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GEOMETRIES ON POLYGONS IN THE UNIT DISC

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ABSTRACT. For a family $\mathscr{C}$ of properly embedded curves in the 2-dimensional disk $\mathbb{D}^2$ satisfying certain uniqueness properties, we consider convex polygons $P \subset \mathbb{D}^2$ and define a metric d on P such that (P,d) is a geodesically complete metric space whose geodesics are precisely the curves $\{c \cap P \mid c \in \mathscr{C}\}$.

Moreover, in the special case $\mathscr{C}$ consists of all Euclidean lines, it is shown that P with this new metric is not isometric to any convex domain in $\mathbb{R}^2$ equipped with its Hilbert metric.

We generalize this construction to certain classes of uniquely geodesic metric spaces homeomorphic to $\mathbb{R}^2$.

1. Introduction

Hilbert's 4$^{\text{th}}$ problem was asking for a characterization of all metrics on a convex subset of Euclidean space for which straight lines are geodesics. Before Hilbert, Beltrami in [1] had already shown that the unit disc in the plane, with the Euclidean chords taken as geodesics of infinite length, is a model of the hyperbolic geometry. However, Beltrami did not give a formula for this distance, and this led Klein in [7] to express the distance in the unit disc in terms of the cross radio. Over the years, Hilbert's fourth problem became a very active research area and Hilbert's metric defined on convex domains using cross ratio played an important role. It was gradually realized that the discovery of all metrics satisfying Hilbert's problem was not plausible. Consequently, each metric resolving Hilbert's problem defines a new geometry worth to be studied. A very important class of such metrics, defined by means of the cross ratio, are referred to as *Hilbert metrics* and play a central role in this research area, see [12] for the origin of Hilbert geometry.

Among the prominent mathematicians worked on the Hilbert's fourth problem, it is worthy to mention Busemann and Pogorelov, see for instance [4], [9], [10]. The ideas of the latter to solve Hilbert's fourth problem came from Busemann, who introduced integral geometry techniques to approach Hilbert's problem.

Hilbert's 4$^{\text{th}}$ problem admits various formulations as well as generalizations. One of them is to find metrics on subsets of the plane with prescribed geodesics. Blaschke and Bol in [2] were the first to consider such problems while Busemann and Salzman [5] focused on the whole Euclidean plane.

Convex polytopes constitute an important class of convex domains whose Hilbert geometry has been extensively studied, see for example [13]. In this work we focus on convex polygons in the unit disk $\mathbb{D}^2$ in $\mathbb{R}^2$. We consider a class $\mathscr{C}$ of continuous curves in $\mathbb{D}^2 \cup \partial\mathbb{D}^2$ satisfying natural assumptions (see properties (C1)-(C3) below) analogous to those satisfied by the class of geodesics in a uniquely geodesic metric space. Then for any convex polygon $P \subset \mathbb{D}^2$ we explicitly construct a metric on P,

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in fact a family of metrics, such that (P,d) is a geodesically complete metric space whose geodesics are precisely the curves $\left\{c\cap P \;\middle|\; c\in\mathscr{C}\right\}$. The construction uses a family of pseudo-metrics on P, one pseudo-metric for each boundary point in $\partial\mathbb{D}^2$. This is done in Section 2.

In Section 3 we show that the same construction works for proper uniquely geodesic metric spaces homeomorphic to $\mathbb{R}^2$. In Section 4 we show that when $\mathscr{C}$ is just the class of the straight lines in $\mathbb{D}^2$, the construction of the metric d on P gives rise to a Hilbert geometry, that is, straight lines are precisely the geodesics with respect to d, and yet, (P,d) as a metric space is not isometric to any convex domain in $\mathbb{R}^2$ equipped with the standard Hilbert metric defined via cross ratio.

2. Definitions and Preliminaries

Let $\mathbb{D}^2$ be the (open) unit disk in $\mathbb{R}^2$ and let $\mathscr{C}$ be a family of continuous and injective maps $I\to\mathbb{D}^2\cup\partial\mathbb{D}^2$ where I is a close interval in $\mathbb{R}$, the endpoints of I are mapped in $\partial\mathbb{D}^2$ and the interior of I is mapped into $\mathbb{D}^2$. Assume that $\mathscr{C}$ satisfies the following properties:

(C1) for any two points $x,y\in\mathbb{D}^2$ there exists a unique curve $c_{xy}\in\mathscr{C}$ containing both x and y. The restriction of c_{xy} with endpoints x and y will be called the segment from x to y and will be denoted by $[x,y]$.

(C2) for any two points $\xi,\eta\in\partial\mathbb{D}^2$ there exists a unique curve $c_{\xi\eta}\in\mathscr{C}$ with endpoints ξ and η. We call $c_{\xi\eta}$ the line from ξ to η and denote it by (ξ,η).

(C3) for any $x\in\mathbb{D}^2$ and $\xi\in\partial\mathbb{D}^2$ there exists a unique curve $c_{x\xi}\in\mathscr{C}$ having ξ as one endpoint and containing x. The restriction of $c_{x\xi}$ with endpoints x and ξ will be called the ray from x to ξ and will be denoted by $[x,\xi)$.

Examples of such families include the geodesic lines in the hyperbolic disk as well as homeomorphic images of these.

From the above properties it easily follows that for any two curves $c,c'\in\mathscr{C}$ either $c\cap c'=\varnothing$ or, $c\cap c'$ is a singleton. In particular, the same holds for any two segments.

Let $\theta_1,\theta_2,\theta_3,\theta_4,\theta_1$ be four points in $\partial\mathbb{D}^2$ in cyclic clockwise order. Each pair of points θ_i,θ_j $i\neq j$ determines exactly two subarcs of $\partial\mathbb{D}^2$. Denote by

- : $\widehat{\theta_1\theta_2}\equiv\widehat{\theta_2\theta_1}$ the subarc not containing θ_3,θ_4,
- : $\widehat{\theta_3\theta_4}\equiv\widehat{\theta_4\theta_3}$ the subarc not containing θ_1,θ_2,
- : $\widehat{\theta_2\theta_4}\equiv\widehat{\theta_4\theta_2}$ the subarc not containing θ_1,
- : $\widehat{\theta_1\theta_3}\equiv\widehat{\theta_3\theta_1}$ the subarc not containing θ_4.

In the sequel we will also deal with arcs determined by a triple of points $\theta_1,\theta_2,\theta_3$ in $\partial\mathbb{D}^2$. In this case $\widehat{\theta_1\theta_2}\equiv\widehat{\theta_2\theta_1}$ is the subarc not containing θ_3 and similarly for $\widehat{\theta_1\theta_3}$, $\widehat{\theta_2\theta_3}$.

Clearly, each $c\in\mathscr{C}$ splits $\mathbb{D}^2$ into two components. We will say that two curves $c,c'\in\mathscr{C}$ intersect transversely (by (C1), at a single point) if c intersects both components of $\mathbb{D}^2$ determined by c'. The following property follows from (C1)-(C3):

(C4) all intersections between curves in $\mathscr{C}$ are transverse.

To see this assume, on the contrary, that the lines (ξ,ξ') and (η,η') intersect at a (single) point $x\in\mathbb{D}^2$ and the intersection is not transverse. We may assume that the cyclic clockwise order of the boundary points of these lines is η,ξ,ξ',η',η. For any points $\theta\in\widehat{\eta\xi}$ and $\theta'\in\widehat{\eta'\xi'}$ we claim

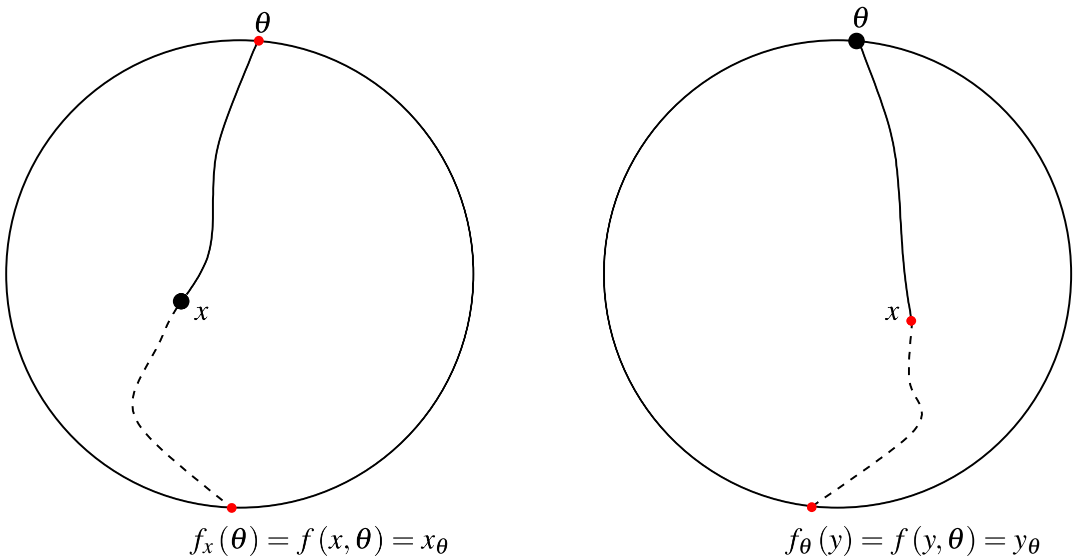

FIGURE 1. The projection maps $f_x : \partial\mathbb{D}^2 \to \partial\mathbb{D}^2$ and $f_\theta : \mathbb{D}^2 \to \partial\mathbb{D}^2$.

that the line (θ,θ') necessarily contains x. To check this observe that the union $c_{x\eta} \cup c_{x\xi}$ separates $\mathbb{D}^2 \cup \partial\mathbb{D}^2$ into two components, one containing θ and the other θ'. If $x \notin (\theta,\theta')$ then (θ,θ') must intersect transversely the interior of either $c_{x\eta}$ or $c_{x\xi}$. Without loss of generality, assume that (θ,θ') intersects transversely the interior of $c_{x\eta}$ at a point, say, z_1. The line (η,η') separates $\mathbb{D}^2 \cup \partial\mathbb{D}^2$ into two components and both θ,θ' belong to one of them. It follows that (θ,θ') must intersect (η,η') at an other point, say, z_2. Then the points z_1, z_2 belong to (θ,θ') as well as to (η,η') which violates property (C1). This shows that $x \in (\theta,\theta')$.
To complete the proof of (C4) consider a point $\theta'' \in \widehat{\eta'\xi'}$ with $\theta'' \neq \theta'$. Then we would have $x \in (\theta,\theta') \cap (\theta,\theta'')$ and, by property (C3), $(\theta,\theta') \equiv (\theta,\theta'')$ which, by (C2), implies that $\theta'' = \theta'$, a contradiction.

We will also need the following property which follows immediately from property (C3):

(C5) for pairwise distinct points $\theta, \theta_1, \dots \theta_m$ in $\partial\mathbb{D}^2$, $(\theta,\theta_i) \cap (\theta,\theta_j) = \emptyset$ if $i \neq j$.

Denote by $\left|\widehat{\theta_i\theta_j}\right|$ the Euclidean length of the arc $\widehat{\theta_i\theta_j}$ and define the ratio $[\theta_1,\theta_2,\theta_3,\theta_4]$ of the points $\theta_1,\theta_2,\theta_3,\theta_4$ by

$$[\theta_1,\theta_2,\theta_3,\theta_4] := \frac{\left|\widehat{\theta_1\theta_3}\right| \left|\widehat{\theta_4\theta_2}\right|}{\left|\widehat{\theta_1\theta_2}\right| \left|\widehat{\theta_4\theta_3}\right|}. \tag{1}$$

We will say that X is a convex subset of $\mathbb{D}^2$ if for any two points $x, y \in X$ the segment $[x,y]$ is entirely contained in X. We say that P is a convex polygon in $\mathbb{D}^2$ if P is an open convex subset of $\mathbb{D}^2$ whose boundary is a finite union of segments.

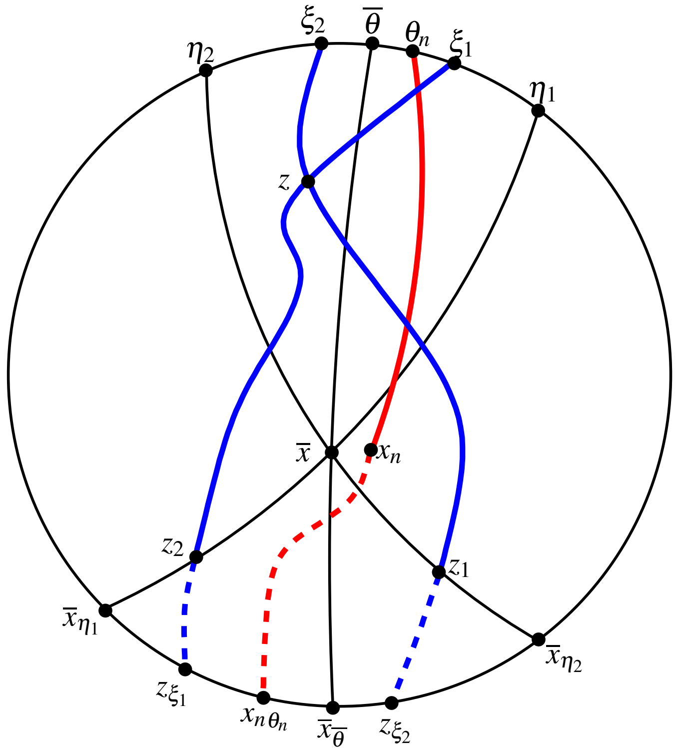

FIGURE 2. Notation for the proof of Lemma 2.

3. A metric with prescribed geodesics

Our goal is to define a metric d on a convex polygon P such that the metric space (P,d) is a geodesically complete metric space whose geodesics are precisely $\left\{c \cap P \,\middle|\, c \in \mathscr{C}\right\}$. We note here that the word geodesic has the classical meaning, namely, the image of an isometric map $\mathbb{R} \to P$.
We will be using the terminology segment, ray and line as introduced in properties (C1), (C2) and (C3).

Define a function

$$f : \mathbb{D}^2 \times \partial\mathbb{D}^2 \to \partial\mathbb{D}^2$$

as follows: for $(x,\theta) \in \mathbb{D}^2 \times \partial\mathbb{D}^2$ there exists, by property (C3), a unique curve $c_{x\theta}$ containing x and having θ as one endpoint. Set $f(x,\theta)$ to be the other endpoint of $c_{x\theta}$.
For fixed $x \in \mathbb{D}^2$ we denote by f_x the induced map

$$f_x : \partial\mathbb{D}^2 \to \partial\mathbb{D}^2 \text{ given by } f_x(\theta) := f(x,\theta),$$

see Figure 1. Similarly, for fixed $\theta \in \partial\mathbb{D}^2$ we have the induced map

$$f_\theta : \mathbb{D}^2 \to \partial\mathbb{D}^2 \text{ given by } f_\theta(x) := f(x,\theta).$$

Notation 1. *We will be writing x_θ instead of $f(x,\theta)$ and we will be calling x_θ the projection from θ of x to the boundary $\partial\mathbb{D}^2$.*

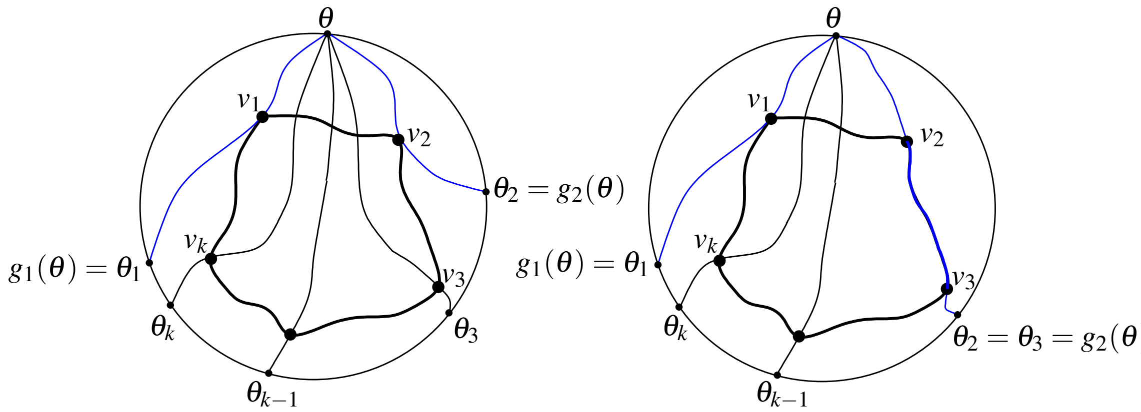

FIGURE 3. The functions g_1, g_2 : on the right is the case where the line (θ, θ_2) contains the vertex v_3 and, hence, the side $[v_2, v_3]$

Lemma 2. *The function $f : \mathbb{D}^2 \times \partial\mathbb{D}^2 \to \partial\mathbb{D}^2$ defined above is continuous.*

Proof. For the reader's convenience, all notation introduced in this proof is included in Figure 2.

Assume f is not continuous at a point $(\overline{x}, \overline{\theta}) \in \mathbb{D}^2 \times \partial\mathbb{D}^2$. Then, there must exist a positive real ε_0 such that

$$\forall n \in \mathbb{N}, \exists (x_n, \theta_n) \text{ satisfying } \left|\wideparen{\overline{\theta}\,\theta_n}\right| < \frac{1}{n}, |\overline{x} - x_n| < \frac{1}{n} \text{ and } \left|\wideparen{\overline{x}_{\overline{\theta}}\, x_{n\theta_n}}\right| \geq \varepsilon_0. \tag{2}$$

where the arc $\wideparen{\overline{\theta}\,\theta_n}$ is the subarc of $\partial\mathbb{D}^2$ not containing $\overline{x}_{\overline{\theta}}$ and $\wideparen{\overline{x}_{\overline{\theta}}\, x_{n\theta_n}}$ the subarc not containing $\overline{\theta}$.

There are exactly two boundary points in $\partial\mathbb{D}^2$ each of which determines, along with $\overline{x}_{\overline{\theta}}$, an arc of Euclidean length ε_0. In other words, there exist η_1, η_2 in $\partial\mathbb{D}^2$ such that the projections $\overline{x}_{\eta_1}, \overline{x}_{\eta_2}$ satisfy

$$\left|\wideparen{\overline{x}_{\overline{\theta}}\,\overline{x}_{\eta_1}}\right| = \varepsilon_0 = \left|\wideparen{\overline{x}_{\overline{\theta}}\,\overline{x}_{\eta_2}}\right| \tag{3}$$

where, $\wideparen{\overline{x}_{\overline{\theta}}\,\overline{x}_{\eta_1}}, \wideparen{\overline{x}_{\overline{\theta}}\,\overline{x}_{\eta_2}}$ are the subarcs not containing $\overline{\theta}$. Let $\wideparen{\eta_1\,\eta_2}$ be the subarc of $\partial\mathbb{D}^2$ which has endpoints $\eta_1\,\eta_2$ and does not contain $\overline{x}_{\eta_1}, \overline{x}_{\eta_2}$. Clearly, by transversality of the intersection of the lines $c_{\overline{x}\overline{\theta}}, c_{\overline{x}\eta_1}, c_{\overline{x}\eta_2}$ the arc $\wideparen{\eta_1\,\eta_2}$ contains $\overline{\theta}$.

Let $\wideparen{\overline{\theta}\,\eta_1}$ be the subarc not containing $\overline{x}_{\eta_1}, \overline{x}_{\eta_2}$ and similarly we specify $\wideparen{\overline{\theta}\,\eta_2}$. Pick arbitrary points $\xi_1 \in \wideparen{\overline{\theta}\,\eta_1}$, $\xi_2 \in \wideparen{\overline{\theta}\,\eta_2}$, $z_1 \in [\overline{x}, \overline{x}_{\eta_2})$ and $z_2 \in [\overline{x}, \overline{x}_{\eta_1})$ with $z_1 \neq \overline{x} \neq z_2$. The rays $[z_1, \xi_2)$ and $[z_2, \xi_1)$ intersect at a point, say, z and $\overline{x}$ is contained in the region R_z bounded by the rays $\left[z, z_{\xi_1}\right), \left[z, z_{\xi_2}\right)$ and the arc $\wideparen{z_{\xi_1}\,z_{\xi_2}}$ where the latter is the subarc not containing ξ_1, ξ_2.

Without loss of generality and by choosing, if necessary, a subsequence we may assume that the sequence $\{\theta_n\} \subseteq \wideparen{\xi_1\,\xi_2}$ and the sequence $\{x_n\}$ is contained in the region R_z. Now for any $(x_n, \theta_n) \in \wideparen{\xi_1\,\xi_2} \times R_z$ we have that $c_{x_n\theta_n}$ intersects both lines $\left(\xi_1, z_{\xi_1}\right), \left(\xi_2, z_{\xi_2}\right)$ at exactly one point and, thus, the projection $x_{n\theta_n}$ is contained in $\wideparen{z_{\xi_1}\,z_{\xi_2}}$. As $\wideparen{z_{\xi_1}\,z_{\xi_2}} \subset \wideparen{\overline{x}_{\eta_1}\,\overline{x}_{\eta_2}}$ we have, by (3), $\left|\wideparen{\overline{x}_{\overline{\theta}}\,x_{n\theta_n}}\right| < \varepsilon_0$ which contradicts (2). ☐

Denote by $v_i, i = 1, \ldots, k$ the vertices of P and fix a point $\theta \in \partial\mathbb{D}^2$. For each i denote by (θ, θ_i) the line containing v_i (see Figure 3). Observe that for $i \neq j$ it may happen that the line (θ, θ_i) contains

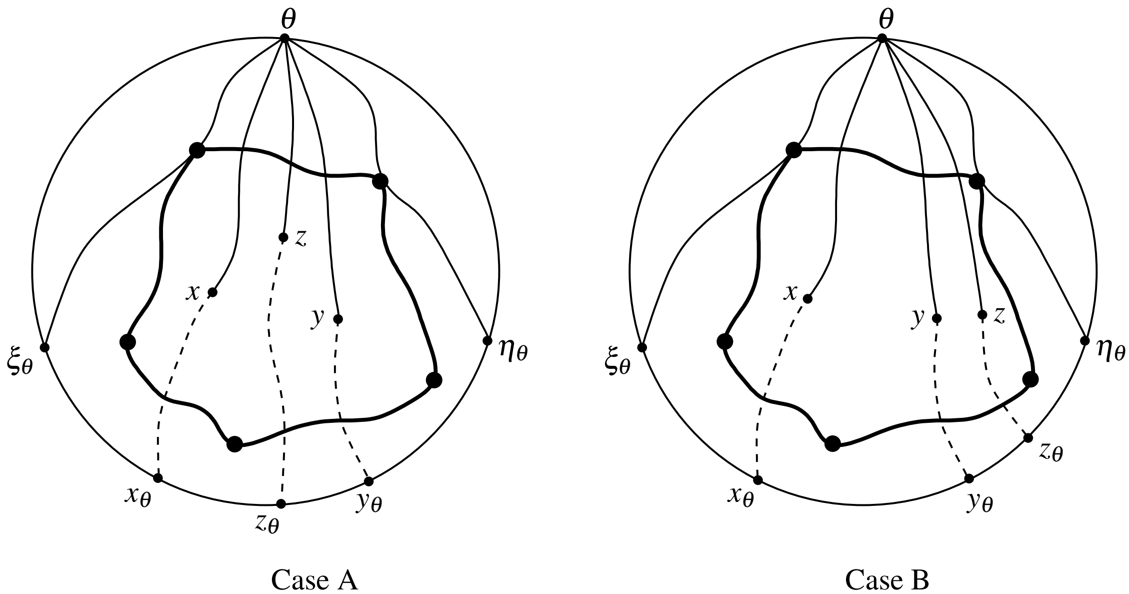

FIGURE 4. The arcs defining the cross ratio $[\eta_\theta, y_\theta, x_\theta, \xi_\theta]$

a vertex $v_j, i \neq j$ in which case $(\theta, \theta_i) = (\theta, \theta_j)$. Otherwise, by property (C5), $(\theta, \theta_i) \cap (\theta, \theta_j) = \emptyset$. In particular, the above lines are at most k and at least 2. Since each (θ, θ_i) separates $\mathbb{D}^2$ into two components there exist exactly two of them, say, (θ, θ_1) and (θ, θ_2) which are outermost in the following sense: one component of $\mathbb{D}^2 \setminus (\theta, \theta_1)$ contains all lines $(\theta, \theta_i), i \neq 1$ and the other component contains none and similarly for (θ, θ_2). Clearly, the line (θ, θ_1) either contains only the vertex v_1 or, by convexity of P, contains a segment $[v_1, v_1']$. In the former case, θ_1 is just $f_{v_1}(\theta)$. The same holds for (θ, θ_2).

The above discussion shows that for each $\theta \in \partial\mathbb{D}^2$ there exist unique points $g_1(\theta), g_2(\theta)$ in $\partial\mathbb{D}^2$ such that the region bounded by the lines $(\theta, g_1(\theta))$, $(\theta, g_2(\theta))$ and the arc $g_1\widehat{(\theta)g_2}(\theta)$ contains P and is minimal with respect to the property of containing P. The arc $g_1\widehat{(\theta)g_2}(\theta)$ is meant to be the subarc with endpoints $g_1(\theta), g_2(\theta)$ not containing θ.

For simplicity, we will be writing ξ_θ and η_θ instead of $g_1(\theta)$ and $g_2(\theta)$ respectively.

Lemma 3. *The maps $g_i : \partial\mathbb{D}^2 \to \partial\mathbb{D}^2, i = 1, 2$ are continuous.*

Proof. If θ is not the endpoint of a line containing a segment of ∂P, then, as explained above, $g_1(\theta)$ is given by some f_{v_1} for a vertex v_1. Therefore, g_1 is continuous at every such θ. Let now θ be the endpoint of a line containing a segment $[v, w]$ of ∂P and $\widehat{\theta^+\theta^-}$ a sufficiently small neighborhood of θ in $\partial\mathbb{D}^2$. Clearly, $f_v(\theta) = f_w(\theta)$ and either $g_1 = f_v$ on $\widehat{\theta^+\theta}$ and $g_1 = f_w$ on $\widehat{\theta^-\theta}$ or, $g_1 = f_w$ on $\widehat{\theta^+\theta}$ and $g_1 = f_v$ on $\widehat{\theta^-\theta}$. In any case g_1 is continuous at every $\theta \in \partial\mathbb{D}^2$ and identically g_2 is. □

For any point $\theta \in \partial\mathbb{D}^2$ we first define a pseudo-metric d_θ on P. Given any two points $x, y \in P$ we may assume, without loss of generality, that the clockwise cyclic order of the four images of θ under the functions f_x, f_y and g_1, g_2 is $g_2(\theta), f_y(\theta), f_x(\theta), g_1(\theta)$. With the simplified notation introduced above these four points are: $\eta_\theta, y_\theta, x_\theta, \xi_\theta$. Define the $\theta-$distance of the points x, y by

$$d_\theta(x,y) := \log[\eta_\theta, y_\theta, x_\theta, \xi_\theta] = \log \frac{\left|\overset{\frown}{\eta_\theta x_\theta}\right| \left|\overset{\frown}{\xi_\theta y_\theta}\right|}{\left|\overset{\frown}{\eta_\theta y_\theta}\right| \left|\overset{\frown}{\xi_\theta x_\theta}\right|}. \tag{4}$$

Lemma 4. *The distance d_θ is a pseudo-metric on P.*

Proof. Clearly d_θ is symmetric. Moreover, if $x \neq y$ then for every θ which is not an endpoint of the curve c_{xy} we have $x_\theta \neq y_\theta$ which implies

$$\frac{\left|\overset{\frown}{\eta_\theta x_\theta}\right|}{\left|\overset{\frown}{\eta_\theta y_\theta}\right|} > 1 \text{ and } \frac{\left|\overset{\frown}{\xi_\theta y_\theta}\right|}{\left|\overset{\frown}{\xi_\theta x_\theta}\right|} > 1 \Longrightarrow d_\theta(x,y) = \log \frac{\left|\overset{\frown}{\eta_\theta x_\theta}\right| \left|\overset{\frown}{\xi_\theta y_\theta}\right|}{\left|\overset{\frown}{\eta_\theta y_\theta}\right| \left|\overset{\frown}{\xi_\theta x_\theta}\right|} \gneqq \log 1 = 0.$$

This shows that $d_\theta(x,y) \neq 0$ if $x \neq y$ and θ is not an endpoint of c_{xy}. Clearly,

$$\text{if } \theta \text{ is an endpoint of } c_{xy} \text{ then } d_\theta(z,w) = 0 \text{ for any pair of points } z, w \in c_{xy}. \tag{5}$$

This is the reason d_θ is just a pseudo-metric and not a metric.
For the triangle inequality, let $x, y, z \in P$ and observe that in the definition of d_θ the points involved are projected onto the boundary circle where the log of the cross ratio obeys the triangle inequality. We carry out the calculation by considering two cases (see Figure 4):

Case A: $z_\theta \in \overset{\frown}{x_\theta y_\theta}$. In this case the triangle inequality is, in fact, equality:

$$\begin{aligned} d_\theta(x,z) + d_\theta(z,y) &= \log[\eta_\theta, z_\theta, x_\theta, \xi_\theta] + \log[\eta_\theta, z_\theta, y_\theta, \xi_\theta] \\ &= \log\left(\frac{\left|\overset{\frown}{\eta_\theta x_\theta}\right| \left|\overset{\frown}{\xi_\theta z_\theta}\right|}{\left|\overset{\frown}{\eta_\theta z_\theta}\right| \left|\overset{\frown}{\xi_\theta x_\theta}\right|} \cdot \frac{\left|\overset{\frown}{\eta_\theta z_\theta}\right| \left|\overset{\frown}{\xi_\theta y_\theta}\right|}{\left|\overset{\frown}{\eta_\theta y_\theta}\right| \left|\overset{\frown}{\xi_\theta z_\theta}\right|} \right) \\ &= \log \frac{\left|\overset{\frown}{\eta_\theta x_\theta}\right| \left|\overset{\frown}{\xi_\theta y_\theta}\right|}{\left|\overset{\frown}{\eta_\theta y_\theta}\right| \left|\overset{\frown}{\xi_\theta x_\theta}\right|} = d_\theta(x,y) \end{aligned} \tag{6}$$

Case B: $z_\theta \notin \overset{\frown}{x_\theta y_\theta}$. We may assume that z_θ is contained in the interior of the arc $\overset{\frown}{y_\theta \eta_\theta}$ (the case $z_\theta \in \overset{\frown}{\xi_\theta x_\theta}$ is treated in an identical manner) which implies that $y_\theta \in \overset{\frown}{x_\theta z_\theta}$. By Case A we have

$$d(x,y) + d(y,z) = d(x,z) \Longrightarrow d(x,y) \leq d(x,z) \leq d(x,z) + d(z,y).$$

□

We now define a metric d on P.

Definition 5. *Consider a countable dense subset $\Theta = \{\theta_i | i \in \mathbb{N}\}$ of $\partial\mathbb{D}^2$ with the following property: for each segment $[v,w]$ in ∂P, the endpoints of the curve c_{vw} are contained in Θ. To each θ_i in Θ assign a positive real w_i such that the series $\sum_i w_i$ converges. For $x, y \in P$ define*

$$d(x,y) := \sum_{i=1}^{\infty} w_i d_{\theta_i}(x,y).$$

Observe that for fixed $x, y \in P$ the function $\theta \to d_\theta(x,y)$ is, by Lemmata 2 and 3, continuous on $\partial\mathbb{D}^2$. It follows that the set $\left\{ d_{\theta_i}(x,y) \;\middle|\; \theta_i \in \Theta \right\}$ is, by compactness of $\partial\mathbb{D}^2$, bounded by some $M > 0$ and *thus*

$$d\,(x,y) \le M \sum_{i=1}^{\infty} w_i < +\infty.$$

Proposition 6. *d is a metric on P.*

Proof. The triangle inequality for d follows from the triangle inequality of all d_θ proven in Lemma 4. Similarly, if $x \neq y$ then $d_\theta\,(x,y) \neq 0$ for every $\theta \in \Theta$ which is not an endpoint of the curve c_{xy} (see the beginning of the proof of Lemma 4). Therefore, d is a metric on P. □

Proposition 7. *The topology induced by d is equivalent to the Euclidean topology.*

Proof. We first show that the Euclidean topology of P is thinner than the topology induced by d. For this, it suffices to show that

$$\forall z_0 \in P \text{ and } \varepsilon > 0, \exists \rho > 0 \text{ such that } |z - z_0| < \rho \Rightarrow d(z,z_0) < \varepsilon. \tag{7}$$

Observe that, by Lemma 2, property (7) holds for the pseudo-metric d_θ, namely, for fixed $\theta \in \partial\mathbb{D}^2$

$$\exists \rho_\theta > 0 \text{ such that } |z - z_0| < \rho_\theta \Rightarrow d_\theta(z,z_0) < \varepsilon. \tag{8}$$

Let $N(z_0)$ be a compact (in the Euclidean topology) neighborhood in P containing z_0. Let $F : \partial\mathbb{D}^2 \times N(z_0) \times N(z_0) \to \mathbb{R}$ be the function given by

$$F(\theta,w,z) = d_\theta(w,z) = \log\left(\frac{|\widehat{\eta_\theta z_\theta}|}{|\widehat{\eta_\theta w_\theta}|}\frac{|\widehat{\xi_\theta w_\theta}|}{|\widehat{\xi_\theta z_\theta}|}\right).$$

By Lemma 3, the projection points $g_1(\theta) = \xi_\theta$ and $g_2(\theta) = \eta_\theta$ depend continuously on θ and by Lemma 2 the same holds for every projection point w_θ. This shows that F is continuous on $\partial\mathbb{D}^2 \times N(z_0) \times N(z_0)$. In particular, (8) holds. By compactness, F is uniformly continuous which implies that ρ_θ in (8) can be chosen independent of θ. It follows that (7) holds.

We proceed with the proof of the proposition by showing the converse. For this it suffices to show that for any sequence $\{z_n\}$ converging to z_0 with respect to the metric d, we have $|z_n - z_0| \to 0$. Assume, on the contrary, that $\{z_n\}$ is a sequence in P with $d\,(z_0,z_n) \to 0$ and $|z_n - z_0| > \varepsilon_0$ for some $\varepsilon_0 > 0$.

By choosing, if necessary, a subsequence we may assume that $\{z_n\}$ converges (in the Euclidean sense) to a point z'. Let θ_0 be a point in $\partial\mathbb{D}^2$ such that the line $c_{z_0\theta_0}$ does not contain z'. Pick a compact Euclidean ball $B(z')$ containing z' such that

$$B(z') \cap c_{z_0\theta_0} = \emptyset. \tag{9}$$

The image $f_{z_0}(B(z'))$ of $B(z')$ under the continuous map $f_{z_0} : \partial\mathbb{D}^2 \to \partial\mathbb{D}^2$ is a compact and connected subset of $\partial\mathbb{D}^2$ which, by (9), does not contain θ_0. It follows that we may pick a compact subinterval $N(\theta_0)$ of $\partial\mathbb{D}^2$ containing θ_0 and disjoint from $f_{z_0}(B(z'))$. By (5) we know that for any $\theta \in \partial\mathbb{D}^2$ and $x \in \mathbb{D}^2$

$$d_\theta(x,z_0) = 0 \Longleftrightarrow x \in c_{z_0\theta}$$

which implies that

$$d_\theta(z,z_0) > 0 \text{ for all } \theta \in N(\theta_0) \text{ and } z \in B(z'). \tag{10}$$

In other words, the restriction of the above defined map $F:\partial\mathbb{D}^2\times N(z_0)\times N(z_0)\to\mathbb{R}$ on the set $B(z')\times N(\theta_0)\times\{z_0\}$ does not attain the value $0\in\mathbb{R}$. By continuity and compactness, let $M>0$ be the minimum of F on $B(z')\times N(\theta_0)\times\{z_0\}$.

Let $\Theta_K=\{\theta_k \mid k=1,2,\dots\}$ be an enumeration of the set $\Theta\cap N(\theta_0)$ for which we proved that

$$d_{\theta_k}(z_n,z_0)>M \text{ for all } z_n \text{ and } \theta_k\in\Theta_K.$$

Then

$$d(z_n,z_0)=\sum_{\theta_i\in\Theta} w_i d_{\theta_i}(z_n,z_0)\geq\sum_{\theta_k\in\Theta_K} w_k d_{\theta_k}(z_n,z_0)>M\sum_k w_k>0$$

which contradicts the assumption $d(z_0,z_n)\to 0$. This completes the proof of the proposition. □

Theorem 8. *The metric space (P,d) is a geodesic metric space whose geodesics are precisely the curves $\{c\cap P \mid c\in\mathscr{C}\}$. In particular, (P,d) is uniquely geodesic.*

Proof. Let $[x,y]$ be the segment with endpoints $x,y\in P$. In other words (see terminology introduced in property (C1)), $[x,y]$ is the restriction of c_{xy} to an appropriate interval I such that $c_{xy}|_I:I\to P$ has endpoints x and y. As all curves in $\mathscr{C}$ are assumed to be continuous and injective with respect to the Euclidean topology of P so is the restriction $c_{xy}|_I$.

Proposition 7 implies that

(11) $$c_{xy}|_I:I\to P \text{ is continuous with respect to the topology induced by } d.$$

We will next show that

(12) $$d(x,z)+d(z,y)=d(x,y) \text{ for every } z\in[x,y].$$

Let $z\in[x,y]$. If θ is not an endpoint of the curve c_{xy}, then $x_\theta\neq y_\theta$ and z is contained in the region bounded by $(\theta,x_\theta),(\theta,y_\theta)$ and the arc $\widehat{x_\theta y_\theta}$ (which is the arc not containing θ). By property (C5) (θ,z_θ) does not intersect neither (θ,x_θ) nor (θ,y_θ). It follows that $z_\theta\in\widehat{x_\theta y_\theta}$. Moreover, the calculation (6) carried out in Case A of Lemma 4 holds verbatim, that is,

(13) $$d_\theta(x,y)=d_\theta(x,z)+d_\theta(z,y).$$

On the other hand, if θ is an endpoint of the curve c_{xy}, then $x_\theta=z_\theta=y_\theta$ and the above inequality holds trivially. It follows that equality (13) holds for all $\theta\in\Theta$ and hence $d(x,y)=d(x,z)+d(z,y)$.

The additive property (12) holds for any three points in $[x,y]$ which implies that

(14) $$c_{xy}|_I:I\to P \text{ has finite length.}$$

The latter property along with (11) assert that $c_{xy}|_I$ can be parametrized by arclength. It is well known (see, for example, Proposition 2.2.7 in [8]) that a curve with arclength parametrization and endpoints x,y is a geodesic segment with respect to a metric d if and only if for every z in the curve we have $d(x,z)+d(z,y)=d(x,y)$. This completes the proof that $[x,y]$ is a geodesic with respect to d.

Last we show that the segment $[x,y]$ is the unique geodesic segment with respect to d joining x,y. Assume, on the contrary, that there exists a geodesic joining x,y which contains a point $z\notin[x,y]$. By the previous discussion, we may assume that $y\notin[x,z]$ and $x\notin[y,z]$. In other words, z is not a point of the curve c_{xy}.

Let ξ_{xy} be one endpoint of c_{xy}. Clearly,

$$d_{\xi_{xy}}(x,y) = 0 \lneqq d_{\xi_{xy}}(x,z).$$

By the triangle inequality for the pseudo-metric $d_{\xi_{xy}}$ we have

$$d_{\xi_{xy}}(x,y) \lneqq d_{\xi_{xy}}(x,z) + d_{\xi_{xy}}(z,y)$$

and

$$d_{\theta}(x,y) \leq d_{\theta}(x,z) + d_{\theta}(z,y) \text{ for all } \theta \neq \xi_{xy}.$$

It follows that $d(x,y) \lneqq d(x,z) + d(z,y)$ which contradicts the fact that z lies on a geodesic joining x,y.

□

Lemma 9. *Let $x \in P$, v a point in the boundary of P and $[x,v]$ the unique segment obtained from $\mathscr{C}$. By the above Theorem, $[x,v) = [x,v] \setminus \{v\}$ can be viewed as a geodesic (with respect to d) ray r_v of P. Then, for any sequence $\{y_n\} \subset [x,v)$ converging to v in the Euclidean sense we have*

$$d(x,y_n) \to \infty \text{ as } n \to \infty.$$

In particular, the geodesic ray r_v is realized by an isometry $r_v : [0,\infty) \longrightarrow P$ and (P,d) a geodesically complete metric space.

Proof. Let θ_0, ξ_{θ_0} be the endpoints of the curve c which is the unique curve determined by the segment of ∂P containing v. By assumption in Definition 5, $\theta_0 \in \Theta$ and it suffices to show that

$$d_{\theta_0}(x,y_n) \to \infty \text{ as } n \to \infty.$$

We claim that $(y_n)_{\theta_0} \to \xi_{\theta_0}$ or, equivalently, $\left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right| \longrightarrow 0$. To see this, assume, on the contrary, that

$$\exists \varepsilon_0 > 0 : \forall n \in \mathbb{N}, \exists N > n \text{ with } \left|\widehat{\xi_{\theta_0}(y_N)_{\theta_0}}\right| \geq \varepsilon_0. \tag{15}$$

Pick a point $\theta_\delta \in \widehat{\xi_{\theta_0}(y_1)_{\theta_0}}$ such that $\left|\widehat{\xi_{\theta_0}\theta_\delta}\right| < \varepsilon_0$. The line $(\theta_0, \theta_\delta)$ must intersect $[x,v]$ at a point, say, y_δ. For all $n \in \mathbb{N}$ such that $y_n \in [y_\delta, v)$ we have $\left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right| < \left|\widehat{\xi_{\theta_0}\theta_\delta}\right| < \varepsilon_0$ contradicting (15).

This shows that $\left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right| \longrightarrow 0$ or, equivalently, $\log \frac{1}{\left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right|} \to \infty$. As $\left|\widehat{\eta_{\theta_0}(y_n)_{\theta_0}}\right| < \left|\widehat{\eta_{\theta_0}\xi_{\theta_0}}\right|$ for all n it follows

$$\begin{aligned} d_{\theta_0}(x,y_n) &= \log \frac{\left|\widehat{\eta_{\theta_0}(y_n)_{\theta_0}}\right| \left|\widehat{\xi_{\theta_0}x_{\theta_0}}\right|}{\left|\widehat{\eta_{\theta_0}x_{\theta_0}}\right| \left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right|} \leq \log \frac{\left|\widehat{\eta_{\theta_0}\xi_{\theta_0}}\right| \left|\widehat{\xi_{\theta_0}x_{\theta_0}}\right|}{\left|\widehat{\eta_{\theta_0}x_{\theta_0}}\right| \left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right|} = \\ &= \log \frac{\left|\widehat{\eta_{\theta_0}\xi_{\theta_0}}\right| \left|\widehat{\xi_{\theta_0}x_{\theta_0}}\right|}{\left|\widehat{\eta_{\theta_0}x_{\theta_0}}\right|} + \log \frac{1}{\left|\widehat{\xi_{\theta_0}(y_n)_{\theta_0}}\right|} \to \infty \end{aligned}$$

as required. □

In a similar manner the following can be seen: let v,w be two points in the boundary of P contained in distinct segments of ∂P. Then $(v,w) := c_{v,w} \cap P$ is a geodesic line in P of infinite length.

4. Generalizations

In this Section let (X,ρ) be a proper geodesic metric space homeomorphic to $\mathbb{R}^2$ with the following uniqueness property:

(G1) for any two points $x,y \in X$ there exists a unique geodesic segment $\sigma_{xy} : I \to X$, where I is an interval in $\mathbb{R}$, with endpoints x and y. We denote σ_{xy} by $[x,y]$. Moreover, every segment $[x,y]$ extends uniquely to a geodesic line (isometry) $\sigma : \mathbb{R} \to X$. We will be saying that the line σ contains the segment $[x,y]$.

Denote by ∂X the boundary at infinity of X defined via asymptotic geodesic rays (see, for example, [3, page 260]). We assume that ∂X, equipped with the topology of uniform convergence on compact sets, is homeomorphic to $\mathbb{S}^1$ and ∂X compactifies X so that $X \cup \partial X$ is homeomorphic to the closed unit disk.

We further assume that $X \cup \partial X$ satisfies the following properties

(G2) for every two points $\xi,\eta \in \partial X$ there exists a unique geodesic line in X joining them.

(G3) for every two points $x \in X$ and $\xi \in \partial X$ there exists a unique geodesic ray in X joining them.

The class of such geodesic metric spaces satisfying properties (G1), (G2) and (G3) includes universal coverings of closed surfaces of genus ≥ 2 with a Riemannian metric of non-positive curvature.

We will say that P is a convex subset of X if for any two points $x,y \in P$ the segment $[x,y]$ is entirely contained in P. We say that P is a convex polygon in X if P is an open bounded convex subset of X whose boundary is a finite union of geodesic segments.

As properties (G1), (G2) and (G3) above are identical with properties (C1), (C2) and (C3) given at the beginning of Section 2, the construction carried out in Section 2 can be applied verbatim with the class $\mathscr{C}$ being the geodesic lines in X. This defines a new metric d on the convex polygon P such that (P,d) is a geodesically complete metric space whose geodesic lines are precisely the curves $\{c \cap P \mid c \text{ geodesic line in } X\}$.

However, a proper geodesic metric space homeomorphic to $\mathbb{R}^2$ need not have a boundary satisfying property (C2). The Euclidean space $\mathbb{R}^2$ itself provides such an example. For this class of metric spaces we carry out in the next Subsection a construction analogous to the one given in Section 2 by deploying a convex polygon containing the given polygon P. In the special case of $\mathbb{R}^2$ we describe, in Subsection 4.2 below, an analogous procedure for putting a metric on a polygon P which does not depend on the choice of a convex polygon containing P.

4.1. *Generalization to geodesic metric spaces.*

Recall that (X,ρ) denotes a proper geodesic metric space homeomorphic to $\mathbb{R}^2$ satisfying properties (G1)-(G3).

Lemma 10. *Given any convex polygon P in X there exists a convex polygon K in X with $P \cup \partial P \subset K$.*

Proof. Denote by $A_1,A_2,\dots A_n$ the vertices of P. For each side $[A_i,A_{i+1}]$ ($i = 1,\dots,n$ with $A_{n+1} \equiv A_1$) consider the geodesic (with respect to the geometry of X) line σ_i containing the side $[A_i,A_{i+1}]$.
Claim: for all i, $\sigma_i \cap P = \varnothing$.

To check this assume σ_i intersects P. Then $P \setminus \sigma_i$ consists of two or more components and denote by P_1 the component whose boundary contains $[A_i,A_{i+1}]$. If P_2 is an other component of $P \setminus \sigma_i$ then at least one of the vertices A_i, A_{i+1} is not contained in ∂P_2. Say, $A_i \notin \partial P_2$ and pick any $x \in P_2$. Then the geodesic segment $[A_i, x]$ must

either, contain $[A_i, A_{i+1}]$ which violates (G2) because $x \notin \sigma_i$

or, intersect σ_i at a point $y \in \sigma_i$ which violates (G1).

Notation: for a geodesic line h not intersecting P we will be writing h^+ for the component of $X \setminus h$ which contains P.

We next construct a convex polygon K containing P. For each of the n sides $[A_i, A_{i+1}]$ of the polygon P consider a geodesic (with respect to the geometry of X) line ℓ_i as follows: choose points $B_i \in \sigma_{i-1}$ and $C_i \in \sigma_{i+1}$ such that

$$[A_{i-1}, B_i] = [A_{i-1}, A_i] \cup [A_i, B_i] \text{ and } [C_i, A_{i+2}] = [C_i, A_{i+1}] \cup [A_{i+1}, A_{i+2}]$$

Set ℓ_i to be the unique geodesic line containing B_i, C_i. Observe that for all i, $\ell_i \cap P = \varnothing$. To check this note that if ℓ_i intersects P then ℓ_i must intersect

either, σ_{i-1} at a point other than $B_i \in \sigma_{i-1} \cap \ell_i$

or, σ_{i+1} at a point other than $C_i \in \sigma_{i+1} \cap \ell_i$.

In both cases we have, by (G1), a contradiction.

Set $K' = \cap_{i=1}^{n} \ell_i^+$. Clearly, K' is convex and contains P. If K' is bounded we set the desired bounded convex polygon K to be K'. If K' is not bounded then each end $K'_j, j = 1, \ldots, k$ of K' is determined by two subrays r_j and r_{j+1} of ℓ_j and ℓ_{j+1} respectively. Pick a line $\ell_{K'_j}$ intersecting both r_j and r_{j+1} and then the intersection

$$K = K' \cap \left(\cap_{j=1}^{k} \ell_{K'_j}^+ \right)$$

is the desired bounded convex polygon containing P. □

Given any convex bounded polygon P in X consider and fix a convex polygon K containing P in its interior. As explained above, such a polygon always exists. The boundary ∂K is homeomorphic to the circle. We may now perform the construction described in Section 3 where the geodesic segments in the metric space (X, ρ) with endpoints on ∂K constitute a collection of curves satisfying properties (C1)-(C4).

The Euclidean length $\left|\widehat{\theta_i\theta_j}\right|$ used to define the pseudo-metric d_θ in P is the only adjustment needed: for points $\theta_1, \theta_2, \theta_3, \theta_4$ in cyclic clockwise order in ∂K, denote by $\widehat{\theta_i\theta_j}$ the piece-wise geodesic curve in ∂K with endpoints θ_i, θ_j and by $\left|\widehat{\theta_i\theta_j}\right|$ its length with respect to the metric ρ of X. Then the pseudo-metric d_θ is given by (4) in an identical way.

The metric space (P, d) obtained in this way is a geodesic metric space whose geodesics are precisely the curves

$$\left\{ \sigma \cap P \;\middle|\; \sigma \text{ is a geodesic line in } (X, \rho) \right\}.$$

In particular, (P, d) is uniquely geodesic (G1) and geodesically complete (G2).

4.2. ***The special case*** $\mathbb{R}^2$**.** Let P be a convex polygon in Euclidean space $\mathbb{R}^2$. We may view $\mathbb{S}^1 \subset \mathbb{R}^2$ as the set of directions in $\mathbb{R}^2$ where each $x \in \mathbb{S}^1$ determines a unique angle $\theta \in [0, 2\pi)$.

For each direction θ there exist exactly two parallel lines, say ℓ_ξ, ℓ_η such that the strip bounded by them contains P and the strip is minimal with respect to this property. Pick any line ℓ perpendicular

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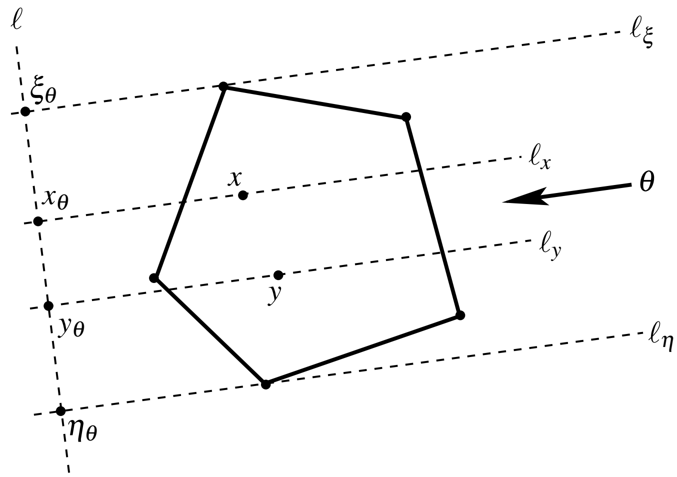

FIGURE 5. The strip which bounds P in the θ direction and the projection points whose distances define d_θ

to ℓ_ξ, ℓ_η and denote by ℓ_x (resp. ℓ_y) the line which contains x (resp. y) and is parallel to ℓ_ξ (see Figure 5). Set

ξ_θ the intersection point $\ell_\xi \cap \ell$
η_θ the intersection point $\ell_\eta \cap \ell$
x_θ the intersection point $\ell_x \cap \ell$
y_θ the intersection point $\ell_y \cap \ell$

We may define d_θ as in Section 2, equation (4)

$$d_\theta(x,y) := \log[\eta_\theta, y_\theta, x_\theta, \xi_\theta] = \log \frac{|\eta_\theta x_\theta|\,|\xi_\theta y_\theta|}{|\eta_\theta y_\theta|\,|\xi_\theta x_\theta|}$$

where $|\;\;|$ stands for Euclidean distance. As in Lemma 4, it can be seen that d_θ is a pseudo-distance on P.

Let Θ be a dense subset of $\mathbb{S}^1$ containing all directions determined by the sides of P. As in Definition 5, to each θ_i in Θ assign a positive real w_i such that the series $\sum_i w_i$ converges. For $x, y \in P$ define

$$d(x,y) := \sum_{i=1}^{\infty} w_i d_{\theta_i}(x,y).$$

Working in an identical way as in Section 2 it can be seen that d is a metric making P a geodesically complete metric space whose geodesic lines are precisely the (open) Euclidean segments in P.

5. Further properties of (P,d)

For a convex domain U in $\mathbb{R}^2$ denote by $d_{\mathcal{H}}$ the Hilbert metric on U for which we refer the reader to [8, Ch.5, Section 6]. Let (P,d) be the metric space obtained in Section 3 with the collection of curves $\mathcal{C}$ being the Euclidean lines.

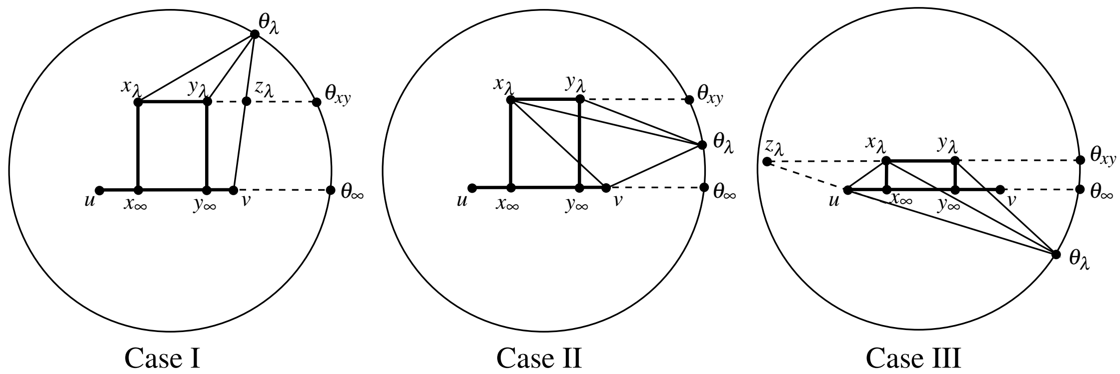

FIGURE 6. The three cases considered in the Claim in the proof of Theorem 11.

Theorem 11. *For any convex domain U in $\mathbb{R}^2$ equipped with the Hilbert metric $d_{\mathscr{H}}$ the metric spaces (P,d) and $(U,d_{\mathscr{H}})$ are not isometric.*

Proof. Before dealing with the proof of the Theorem, we will state and prove a claim concerning geodesic rays in P which have endpoints in the same side of P.

Let $[u,v]$ be a side of P, where u,v are vertices in ∂P, and consider two parallel (in the Euclidean sense) geodesic rays $[x,x_\infty]$ and $[y,y_\infty]$ with $x_\infty, y_\infty \in [u,v]$ and $y_\infty \in [x_\infty, v]$ Pick sequences $\{x_n\}_{n\in\mathbb{N}} \subset [x,x_\infty]$ and $\{y_n\}_{n\in\mathbb{N}} \subset [y,y_\infty]$ such that for all $n \in \mathbb{N}$ the segment $[x_n,y_n]$ is parallel to $[u,v]$ and, in addition, $x_n \to x_\infty$ and $y_n \to y_\infty$. In other words,

$$\lim_{n\to\infty} d(x,x_n) = \infty = \lim_{n\to\infty} d(y,y_n).$$

Claim: The set $\left\{ d(x_n,y_n) \,\middle|\, n \in \mathbb{N} \right\}$ is bounded.

Proof of Claim. As $d(x_n,y_n) = \sum_{k=1}^{\infty} w_k d_{\theta_k}(x_n,y_n)$ it suffices to show that

$$\left\{ d_{\theta_k}(x_n,y_n) \,\middle|\, k,n \in \mathbb{N} \right\}$$

is bounded. Assume that it is not. Then there must exist a sequence

$$\left\{ d_{\theta_{k(\lambda)}}\left(x_{n(\lambda)},y_{n(\lambda)}\right) \right\}_{\lambda=1}^{\infty} \tag{16}$$

converging to ∞ as $\lambda \to \infty$. To simplify notation we write $d_{\theta_\lambda}(x_\lambda,y_\lambda)$ instead of $d_{\theta_{k(\lambda)}}\left(x_{n(\lambda)},y_{n(\lambda)}\right)$. The Euclidean line extending the side $[u,v]$ intersects the boundary of the unit disk in two points denoted by θ_∞ and $\theta_{-\infty}$. Clearly, $\left\{ d_{\theta_\lambda}(x_\lambda,y_\lambda) \right\}$ is bounded for all θ_λ away from θ_∞ and $\theta_{-\infty}$. We will examine $\left\{ d_{\theta_\lambda}(x_\lambda,y_\lambda) \right\}$ for θ_λ close to θ_∞ and an identical argument will work for $\theta_{-\infty}$. We may assume that the segment $[u,\theta_\infty]$ contains v.

We distinguish 3 cases as depicted in Figure 6. Denote by θ_{xy} the intersection of $\partial\mathbb{D}^2$ with the extension of $[x_\lambda,y_\lambda]$ such that $y_\lambda \in [x_\lambda,\theta_{xy}]$.

Case I: $\theta_{xy} \in \widehat{\theta_\lambda\theta_\infty}$.

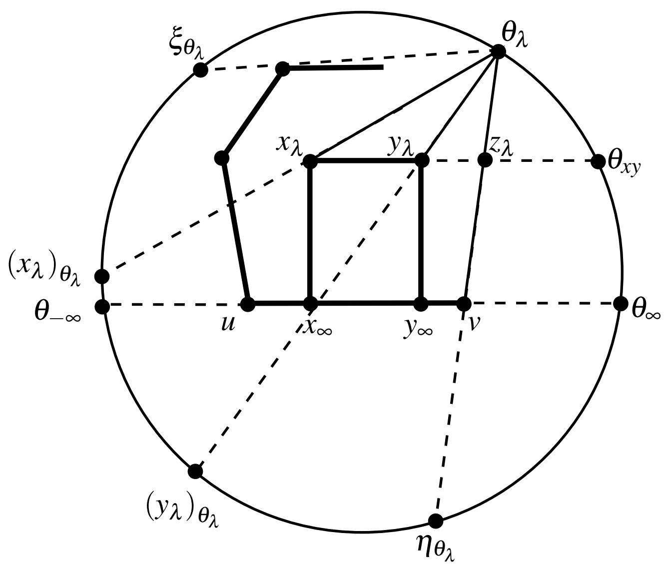

FIGURE 7. Relevant notation in Case I of the Claim in the proof of Theorem 11.

Case II: $\theta_\lambda \in \widehat{\theta_{xy}\theta_\infty}$.
Case III: $\theta_\infty \in \widehat{\theta_\lambda \theta_{xy}}$.
We discuss in detail Case I (see Figure 7). The distance $d_{\theta_\lambda}(x_\lambda, y_\lambda)$ is by definition (see equations (1) and (4)) the sum of the logarithms of two ratios

$$d_{\theta_\lambda}(x_\lambda, y_\lambda) = \log \frac{\left|\eta_{\theta_\lambda}\widehat{(x_\lambda)_{\theta_\lambda}}\right|}{\left|\eta_{\theta_\lambda}\widehat{(y_\lambda)_{\theta_\lambda}}\right|} + \log \frac{\left|\xi_{\theta_\lambda}\widehat{(y_\lambda)_{\theta_\lambda}}\right|}{\left|\xi_{\theta_\lambda}\widehat{(x_\lambda)_{\theta_\lambda}}\right|}$$

For large enough λ both projections $(x_\lambda)_{\theta_\lambda}$ and $(y_\lambda)_{\theta_\lambda}$ approach $\theta_{-\infty}$ and therefore the second summand is bounded for all large enough λ. We will reach a contradiction by showing that the first summand or, equivalently, the ratio $x_\lambda\widehat{\theta_\lambda} z_\lambda / z_\lambda \widehat{\theta_\lambda} y_\lambda$ of the corresponding angles is bounded, where z_λ is the point where the extension of $[x_\lambda, y_\lambda]$ meets $[\theta_\lambda, v]$. As $\sin a < a < 2\sin a$ for small enough a we may work with the ratio

$$\frac{\sin\left(x_\lambda \widehat{\theta_\lambda} z_\lambda\right)}{\sin\left(z_\lambda \widehat{\theta_\lambda} y_\lambda\right)}$$

Using the law of sines for the triangles $x_\lambda \theta_\lambda z_\lambda$ and $y_\lambda \theta_\lambda z_\lambda$, we obtain

$$\frac{\sin\left(x_\lambda \widehat{\theta_\lambda} z_\lambda\right)}{\sin\left(z_\lambda \widehat{\theta_\lambda} y_\lambda\right)} = \frac{|x_\lambda z_\lambda|\ |y_\lambda \theta_\lambda|}{|y_\lambda z_\lambda|\ |x_\lambda \theta_\lambda|}.$$

Clearly, the right hand side of the above equality converges to $\frac{|x_\infty v|}{|y_\infty v|}\frac{|y_\infty \theta_\infty|}{|x_\infty \theta_\infty|}$ as $\lambda \to \infty$, which is a positive real number depending on the Euclidean distances of the boundary points x_∞, y_∞, v and θ_∞. Therefore we have shown that the set

$$\left\{d_\lambda\left(x_\lambda, y_\lambda\right) \mid \lambda \text{ satisfies Case I}\right\}$$

is bounded.

In Case II, adopting the same reasoning, we will reach a contradiction by showing that the ratio $y_\lambda \widehat{\theta_\lambda} v / x_\lambda \widehat{\theta_\lambda} v$ is bounded. As

$$\frac{y_\lambda \widehat{\theta_\lambda} v}{x_\lambda \widehat{\theta_\lambda} v} = \frac{x_\lambda \widehat{\theta_\lambda} v + x_\lambda \widehat{\theta_\lambda} y_\lambda}{x_\lambda \widehat{\theta_\lambda} v} = 1 + \frac{x_\lambda \widehat{\theta_\lambda} y_\lambda}{x_\lambda \widehat{\theta_\lambda} v}$$

it suffices to bound the ratio

$$\frac{\sin\left(x_\lambda \widehat{\theta_\lambda} y_\lambda\right)}{\sin\left(x_\lambda \widehat{\theta_\lambda} v\right)}.$$

Using again the law of sines for the triangles $x_\lambda \theta_\lambda y_\lambda$ and $x_\lambda \theta_\lambda v$ we obtain

$$\frac{\sin\left(x_\lambda \widehat{\theta_\lambda} y_\lambda\right)}{\sin\left(x_\lambda \widehat{\theta_\lambda} v\right)} = \frac{\sin\left(y_\lambda \widehat{x_\lambda} \theta_\lambda\right)}{\sin\left(v \widehat{x_\lambda} \theta_\lambda\right)} \frac{|x_\lambda y_\lambda|}{|y_\lambda \theta_\lambda|} \frac{|\theta_\lambda v|}{|x_\lambda v|}$$

$$(17) \qquad < \frac{\sin\left(x_\lambda \widehat{\theta_\lambda} v\right)}{\sin\left(v \widehat{x_\lambda} \theta_\lambda\right)} \frac{|x_\lambda y_\lambda|}{|y_\lambda \theta_\lambda|} \frac{|\theta_\lambda v|}{|x_\lambda v|} \overset{(*)}{=} \frac{|x_\lambda v|}{|\theta_\lambda v|} \frac{|x_\lambda y_\lambda|}{|y_\lambda \theta_\lambda|} \frac{|\theta_\lambda v|}{|x_\lambda v|} = \frac{|x_\lambda y_\lambda|}{|y_\lambda \theta_\lambda|}$$

where the equality $(*)$ follows from the law of sines for the triangle $x_\lambda \theta_\lambda v$ and the inequality follows from the fact the angle $x_\lambda \widehat{\theta_\lambda} v$ is always (in Case II) strictly larger than $y_\lambda \widehat{x_\lambda} \theta_\lambda$.

The right hand side of (17) clearly converges to $\frac{|x_\infty y_\infty|}{|y_\infty \theta_\infty|}$ as $\lambda \to \infty$, and as before, it follows that the set

$$\left\{d_\lambda\left(x_\lambda, y_\lambda\right) \mid \lambda \text{ satisfies Case II}\right\}$$

is bounded.

In Case III observe that the extension of the segment $[x_\lambda, y_\lambda]$ intersects $[\theta_\lambda, u]$ at a point z_λ which, for sufficiently large λ, lies inside the unit disk. As in Case I, we use the law of sines for the triangles $y_\lambda \theta_\lambda z_\lambda$ and $x_\lambda \theta_\lambda z_\lambda$ to obtain

$$\frac{\sin\left(y_\lambda \widehat{\theta_\lambda} z_\lambda\right)}{\sin\left(x_\lambda \widehat{\theta_\lambda} z_\lambda\right)} = \frac{|y_\lambda z_\lambda|}{|x_\lambda z_\lambda|} \frac{|x_\lambda \theta_\lambda|}{|y_\lambda \theta_\lambda|} \longrightarrow \frac{|y_\infty u|}{|x_\infty u|} \frac{|x_\infty \theta_\infty|}{|y_\infty \theta_\infty|} \text{ as } \lambda \to \infty.$$

This completes the proof of the Claim.

We return now to the proof of Theorem 11. Assume, on the contrary, that $F : (P, d) \to (U, d_{\mathscr{H}})$ is an isometry. We write a' for the image $F(a)$ of a point $a \in P$. The images of the geodesic rays $[x, x_\infty]$ and $[y, y_\infty]$ under F, denoted by $[x', x'_\infty]$ and $[y', y'_\infty]$ respectively, are clearly geodesic rays in U determining boundary points $x'_\infty, y'_\infty \in \partial U$. Moreover,

$$\lim_{n\to\infty} d_{\mathscr{H}}\left(x', x'_n\right) = \infty = \lim_{n\to\infty} d_{\mathscr{H}}\left(y', y'_n\right).$$

Let $[x'_\infty, y'_\infty]$ be the Euclidean segment joining x'_∞ and y'_∞.

If (x'_∞, y'_∞) is contained in U then $d_{\mathscr{H}}(x'_n, y'_n) \to \infty$ as $n \to \infty$, a contradiction by the Claim.

If $(x'_\infty, y'_\infty) \nsubseteq U$, then by convexity of U, $(x'_\infty, y'_\infty) \subset \partial U$, that is, ∂U contains at least one segment. In an identical way, we may perform the same construction starting with the side $[v,q]$ adjacent to $[u,v]$ and geodesic rays $[z, z_\infty]$ and $[w, w_\infty]$ with $z_\infty, w_\infty \in [v,q]$. It follows that (z'_∞, w'_∞) determines again a segment in ∂U. If $x'_\infty, y'_\infty, z'_\infty, w'_\infty$ were collinear then the geodesic line $(y_\infty, z_\infty) \subset P$ would have an image $(y'_\infty, z'_\infty) \subset U$ connecting the points y'_∞, z'_∞ contained in a segment in ∂U, which is impossible. It follows that ∂U contains two distinct segments and, thus, the metric space $(U, d_{\mathscr{H}})$ is not uniquely geodesic, a contradiction by Proposition 8. □

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References

[1] E. Beltrami, Saggio di Interpretazione della geometria non-Euclidea. Giornaledi Matematiche vol. VI (1868). Beltrami's Works, Vol. I, pp. 374–405. French translation: Essai d'interprétation de la géométrie noneucilienne, by J. Hoüel, Annales ENS, 1re Sér. t. 6 (1869), p. 251-288. English translation: Essay on the interpretation of non-Euclidean geometry, by J. Stillwell, in [11] pp. 7–34.

[2] W. Blaschke and G. Bol, *Geometrie der Gewebe,* Grundlehren Math. Wiss. 49, Springer-Verlag, Berlin 1938.

[3] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag, Berlin, 1999.

[4] H. Busemann, *On Hilbert fourth problem,* Uspechi Mat. Nauk 21 (1966), no. 1 (127), 155-164.

[5] H. Busemann and H. Salzman, *Metric collineations and inverse problems,* Math. Z. 87 (1965), 214–240.

[6] B. Colbois, C. Vernicos, and P. Verovic, *Hilbert geometry for convex polygonal domains,* Journal of Geometry. 100 (2011), 37–64.

[7] F. Klein, Uber die sogenannte Nicht-Euklidische Geometrie (erster Aufsatz). Math. Ann. IV (1871), 573–625. English translation by J. Stillwell in [11].

[8] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature,* IRMA Lectures in Mathematics and Theoretical Physics 6, European Mathematical Society, 2013.

[9] A. V. Pogorelov, Hilbert's fourth problem. Translated by R. A. Silverman, Edited by I. Kra, in cooperation with E. Zaustinskyi, V. H. Winston & Sons, Washington D. C., 1979. Original by Nauka.

[10] A. V. Pogorelov, A complete solution of Hilbert's fourth problem, Sov. Math. 14 (1973), pp. 46–49.

[11] J. Stillwell, Sources of hyperbolic geometry, History of Mathematics series, Vol. 10, AMS-LMS, 1996.

[12] M. Troyanov, *On the origin of Hilbert geometry*, Handbook of Hilbert Geometry, Lectures in Mathematics and Theoretical Physics 22, A. Papadopoulos and M. Troyanov editors, EMS 2014.

[13] C. Vernicos, *On the Hilbert geometry of convex polytopes*, Handbook of Hilbert Geometry, Lectures in Mathematics and Theoretical Physics 22, A. Papadopoulos and M. Troyanov editors, EMS 2014.

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