϵ of Λ_1 and Λ_2 respectively such that

$$\sum_{j\in J_2} y_j - \sum_{j\in J_1} x_j > 2\epsilon,$$

for any finite sets J_1 and J_2 containing J_1^ϵ and J_2^ϵ respectively. That being said, there exists a finite subset J_3^ϵ of Λ_3 such that

$$\sum_{j\in J_3} |z_j| < \epsilon,$$

for any finite subset J_3 of Λ_3 that is disjoint from J_3^ϵ. As a consequence, we can write

$$\begin{aligned}\sum_{j\in J_2\sqcup J_3} (y+z)_j - \sum_{j\in J_1\sqcup J_3'} (x+z)_j &= \sum_{j\in J_2} y_j + \sum_{j\in J_3} z_j - \sum_{j\in J_1} x_j - \sum_{j\in J_3'} z_j \\ &\geq \sum_{j\in J_2} y_j - \sum_{j\in J_1} x_j - \sum_{j\in J_3\Delta J_3'} |z_j| > \epsilon,\end{aligned}$$

for any finite set J_3' containing J_3^ϵ. ☐

9 Completeness of the real numbers

Let us introduce a notation specific to this section to distinguish between a sequence of aggregates and a single aggregate, i.e. an element $(x_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{A}$. A sequence of aggregates, considered as a sequence on $\mathbb{A}$ or $\mathbb{R}$, will be consistently underlined: for such a sequence $(\underline{x}_k)_{k\in\mathbb{N}}$, each $\underline{x}_k$ represents an aggregate. In other words, for each $k \in \mathbb{N}$, one has $\underline{x}_k = (x_\lambda^{(k)})_{\lambda\in\Lambda}$ for some $(x_\lambda^{(k)})_{\lambda\in\Lambda} \in \mathbb{A}$.

In this section, given a real number x such that $x > 0$ (resp. $x < 0$), we will implicitly assume, by virtue of Proposition 7.4, that the selected representative $(x_\lambda)_{\lambda\in\Lambda}$ of the equivalence class satisfies $x_\lambda > 0$ (resp. $x_\lambda < 0$) for any $\lambda \in \Lambda$. Similarly, we will designate $(x_j)_{j\in\mathbb{N}}$ with $x_j = 0$ for $j \in \mathbb{N}$ as the representative of $0 \in \mathbb{R}$. It is easy to show that one must choose the adequate representative when dealing with limits.

Remark 9.1. Let us consider the sequence $(\underline{x}_k)_{k\in\mathbb{N}}$ on $\mathbb{A}$, where $\underline{x}_k$ is the element $(x_j^{(k)})_{j\in J_k}$ of $\mathbb{A}$ defined by $J_k = \{1, \dots, 2k\}$ and $x_j^{(k)} = (-1)^j$. Since $\underline{x}_k$ represents 0 on $\mathbb{A}$ for $k \in \mathbb{N}$, a suitable notion of limit should ensure that the sequence $(\underline{x}_k)_{k\in\mathbb{N}}$ converges to 0. However, since $\sum_{j\in J_k} |x_k^{(k)}| = 2k$, defining such a limit in $\mathbb{A}$ from $(\underline{x}_k)_{k\in\mathbb{N}}$ is not straightforward.

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We first need a notion of absolute value on $\mathbb{R}$ to define a distance. Thanks to our assumption on the representatives (they are considered as either nonnegative or nonpositive), we can introduce the following definition.

Definition 9.2. Given an element $(x_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{R}$, the absolute value of $(x_\lambda)_{\lambda\in\Lambda}$ is the element $|(x_\lambda)_{\lambda\in\Lambda}|$ of $\mathbb{R}$ defined by

$$|(x_\lambda)_{\lambda\in\Lambda}| = \begin{cases} (x_\lambda)_{\lambda\in\Lambda} & \text{if } (x_\lambda)_{\lambda\in\Lambda} \text{ is nonnegative} \\ -(x_\lambda)_{\lambda\in\Lambda} & \text{otherwise} \end{cases}.$$

It is easy to check that the triangle inequality is satisfied.

Lemma 9.3. For any elements x and y of $\mathbb{R}$, we have $|x+y| \leq |x| + |y|$.

Proof. Let us write $x = (x_\lambda)_{\lambda\in\Lambda_1}$ and $y = (y_\lambda)_{\lambda\in\Lambda_2}$. We have

$$-|\sum_{j\in J_1} x_j| - |\sum_{j\in J_2} y_j| \leq \sum_{j\in J_1} x_j + \sum_{j\in J_2} y_j \leq |\sum_{j\in J_1} x_j| + |\sum_{j\in J_2} y_j|,$$

for any finite subset J_1 of Λ_1 and any finite subset J_2 of Λ_2, thus

$$|\sum_{j\in J_1} x_j| + |\sum_{j\in J_2} y_j| - |\sum_{j\in J_1} x_j + \sum_{j\in J_2} y_j| \geq 0,$$

which is sufficient to conclude if J_1 and J_2 are supposed large enough. □

We will also need the following lemmata to prove the completeness of $\mathbb{R}$.

Lemma 9.4. If $(\underline{x}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence on $\mathbb{R}$ that does not converge to 0, then one (and only one) of the following statements holds:

- There exists $k_* \in \mathbb{N}$ such that $\underline{x}_k > 0$ for any $k \geq k_*$.
- There exists $k_* \in \mathbb{N}$ such that $\underline{x}_k < 0$ for any $k \geq k_*$.

Proof. Suppose we have a Cauchy sequence $(\underline{x}_k)_{k\in\mathbb{N}}$, where $\underline{x}_k \geq 0$ and $\underline{x}_{k'} \leq 0$ for infinitely many indices k and k' in $\mathbb{N}$. Given a rational number $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|\underline{x}_p - \underline{x}_q| < \epsilon$ for any $p, q \geq k_0$. Choosing $\underline{x}_p$ and $\underline{x}_q$ such that $\underline{x}_p \geq 0$ and $\underline{x}_q \leq 0$, we have

$$|\underline{x}_p| \leq |\underline{x}_p - \underline{x}_q| < \epsilon.$$

Thus, we can construct a subsequence of $(\underline{x}_k)_{k\in\mathbb{N}}$ that converges to 0, implying that the sequence $(\underline{x}_k)_{k\in\mathbb{N}}$ itself converges to 0. □

Lemma 9.5. Given $k \in \mathbb{N}$, let $\underline{x}_k = (x_\lambda^{(k)})_{\lambda\in\Lambda_k} \in \mathbb{R}$ be such that the sequence $(\underline{x}_k)_{k\in\mathbb{N}}$ is Cauchy; the set

$$E = \bigcup_{k\in\mathbb{N}} \{\sum_{j\in J} |x_j^{(k)}| : J \subset \Lambda_k,\ J \text{ finite}\}$$

is bounded.

Proof. Since $(\underline{x}_k)_{k\in\mathbb{N}}$ is Cauchy, from Lemma 9.4, there exists $k_* \in \mathbb{N}$ such that either $\underline{x}_k \geq 0$ for any $k \geq k_*$ or $\underline{x}_k \leq 0$ for any $k \geq k_*$; without loss of generality, we can suppose that $\underline{x}_k \geq 0$ for any k. Let $k_0 \in \mathbb{N}$ be such that $|\underline{x}_p - \underline{x}_q| < 1$ for any $p, q \in \mathbb{N}$ satisfying $p, q \geq k_0$. Therefore, we have $|\underline{x}_k| < |\underline{x}_{k_0}| + 1$ for any $k \geq k_0$. There thus exists a rational constant $C_0 > 0$ such that

$$\sum_{j\in J} |x_j^{(k)}| = |\sum_{j\in J} x_j^{(k)}| \leq C_0,$$

for any finite subset J of Λ_k and any $k \geq k_0$, where we have used the fact that $(x_\lambda^{(k)})_{\lambda\in\Lambda_k}$ is nonnegative. Now, if, given $k \in \mathbb{N}$, C_k is a rational upper bound for

$$\{\sum_{j\in J} |x_j^{(k)}| : J \subset \Lambda_k,\ J \text{ finite}\},$$

$\max_{0\leq k\leq k_0} C_k$ is a rational upper bound for the set E. □

Proposition 9.6. Any Cauchy sequence $(\underline{x}_k)_{k\in\mathbb{N}}$ on $\mathbb{R}$ converges to an element of $\mathbb{R}$.

Proof. From the Cauchy sequence $(\underline{x}_k)_{k\in\mathbb{N}}$, we can extract a subsequence $(\underline{x}_{l(k)})_{k\in\mathbb{N}}$ for which we have

$$|\underline{x}_{l(k)} - \underline{x}_{l(k+1)}| < 2^{-k}.$$

Let us set $\Lambda'_0 = \Lambda_{l(1)}$, $\Lambda'_k = \Lambda_{l(k+1)} \sqcup \Lambda_{l(k)}$ for $k \in \mathbb{N}$ and $\underline{\delta}_k = (\delta_j^{(k)})_{j\in\Lambda'_k}$, with

$$(\delta_j^{(k)})_{j\in\Lambda'_k} = \underline{x}_{l(k+1)} - \underline{x}_{l(k)},$$

so that we have $\underline{x}_{l(k+1)} = \underline{x}_{l(1)} + \sum_{j=1}^{k} \underline{\delta}_j$. Let us remark that $(\underline{\delta}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence that converges to 0. Define

$$\Lambda = \bigsqcup_{k\in\mathbb{N}_0} \Lambda'_k = \bigcup_{k\in\mathbb{N}_0} \Lambda'_k \times \{k\}$$

and let $(\ell_\lambda)_{\lambda\in\Lambda}$ be the element of $\mathcal{A}$ given by

$$(\ell_\lambda)_{\Lambda'_k\times\{k\}} = \begin{cases} \underline{x}_{l(1)} & \text{if } k = 0 \\ \underline{\delta}_k & \text{if } k > 0 \end{cases}.$$

The previous lemma ensures that $(\ell_\lambda)_{\lambda\in\Lambda}$ belongs to $\mathbb{R}$.

The relation $|(\ell_\lambda)_{\lambda\in\Lambda} - \underline{x}_{l(k)}| \leq \sum_{j=k+1}^{\infty} |\underline{\delta}_j| < 2^{1-k}$ shows that the sequence $(\underline{x}_{l(k)})_{k\in\mathbb{N}}$ converges to $(\ell_\lambda)_{\lambda\in\Lambda}$. We have thus built a convergent subsequence of the Cauchy sequence, which is sufficient to conclude. □

Weierstraß did present the key results for establishing Proposition 9.6.

Remark 9.7. At the end of Chapter 1 of his course [41], Weierstraß demonstrates that a series of positive numerical values converges if and only if it is bounded. By reducing the study of the convergence of a Cauchy subsequence to that of a geometric series, this result, generalized to non-positive numerical values, can be instrumental in establishing completeness. Another technique involves invoking the Bolzano-Weierstraß theorem, demonstrated in Chapter 9 of [41].

One can show that $\mathbb{Q}$ is dense in $\mathbb{R}$ and thus recover the approach of Méray and Cantor for defining the real numbers, see Section 11.

Proposition 9.8. Given a real number x, there exists a sequence on $\mathbb{Q}$ that converges to x. For $x > 0$ (resp. $x < 0$), this sequence can be supposed increasing (resp. decreasing).

Proof. This is a direct consequence of the proof of Proposition 7.4 □

10 The field of real numbers

To define the multiplication of two real numbers in a practical manner, we can use the Cauchy product and Mertens' theorem. This perspective closely aligns with Weierstraß's construction, and is consistent with the interpretation of absolutely convergent series [39].

Definition 10.1. The product of two elements $(x_j)_{j\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$ of $\mathbb{A}$ is given by the element $(z_j)_{j\in\mathbb{N}_0}$ defined by

$$z_j = \sum_{k=0}^{j} x_k y_{j-k}. \tag{5}$$

This product will be denoted by $(x_j)_{j\in\mathbb{N}_0} \times (y_j)_{j\in\mathbb{N}_0}$.

In this section, we will assume that the indexing set defining the real numbers is $\mathbb{N}_0$, although this is not necessary.

Remark 10.2. The product has been defined for real numbers of the form $(x_j)_{j\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$, i.e. where the indexing set is necessarily $\mathbb{N}_0$. Given two elements $(x_j)_{j\in\Lambda_1}$ and $(y_j)_{j\in\Lambda_2}$ of $\mathbb{A}$, we can define their product as the element $(z_j)_{j\in\Lambda_1}$ of $\mathbb{A}$ defined by

$$z_{\sigma_1^{-1}(j)} = \sum_{k=0}^{j} x_{\sigma_1(k)} y_{\sigma_2(j-k)},$$

where σ_1 is a bijection from $\mathbb{N}_0$ to Λ_1 and σ_2 is a bijection from $\mathbb{N}_0$ to Λ_2.

We will implicitly use the properties of the Cauchy product [18, 22]; as explained in the following remark, we can use a more natural approach by leveraging the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark 10.3. One can avoid any appeal to the theory of Cauchy products by defining the product of two real numbers x and y from $\mathbb{R}$ as the limit (in $\mathbb{R}$) of the sequence $(r_j s_j)_{j\in\mathbb{N}}$, where $(r_j)_{j\in\mathbb{N}}$ and $(s_j)_{j\in\mathbb{N}}$ are two sequences on $\mathbb{Q}$ that converge (in $\mathbb{R}$) to x and y respectively. More technically, let $\mathbb{F}_j$ be the subset of $\mathbb{R}$ made from the elements $(x_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{A}$ for which the cardinality of Λ is equal to $j \in \mathbb{N}$ at most. As $\mathbb{F}_j$ is a subset of $\mathbb{Q}$, we naturally have a product on $\mathbb{F}_j$ and we can define the product on $\mathbb{R}$ by identifying $\mathbb{R}$ with the inverse limit of the sets $\mathbb{F}_j$ [2].

Proposition 10.4. If $(x_j)_{j\in\mathbb{N}_0}$ represents the rational number r_1 and $(y_j)_{j\in\mathbb{N}_0}$ represents the rational number r_2, then $(x_j)_{j\in\mathbb{N}_0} \times (y_j)_{j\in\mathbb{N}_0}$ represents the rational number $r_1 r_2$.

Proof. This is a direct consequence of the properties of the Cauchy product. □

Proposition 10.5. If x, x', y and y' are elements of $\mathbb{A}$ such that $x \sim x'$ and $y \sim y'$, then we have $x \times y \sim x' \times y'$.

Proof. Let $(r_j)_{j\in\mathbb{N}_0}$ be a sequence on $\mathbb{Q}$ that converges to x in $\mathbb{R}$ and $(s_j)_{j\in\mathbb{N}_0}$ be a sequence on $\mathbb{Q}$ that converges to y in $\mathbb{R}$. By Mertens' theorem, $r_j s_j = r_j \times s_j$ in $\mathbb{R}$ and, since the Cauchy product is continuous, $(r_j s_j)_{j\in\mathbb{N}_0}$ converges to $x \times y$. One can conclude, since $(r_j)_{j\in\mathbb{N}_0}$ and $(s_j)_{j\in\mathbb{N}_0}$ also converge in $\mathbb{R}$ to x' and y' respectively. □

In practice, in order to easily compute the product of two real numbers, the choice of the representative of the equivalence class is crucial. For example, one can systematically choose the decimal representation. This approach can be directly formalized [38].

Let us consider the multiplicative inverse of an element $(x_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}$. Let us suppose that $(x_j)_{j\in\mathbb{N}_0} > 0$, so that we can assume to have $x_j > 0$ for any $j \in \mathbb{N}_0$. Let $z_j = 9/10^{j+1}$ for $j \in \mathbb{N}_0$, so that $(z_j)_{j\in\mathbb{N}_0} = 1$ in $\mathbb{R}$. Since the system defining every z_j made of equations of type (5) is lower triangular, there always exists an element $(y_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}$ such that $(x_j)_{j\in\mathbb{N}_0} \times (y_j)_{j\in\mathbb{N}_0} = 1$ in $\mathbb{R}$.

As an example, let us consider the simple case where $(x_j)_{j\in\mathbb{N}_0} = 1/3$. Using the decimal representation, we have $x_j = 3/10^{j+1}$. The equation defining z_0 is thus

$$\frac{9}{10} = \frac{3}{10} y_0,$$

which gives $y_0 = 3$. For the second equation, we get

$$\frac{9}{100} = \frac{3}{10} y_1 + \frac{3}{100} 3,$$

and thus $y_1 = 0$. From there, a simple recurrence shows that the solution of the system is $(y_j)_{j\in\mathbb{N}_0} = 3$, with $y_0 = 3$ and $y_j = 0$ for $j > 0$. For $(x_j)_{j\in\mathbb{N}_0} = 1/4$, this method is less natural. Let us take $x_0 = 2/10$, $x_1 = 5/100$ and $x_j = 0$ for $j > 1$. The first equation is

$$\frac{9}{10} = \frac{2}{10} y_0,$$

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which gives $y_0 = 9/2$. For the subsequent equations, we get

$$\frac{9}{10^{j+1}} = \frac{2}{10} y_j + \frac{5}{100} y_{j-1},$$

which gives the recurrence relation

$$y_j = \frac{1}{10^j}\frac{9}{2} - \frac{1}{4} y_{j-1},$$

for $j > 1$. It is easy to check that $\sum_{j\in\mathbb{N}_0} y_j = 4$: in $\mathbb{Q}$, we have

$$\sum_{j=0}^{\infty} y_j = \frac{9}{2} + \sum_{j=1}^{\infty} y_j = \frac{9}{2} + \frac{1}{2} - \frac{1}{4}\sum_{j=0}^{\infty} y_j,$$

which gives

$$\frac{5}{4}\sum_{j=0}^{\infty} y_j = 5.$$

Obviously, in practice, it is far easier to consider $1/4$ as a rational number for computing its multiplicative inverse, but the method proposed here shows that every strictly positive number has a multiplicative inverse. Of course, this also shows that any strictly negative real number has a multiplicative inverse.

Proposition 10.6. The set $\mathbb{R}$ equipped with the operations of addition, multiplication and the order defined above is an ordered field.

Proof. Let $(x_j)_{j\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$ be two nonnegative real numbers of $\mathbb{R}$. Since we can assume to have $x_j \geq 0$ and $y_j \geq 0$ for any $j \in \mathbb{N}_0$, the product $x \times y$ is also nonnegative. The distributivity follows from the properties of the Cauchy product. □

Let us end this section by presenting how Weierstraß defines the division in [41]. He cleverly relies on aliquot parts.

Remark 10.7. To demonstrate that, given two numbers a and b, there always exists a number c such that $c \times b = a$, Weierstraß first observes that it suffices to prove that every positive number b has a multiplicative inverse. To achieve this, he initially selects an integer m such that $m - 1 < b \leq m$ and sets $b_1 = m - b$. He then demonstrates that

$$\sum_{j=0}^{\infty} b_1^j \frac{1}{m^{j+1}} \leq \frac{1}{1-b_1} \tag{6}$$

converges and induces the inverse of b. For example, for $1/n$ ($n \in \mathbb{N}$), we get $m = 1$, $b_1 = \frac{n-1}{n}$ and (6) gives

$$\sum_{j=0}^{\infty} (\frac{n-1}{n})^j = \frac{1}{1-\frac{n-1}{n}} = n.$$

11 Comparison with alternative constructions

The definition of the real numbers $\mathbb{R}$ is equivalent to the classical constructions of Cantor and Weierstraß.

Let $\mathbf{R}$ be the set of real numbers as defined by Cantor. As $\mathbb{R}$ is inherently Archimedean, it is isomorphic to $\mathbf{R}$ [28]. We can explicitly construct such an isomorphism.

Remark 11.1. In this remark, we exclusively examine absolutely converging series, with $\mathbb{Q}$ regarded as a subset of $\mathbf{R}$ rather than as a subset of $\mathbb{R}$. The isomorphism theorem (see [28] for example) allows extending the function

$$\varphi : \mathbb{Q} \to \mathbb{R} \quad x = \sum_{j=0}^{\infty} x_j \mapsto (x_j)_{j\in\mathbb{N}_0},$$

to an isomorphism between $\mathbf{R}$ and $\mathbb{R}$, utilizing the density of $\mathbb{Q}$ in $\mathbf{R}$.

It can also be verified that we have

$$\varphi^{-1} : \mathbb{R} \to \mathbf{R} \quad (x_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{j=0}^{\infty} x_{\sigma(j)},$$

where σ is any one-to-one mapping from $\mathbb{N}_0$ to Λ.

Now, let $\mathbb{R}_W$ be the set of real numbers as defined by Weierstraß in [41], where the aliquot part $1/n$ corresponds to a rational number. Again, $\mathbb{R}$ is isomorphic to $\mathbb{R}_W$ for the same reason as before. Indeed, Weierstraß's construction is equivalent to the approach proposed here, in the sense that $[(x_j)_{j\in\mathbb{N}_0}]_\mathbb{R} = [(x_j)_{j\in\mathbb{N}_0}]_{\mathbb{R}_W}$, where $[x]_\mathbb{R}$ (resp. $[x]_{\mathbb{R}_W}$) denotes the equivalence class of x in $\mathbb{R}$ (resp. in $\mathbb{R}_W$). To be more precise, if $x \in \mathbf{R}$ is represented as

$$x = \sum_{j\in\mathbb{N}_0} \frac{x_j}{10^j},$$

with $x_j \in \{0, \ldots, 9\}$, then both $[(x_j 10^{-j})_{j\in\mathbb{N}_0}]_\mathbb{R}$ and $[(x_j 10^{-j})_{j\in\mathbb{N}_0}]_{\mathbb{R}_W}$ represent the number x [26].

Remark 11.2. It is shown in [26, 39] that each real number of $\mathbb{R}_W$ contains in its equivalence class a decimal representation. Therefore, if

$$x = \sum_{j=0}^{\infty} \frac{x_j}{10^j}$$

is a decimal representation of $x \in \mathbf{R}$, then $(x_j 10^{-j})_{j\in\mathbb{N}_0}$ defines a numerical value (eventually containing infinitely many elements) and conversely, every numerical value is equivalent to a numerical value $(x_j 10^{-j})_{j\in\mathbb{N}_0}$ which is the decimal representation of a number $x \in \mathbf{R}$. In other words, to every number

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$(x_j 10^{-j})_{j\in\mathbb{N}_0}$ of $\mathbb{R}_W$, we can associate a number $x \in \mathbf{R}$ and vice versa. As before, and for the same reasons,

$$\varphi_W : \mathbf{R} \to \mathbb{R}_W \quad \sum_{j=0}^{\infty} \frac{x_j}{10^j} \mapsto (x_j 10^{-j})_{j\in\mathbb{N}_0}$$

is an isomorphism. As a consequence, the canonical application

$$\iota : \mathbb{R} \to \mathbb{R}_W \quad (x_j 10^{-j})_{j\in\mathbb{N}_0} \mapsto (x_j 10^{-j})_{j\in\mathbb{N}_0}$$

is an isomorphism as well.

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