Coefficient Functionals and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators

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Abstract

This paper examines the coefficient problems for the class of semigroup generators, a topic in complex dynamics that has recently been studied in context of geometric function theory. Further, sharp bounds of coefficient functional such as second order Hankel determinant, third order Toeplitz and Hermitian-Toeplitz determinants are derived. Additionally, the sharp growth estimates and the bounds of difference of successive coefficients are determined, which are used to prove the Bohr and the Bohr- Rogosinski phenomenon for the class of semigroup generators.

Keywords: Holomorphic generators; Hankel determinant; Toeplitz determinant; Zalcman functional; Successive coefficient difference; Bohr and Bohr-Rogosinski radius.
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1 Introduction

Let $\mathcal{H}$ be the class of holomorphic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $A \subset \mathcal{H}$ containing functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

By $\mathcal{B}$, we denote the class of holomorphic self mappings from $\mathbb{D}$ to $\mathbb{D}$. A family $\{u_t(z)\}_{t\geq 0} \subset \mathcal{B}$ is called a one parameter continuous semigroup if (i) $\lim_{t\to 0} u_t(z) = z$, (ii)$u_{t+s}(z) = u_t(z) \circ u_s(z)$, and (iii) $\lim_{t\to s} u_t(z) = u_s(z)$ for each $z \in \mathbb{D}$ hold.

Berkson and Porta [3] showed that each one parameter semigroup is locally differentiable in parameter $t \geq 0$ and moreover, if

$$\lim_{t\to 0} \frac{z - u_t(z)}{t} = f(z),$$

which is a holomorphic function, then $u_t(z)$ is the solution of the the Cauchy problem

$$\frac{\partial u_t(z)}{\partial t} + f(u_t(z)) = 0, \quad u_0(z) = z.$$

The function f is called the holomorphic generator of semigroup $\{u_t(z)\} \subset \mathcal{B}$. The class of all holomorphic generators is denoted by $\mathcal{G}$. Also, note that each element of $\{u_t(z)\}$ generated by $f \in \mathcal{G}$ is univalent function while f is not necessarily univalent [14]. Various properties of generators and semigroup generated by them are discussed in [3, 6, 13, 16, 14, 39]. Berkson and Porta [3] proved:

Theorem 1.1. *The following assertions are equivalent:*

(a) $f \in \mathcal{G}$;

(b) $f(z) = (z - \sigma)(1 - z\bar{\sigma})p(z)$ *with some* $\sigma \in \overline{\mathbb{D}}$ *and* $p \in \mathcal{H}$, $\mathrm{Re}(p(z)) \geq 0$.

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The point $\sigma \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ is called the Denjoy–Wolff point of the semigroup generated by f. By Denjoy-Wolff theorem [39] for continuous semigroup, if any element of the semigroup generated by f is neither an elliptic automorphism of $\mathbb{D}$ nor the identity map for at least one $t \in [0, \infty)$, then there is a unique point $\sigma \in \overline{\mathbb{D}}$ such that $\lim_{t\to\infty} u(t, z) = \sigma$ uniformly, for each $z \in \mathbb{D}$. We denote the class of holomorphic generators with Denjoy-Wolff point σ by $\mathcal{G}[\sigma]$. For $\sigma = 0$, we obtain the following subclass

$$\mathcal{G}[0] = \{f \in \mathcal{G} : f(z) = zp(z), \ \ \operatorname{Re} p(z) \geq 0\}.$$

Bracci et al. [7] considered the class $\mathcal{G}_0 = \mathcal{G}[0] \cap \mathcal{A}$. In the study of non-autonomous problem such as Loewner theory, the class $\mathcal{G}_0$ plays a significant role [8, 12]. Various subclasses of $\mathcal{G}_0$ with parameter such that $\mathcal{R}$ is the smallest one were recently studied (also called filtration), where

$$\mathcal{R} = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0\}$$

is the class of functions with bounded turning (see [7, 16, 40]). In particular, for $\beta \in [0, 1]$, the class

$$\mathcal{A}_\beta = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\beta \frac{f(z)}{z} + (1 - \beta) f'(z) \right) > 0 \right\} \tag{1.2}$$

is a subclass of $\mathcal{G}_0$. In [7], the authors proved that $\mathcal{A}_{\beta_1} \subsetneq \mathcal{A}_{\beta_2} \subsetneq \mathcal{G}_0$ for $0 \leq \beta_1 < \beta_2 < 1$ and whenever $f \in \mathcal{A}_\beta$,

$$\operatorname{Re} \frac{f(z)}{z} \geq \int_0^1 \frac{1 - t^{1-\beta}}{1 + t^{1-\beta}} dt.$$

Clearly, when $\beta = 0$, the class $\mathcal{A}_\beta$ reduces to the class $\mathcal{R}$. Elin et al. [15] solved the radii problems for the class $\mathcal{A}_\beta$. They found the radii $r \in (0, 1)$ for $f \in \mathcal{A}_\beta$ such that $f(rz)/r$ belong to the class of starlike functions, denoted by $\mathcal{S}^*$, and some other subclasses of starlike functions. This problem arises from the fact that neither $\mathcal{S}^* \subset \mathcal{A}_\beta$ nor $\mathcal{A}_\beta \subset \mathcal{S}^*$. Generalizing this work, Giri and Kumar [18] obtained r such that $f(rz)/r$ belong to a unified subclass of starlike functions $\mathcal{S}^*(\varphi)$, where φ is a univalent function mapping the unit disk in certain specific domain in the right half plane.

For the class $\mathcal{A}_\beta$, coefficient problems, growth estimates and others were still open. In this paper, we focus on these problems. We find the bound of n^{th} Taylor series coefficient of $f \in \mathcal{A}_\beta$ and certain coefficient functionals such as second Hankel determinant, third order Toeplitz and Hermitian Toeplitz determinant, and Zalcman functional. Later, Bohr and Bohr-Rogosinski phenomenon with growth estimates are also discussed for the same class.

In 1914, Bohr [5] proved that, if $\omega(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{B}$, then $\sum_{n=0}^{\infty} |c_n| r^n \leq 1$ for all $z \in \mathbb{D}$ with $|z| = r \leq 1/3$. The constant $1/3$ is known as Bohr radius and it can not be improved. Different generalizations of the Bohr inequality are taken into consideration [42, 30]. We say that, the class $\mathcal{A}_\beta$ satisfies the Bohr phenomenon if there exists r_b such that

$$|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\mathbb{D}))$$

holds in $|z| = r \leq r_b$, where $\partial f(\mathbb{D})$ is the boundary of image domain of $\mathbb{D}$ under f and d denotes the Euclidean distance between $f(0)$ and $\partial f(\mathbb{D})$.

Muhanna [35] showed that the Bohr phenomenon holds for the class of univalent functions and the class of convex functions, when $|z| = r \leq 3 - 2\sqrt{2}$ and $|z| = r \leq 1/3$ respectively. We refer to the survey article [34] for further details on this topic. There is also the concept of Rogosinski radius along with the Bohr radius, although a little is known about Rogosinski radius in comparison to Bohr radius [20, 25, 38]. It says that, if $\omega(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{B}$, then

$$\sum_{n=0}^{N-1} |c_n||z|^n \leq 1 \quad (N \in \mathbb{N})$$

in the disk $|z| = r \leq 1/2$. The radius $1/2$ is called the Rogosinski radius. Kayumov et al. [21] considered the following expression, called Bohr-Rogosinski sum,

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n$$

and found the radius r_N such that $R_N^f(z) \leq 1$ in $|z| = r \leq r_N$ for the Cesáro operators on the space of bounded analytic functions. The largest such r_N is called the Bohr-Rogosinski radius. Here, we say that:

Definition 1.2. The class $\mathcal{A}_\beta$ satisfies the Bohr-Rogosinski phenomenon if there exist r_N such that

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega)), \quad m, N \in \mathbb{N}$$

holds in $|z| = r \leq r_N$.

Section 5 is devoted to find the r_b and r_N for the class $\mathcal{A}_\beta$.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, the m^{th} Hankel, Toeplitz and Hermitian Toeplitz determinant for $m \geq 1$ and $n \geq 0$ are respectively given by

$$H_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix},$$

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix}, \tag{1.3}$$

$$T_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n+m-1} & \bar{a}_{n+m-2} & \cdots & a_n \end{vmatrix}, \tag{1.4}$$

where $\bar{a}_n = \overline{a_n}$. Toeplitz matrices have constant entries along their diagonals, while Hankel matrices have constant entries along their reverse diagonals. In particular,

$$H_2(n)(f) = a_n a_{n+2} - a_{n+1}^2, \;\; T_3(1)(f) = 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2$$

and $T_{3,1}(f) = 1 - 2|a_2|^2 + 2\operatorname{Re}(a_2^2 \bar{a}_3) - |a_3|^2$. Finding the sharp bound of $|H_2(2)(f)|$ for the class $\mathcal{S}$ and its subclasses has always been the focus of many researchers. Although, investigations concerning Toeplitz and Hermitian Toeplitz are recently introduced in [2, 11], a summary of some of the more significant results is given in [41]. For more work in this direction (see [24, 23, 27, 19, 36]).

In 1999, Ma [31] proposed a conjecture for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ that

$$|J_{m,n}| := |a_n a_m - a_{n+m-1}| \leq (m-1)(n-1).$$

He proved this conjecture for the class of starlike functions and univalent functions with real coefficients. It is also called generalized Zalcman conjecture as it generalizes the Zalcman conjecture $|a_n^2 - a_{2n-1}| \leq (2n-1)^2$ for $f \in \mathcal{S}$. Recently, bound of $|J_{2,3}|$ are obtained for various subclasses of $\mathcal{A}$ [1, 10]. In section 2 and 3, we obtain the sharp bound of $|H_2(2)(f)|$, $|T_3(1)(f)|$ and $|J_{2,3}(f)|$ for $f \in \mathcal{A}_\beta$.

2 Hankel Determinant and Zalcman Functional

Theorem 2.1. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then*

$$|a_n| \le \frac{2}{n-\beta(n-1)}. \tag{2.1}$$

Further, this inequality is sharp for each n.

Proof. Let $f \in \mathcal{A}_\beta$ is given by (1.1), then we have

$$\beta\frac{f(z)}{z} + (1-\beta)zf'(z) = p(z) \qquad (z \in \mathbb{D}),$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ such that $\operatorname{Re} p(z) > 0$ is a member of the Carathéodory class $\mathcal{P}$. Upon comparing the coefficients of same powers on either side with the series expansion of f and p yields

$$(n-(n-1)\beta)a_n = p_{n-1} \tag{2.2}$$

for $n = 2, 3, 4, \cdots$, which gives the needed bound of $|a_n|$ using the Carathéodory coefficient bounds $|p_n| \le 2$ (see [12]). The function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = z\Bigg(-1 + 2\bigg({}_2F_1\bigg[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, z\bigg]\bigg)\Bigg) = z + \sum_{n=2}^{\infty} \frac{2}{n-(n-1)\beta} z^n \tag{2.3}$$

satisfies the condition $\operatorname{Re}\Big(\beta\tilde{f}(z)/z + (1-\beta)\tilde{f}'(z)\Big) > 0$, hence $\tilde{f}$ is a member of $\mathcal{A}_\beta$, where ${}_2F_1$ denotes the Gauss hypergeometric function. Equality in (2.1) occurs for $\tilde{f}$, which proves the sharpness of the bound.

Corollary 2.2. *If $f \in \mathcal{A}_\beta$, then for any real $\mu \ge 0$*

$$|a_n a_{n+2} - \mu a_{n+1}^2| \le \frac{4}{(n-(n-1)\beta)(n+2-(n+1)\beta)} + \frac{4\mu}{(n+1-n\beta)^2}.$$

The bound is sharp.

Proof. Since $|a_n a_{n+2} - \mu a_{n+1}^2| \le |a_n||a_{n+2}| + \mu|a_{n+1}|^2$. The bound simply follows from (2.1). To see the sharpness, consider

$$\tilde{f}_1(z) = z\left(-1 + 2\left({}_2F_1\left[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, iz\right]\right)\right) = z + \sum_{n=2}^{\infty} \frac{2i^{n-1}}{(n-(n-1)\beta)} z^n. \tag{2.4}$$

It can be easily seen that $\tilde{f}_1(z)$ satisfy (1.2), thus $\tilde{f}_1 \in \mathcal{A}_\beta$.

For $\mu = 1$, Corollary 2.2 gives the following sharp bound:

Corollary 2.3. *If $f \in \mathcal{A}_\beta$ is of the form (2.2), then*

$$|H_2(n)(f)| \le \frac{4\left((2n^2-1)\beta^2 - (4n^2+4n-2)\beta + 2n^2+4n+1\right)}{(n-(n-1)\beta)(n+2-(n+1)\beta)(n+1-n\beta)^2}.$$

For $n = 2$ and 3, the following sharp bound of second order Hankel determinant follows:

Corollary 2.4. *If $f \in \mathcal{A}_\beta$ is of the form (2.2), then*

$$|H_2(2)(f)| \le \frac{4(7\beta^2 - 22\beta + 17)}{(4-3\beta)(3-2\beta)^2(2-\beta)}, \quad |H_2(3)(f)| \le \frac{4(17\beta^2 - 46\beta + 31)}{(5-4\beta)(4-3\beta)^2(3-2\beta)}.$$

Theorem 2.5. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then*

$$|J_{2,3}(f)| \leq \frac{2}{4-3\beta}.$$

The bound is sharp.

Proof. Let $f \in \mathcal{A}_\beta$ is given by (1.1), then from (2.2), we have

$$|J_{2,3}(f)| = |a_2 a_3 - a_4| = \left| \frac{p_1 p_2}{(3-2\beta)(2-\beta)} - \frac{p_3}{4-3\beta} \right|. \tag{2.5}$$

For $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, Libera et al. [29] proved that

$$\begin{aligned} 2p_2 &= p_1^2 + x(4-p_1^2), \\ 4p_3 &= p_1^3 + 2xp_1(4-p_1^2) - x^2 p_1(4-p_1^2) + 2z(1-|x|^2)(4-p_1^2), \end{aligned} \tag{2.6}$$

where $|x| \leq 1$ and $|z| \leq 1$. Substituting these values of p_2 and p_3 in (2.5), we obtain

$$\begin{aligned} |J_{2,3}(f)| = \Bigg| \frac{p_1^3}{4}\left(\frac{2}{2\beta^2-7\beta+6} + \frac{1}{3\beta-4}\right) - \frac{p_1(4-p_1^2)(1-\beta)^2 x}{(2-\beta)(3-2\beta)(4-3\beta)} \\ + \frac{p_1(4-p_1^2)x^2}{4(4-3\beta)} - \frac{(4-p^2)(1-|x|^2)z}{2(4-3\beta)} \Bigg|. \end{aligned}$$

Since the class $\mathcal{P}$ is rotationally invariant and it is an easy exercise to check that the class $\mathcal{A}_\beta$ is also rotationally invariant, therefore, without losing generality, we can take $p_1 = p \in [0,2]$. Now, applying the triangle inequality with $|x| = \rho$, we obtain

$$\begin{aligned} |J_{2,3}(f)| \leq & \frac{p^3}{4}\left(\frac{2}{2\beta^2-7\beta+6} + \frac{1}{3\beta-4}\right) + \frac{p(4-p^2)(1-\beta)^2\rho}{(2-\beta)(3-2\beta)(4-3\beta)} + \frac{4-p^2}{2(4-3\beta)} \\ & + \rho^2\left(\frac{p(4-p^2)}{4(4-3\beta)} - \frac{(4-p^2)}{2(4-3\beta)}\right) =: F(p,\rho). \end{aligned}$$

To determine the maximum value of $F(p,\rho)$, first we find out the stationary points, given by the roots of $\partial F/\partial p = 0$ and $\partial F/\partial \rho = 0$, where

$$\begin{aligned} \frac{\partial F(p,\rho)}{\partial p} &= \frac{3p^2(r^2(2\beta^2-7\beta+6) + 4r(1-\beta)^2 + 2\beta^2-\beta-2)}{4(-2+\beta)(-3+2\beta)(-4+3\beta)} + p\left(\frac{r^2}{4-3\beta} - \frac{1}{4-3\beta}\right) \\ &\quad + \frac{r^2}{4-3\beta} + \frac{4r(1-\beta)^2}{(4-3\beta)(3-2\beta)(2-\beta)}. \\ \frac{\partial F(p,\rho)}{\partial \rho} &= 2r\left(\frac{p(4-p^2)}{4(4-3\beta)} + \frac{-4+p^2}{2(4-3\beta)}\right) + \frac{p(4-p^2)(1-\beta)^2}{(4-3\beta)(3-2\beta)(2-\beta)}. \end{aligned}$$

A simple calculation shows that for $p \in [0,2]$ and $r \in [0,1]$, the stationary point is $(0,0)$ and

$$\left(\frac{\partial^2 F}{\partial p^2}\frac{\partial^2 F}{\partial \rho^2} - \frac{\partial^2 F}{\partial \rho \partial p}\right)_{(p,\rho)=(0,0)} = \frac{4(8-11\beta+4\beta^2)}{(3-2\beta)^2(2-\beta)^2(4-3\beta))} > 0 \quad \text{for all } \beta \in [0,1].$$

Thus $F(p,\rho)$ attains either maximum or minimum at $(p,\rho) = (0,0)$. Since, we have

$$\left(\frac{\partial^2 F}{\partial p^2}\right)_{(0,0)} = \frac{-1}{4-3\beta} < 0, \quad \left(\frac{\partial^2 F}{\partial \rho^2}\right)_{(0,0)} = \frac{-4}{4-3\beta} < 0 \quad \text{for all } \beta \in [0,1].$$

Therefore, $F(p,\rho)$ attain its maximum value at $(p,\rho) = (0,0)$, which is $2/(4-3\beta)$.

Now, to prove the sharpness of the bound, consider the function $\tilde{f}_2 : \mathbb{D} \to \mathbb{C}$ given by

$$\alpha \frac{\tilde{f}_2(z)}{z} + (1-\alpha)\tilde{f}_2'(z) = \frac{1+z^3}{1-z^3}. \tag{2.7}$$

If $\tilde{f}_2(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $a_2 = a_3 = 0$ and $a_4 = 2/(4-3\beta)$, thus $|J_{2,3}(f)| = 2/(4-3\beta)$.

3 Toeplitz and Hermitian-Toeplitz Determinant

Theorem 3.1. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then*

(i) $|T_{2,n}(f)| \leq 4\left(\dfrac{1}{(n-\beta(n-1))^2} + \dfrac{1}{(n+1-n\beta)^2}\right)$,

(ii) $|T_{3,1}(f)| \leq \dfrac{4\beta^4 - 28\beta^3 + 101\beta^2 - 196\beta + 140}{(3-2\beta)^2(\beta-2)^2}$.

The bounds are sharp.

Proof. From (1.3), it follows that

$$|T_{2,n}(f)| = |a_n^2 - a_{n+1}^2| \leq |a_n|^2 + |a_{n+1}|^2.$$

Using the bound of $|a_n|$ from (2.1), required bound of $|T_{2,n}(f)|$ follows directly and equality case holds for the function $\tilde{f}_1$ given by (2.4).

Now we proceed for $|T_{3,1}(f)|$. Again from (1.3), we have

$$|T_{3,1}(f)| = |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|. \tag{3.1}$$

By (2.2),

$$|a_3 - 2a_2^2| = \frac{1}{3-2\beta}\left|p_2 - \frac{2(3-2\beta)}{(2-\beta)^2} p_1^2\right|.$$

Applying the well known result $|p_2 - \mu p_1^2| \leq 4\mu - 2$ for $\mu > 1$ (see [32]), we obtain

$$|a_3 - 2a_2^2| \leq \frac{8}{(2-\beta)^2} - \frac{2}{3-2\beta}.$$

Using this bound of $|a_3 - 2a_2^2|$ and the bounds of $|a_2|$, $|a_3|$ from (2.1) in (3.1), required bound of $|T_{3,1}(f)|$ follows. Sharpness of the bound of $|T_{3,1}(f)|$ follows from the function $\tilde{f}_1$.

Remark 3.1. The bounds of $|T_{2,n}(f)|$ and $|T_{3,1}(f)|$ for the class $\mathcal{R}$ follow from Theorem 3.1, when $\beta = 0$ [2, Theorem 2.12].

Theorem 3.2. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then*

$$T_{3,1}(f) \leq \begin{cases} \dfrac{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}{(3-2\beta)^2(2-\beta)^2}; & \frac{10-\sqrt{10}}{9} \leq \beta \leq 1, \\ 1; & 0 \leq \beta \leq \frac{10-\sqrt{10}}{9}. \end{cases} \tag{3.2}$$

The bounds are sharp.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_\beta$, Theorem 2.1 yields

$$|a_2| \leq \frac{2}{2-\beta} \quad \text{and} \quad |a_3| \leq \frac{2}{3-2\beta}.$$

Hence $|a_2| \in [0,2]$ and $|a_3| \in [0,2]$ for $\beta \in [0,1]$. From (1.4), we have

$$\begin{aligned} T_{3,1}(f) &= 1 + 2\,\mathrm{Re}(a_2^2 \bar{a}_3) - 2|a_2|^2 - |a_3|^2 \\ &\leq 1 + 2|a_2|^2|a_3| - 2|a_2|^2 - |a_3|^2 =: g(|a_3|), \end{aligned}$$

where $g(x) = 1 + 2|a_2|^2 x - 2|a_2|^2 - x^2$ with $x = |a_3| \in [0,2]$. Since $g'(x) = 0$ at $x_0 := |a_2|^2$ and $g''(x_0) < 0$, therefore $g(x)$ attains its maximum value at $x = x_0$, whenever $|a_2|^2$ belongs to the range of x, that means $|a_2|^2 \leq 2$. Thus

$$T_{3,1}(f) \leq g(|a_2|^2) = (|a_2|^2 - 1)^2$$

$$\leq 1 \quad \text{when} \quad |a_2|^2 \leq 2,$$
$$= 1 \quad \text{when} \quad 0 \leq \beta \leq 2-\sqrt{2}.$$

Now, the other case, when $|a_2|^2$ does not lie in the range of x, that is $|a_2|^2 > 2$ or $2-\sqrt{2} \leq \beta \leq 1$, then

$$\begin{aligned} T_{3,1}(f) &\leq \max g(x) = g\left(\frac{2}{3-2\beta}\right) \\ &= 1 - 2a_2^2 - \frac{4}{(3-2\beta)^2} + \frac{4a_2^2}{3-2\beta} \\ &\leq \frac{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}{(3-2\beta)^2(2-\beta)^2}. \end{aligned}$$

Using all these above arguments, we obtain

$$T_{3,1}(f) \leq \begin{cases} 1, & 0 \leq \beta \leq \beta_0; \\ \frac{4\beta^4-28\beta^3+37\beta^2-4\beta-4}{(3-2\beta)^2(2-\beta)^2}, & \beta_0 \leq \beta \leq 1, \end{cases}$$

where $\beta_0 = (10-\sqrt{10})/9$ is the root of the equation $9\beta^2 - 20\beta + 10 = 0$.

The sharpness of the bound follows from $f(z) = z$ when $0 \leq \beta \leq (10-\sqrt{10})/9$. However, for $(10-\sqrt{10})/9 \leq \beta \leq 1$, equality in (3.2) holds for the function $\tilde{f}$ given in (2.3).

Remark 3.2. For $\beta = 0$ in Theorem 3.2, we obtain $T_{3,1}(f) \leq 1$ for $f \in \mathcal{R}$ [23, Example 2.4].

Theorem 3.3. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then*

$$T_{3,1}(f) \geq 1 - \frac{4\beta - 9}{\beta^4 - 4\beta^3 + 2\beta^2 + 8\beta - 8}.$$

The bound is sharp.

Proof. Let $f \in \mathcal{A}_\beta$, then from (2.2), we have

$$a_2 = \frac{p_1}{2-\beta}, \quad a_3 = \frac{p_2}{3-2\beta}.$$

Now, by replacing p_2 in terms of p_1 using (2.6), we get

$$2\operatorname{Re}(a_2^2\bar{a}_3) = \frac{p_1^4 + p_1^2(4-p_1^2)\operatorname{Re}\zeta}{(3-2\beta)(2-\beta)^2}, \quad -|a_2|^2 = \frac{|p_1|^2}{(2-\beta)^2},$$
$$-|a_3|^2 = -\frac{p_1^4 + (4-p_1^2)^2|\zeta|^2 + 2p_1^2(4-p_1^2)\operatorname{Re}\zeta}{4(3-2\beta)^2}.$$

A simple computation yields that

$$\begin{aligned} T_{3,1}(f) = 1 + \frac{1}{4(3-2\beta)^2(2-\beta)^2}\Big(&p_1^4(8-4\beta-\beta^2) - 8p_1^2(3-2\beta)^2 \\ &- (4-p_1^2)^2(2-\beta)^2|\zeta|^2 + 2p_1^2(4-p_1^2)(2-\beta^2)\operatorname{Re}\zeta\Big) \\ =:\ & g(p_1, \zeta, \operatorname{Re}(\zeta)). \end{aligned}$$

Since the classes $\mathcal{A}_\beta$ and $\mathcal{P}$ are rotationally invariant, we can take $p = p_1 \in [0,2]$. Using $\operatorname{Re}(\zeta) \geq -|\zeta|$ with notation $|\zeta| = y$, we have $g(p_1, |\zeta|, \operatorname{Re}\zeta) \geq g_1(p, y)$, where

$$g_1(p,y) = 1 + \frac{1}{4(3-2\beta)^2(2-\beta)^2}\Big(p^4(8-4\beta-\beta^2) - 8p^2(3-2\beta)^2$$

$$-(4-p^2)^2(2-\beta)^2y^2-2p^2(4-p^2)(2-\beta^2)y\Big).$$

Also, note that

$$\frac{\partial g_1(p,y)}{\partial y}=-\frac{2(4-p^2)^2y(2-\beta)^2+2p^2(4-p^2)(2-\beta^2)}{4(3-2\beta)^2(2-\beta)^2}<0$$

for all $p\in[0,2]$ and $\beta\in[0,1]$. Hence $g_1(p,y)$ is a decreasing function of y with $g_1(p,y)\leq g_1(p,1)=:g_2(p)$. Minimum of $g_2(p)$ is the lower bound of $\det T_{3,1}(f)$. The equation $g_2'(p)=0$ gives the following critical points

$$p^{(1)}=0,\quad p^{(2)}=\pm\sqrt{\frac{(2\beta^2-8\beta+7)}{(2-\beta^2)}}.$$

Using the basic calculus rule, it can be easily observed that the function $g_2(p)$ attains its minimum value at $p^{(2)}$ as $g''(p^{(2)})>0$ for all $\beta\in[0,1]$. Thus

$$\det T_{3,1}(f)\geq g_2(p^{(2)})=1-(4\beta-9)/(\beta^4-4\beta^3+2\beta^2+8\beta-8).$$

To show the sharpness consider the function $\tilde{f}_3\in\mathcal{A}$ given by

$$\beta\frac{\tilde{f}_3(z)}{z}+(1-\beta)\tilde{f}_3'(z)=\frac{1-z^2}{1-z\sqrt{(2\beta^2-8\beta+7)/(2-\beta^2)}+z^2}.$$

For $\tilde{f}_3(z)=z+\sum_{n=2}^{\infty}a_nz^n$, we have

$$a_2=\frac{1}{2-\beta}\sqrt{\frac{2\beta^2-8\beta+7}{2-\beta^2}},\quad a_3=\frac{1-2\beta}{2-\beta^2}$$

and $T_{3,1}(\tilde{f}_3)=1-(4\beta-9)/(\beta^4-4\beta^3+2\beta^2+8\beta-8)$.

Remark 3.3. For $\beta=0$ in Theorem 3.2, we obtain $\det T_{3,1}(f)\geq -1/8$ for $f\in\mathcal{R}$ [23, Example 2.4].

4 Coefficient Difference

Robertson [37] proved that $3|a_{n+1}-a_n|\leq(2n+1)|a_2-1|$ for the class of convex functions. Recently, Li and Sugawa [28] obtained the bound of $|a_{n+1}-a_n|$ for particular choices of n for the class of convex function with fixed second coefficient. In this section, we find the the bound of $|a_{n+1}^N-a_n^N|$ $(N\in\mathbb{N})$ depending on the second coefficient for $f\in\mathcal{A}_\beta$. In fact, it is more convenient to express our result in terms of $p_1=p$, applying the correspondence

$$(2-\beta)a_2=p_1=p.$$

To make the results more legible, we define the class $\mathcal{A}_\beta(p)$, $p\in[-2,2]$ as follows

$$\mathcal{A}_\beta(p)=\{f\in\mathcal{A}_\beta : f''(0)=p\}.$$

Clearly,

$$\bigcup_{-2\leq p\leq 2}\mathcal{A}_\beta(p)\subset\mathcal{A}_\beta\quad\text{and}\quad\bigcup_{-2\leq p\leq 2}\mathcal{A}_\beta(p)\neq\mathcal{A}_\beta.$$

The following lemmas are used to establish our main results.

Lemma 4.1. *[9] If $p(z)=1+\sum_{n=1}^{\infty}p_nz^n\in\mathcal{P}$, then the following estimate holds:*

$$|p_{n+1}^N-p_n^N|\leq 2^N\sqrt{2-2^{1-N}\operatorname{Re}(p_1^N)}\quad(N\in\mathbb{N}).$$

Equality holds for the function $(1+e^{i\alpha}z)/(1-e^{i\alpha}z)$, where $\alpha=\cos^{-1}(b/2)$ and $\operatorname{Re}p_1=2b$.

Lemma 4.2. *[26] Fix $\zeta \in \bar{\mathbb{D}}$. If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, then*

$$|\xi p_{n+1} - p_n| \leq \frac{2(1-|\xi|^n)\left(1+|\xi|^2 - \mathrm{Re}(\xi p_1)\right)}{1-|\xi|} + |2 - \xi p_1||\xi|^n \quad for \ |\xi| < 1.$$

The bounds are sharp for $p(z) = (1+z)/(1-z)$.

According to Komatu [22], if $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ both are the members of $\mathcal{P}$, then the weighted Hadamard product, $f * g$, also belongs to $\mathcal{P}$, where

$$f * g = 1 + \sum_{n=1}^{\infty} \frac{p_n q_n}{2} z^n.$$

Let us define $F_j(z) = F_{j-1} * p(z)$ for $j \in \mathbb{N}$ with $F_0(z) = p(z)$, then using the above result, we have $F_j \in \mathcal{P}$. Particulary, for $N \in \mathbb{N}$, the function

$$F_{N-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{p_n^N}{2^{N-1}} z^n \in \mathcal{P}.$$

Replacing $p(z)$ in Lemma 4.2 by F_{N-1}, the result is as follows:

Lemma 4.3. *Fix $\xi \in \bar{\mathbb{D}}$ and $N \in \mathbb{N}$. If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then*

$$|\xi p_{n+1}^N - p_n^N| \leq \frac{2(1-|\xi|^n)\left(2^{N-1} + 2^{N-1}|\xi|^2 - \mathrm{Re}(\xi p_1^N)\right)}{1-|\xi|} + |2^N - \xi p_1^N| \cdot |\xi|^n \qquad for \ |\xi| < 1,$$

Equality holds for the function $(1+z)/(1-z)$.

Theorem 4.1. *If $f \in \mathcal{A}_\beta(p)$, then the following inequalities hold:*

$$|a_{n+1}^N - a_n^N| \leq \begin{cases} \dfrac{2(\sigma^n - \mu^n)(2^{N-1}\sigma^2 + 2^{N-1}\mu^2 - \sigma\mu p^N)}{(\sigma-\mu)\sigma\mu^{n+1}} + \dfrac{\sigma^n |2^N \mu - \sigma p^N|}{\sigma \mu^{n+1}}; & \beta \in [0,1), \\ \dfrac{2^N\sqrt{2 - 2^{1-N} p^N}}{\sigma}; & \beta = 1, \end{cases} \tag{4.1}$$

where $\sigma = (n-(n-1)\beta)^N$ and $\mu = (n+1-n\beta)^N$. Bounds for $\beta \in [0,1)$ is sharp for $p = 2$ whereas for $\beta = 1$, bound is sharp for odd N and $p = -2$.

Proof. For $f \in \mathcal{A}_\beta(p)$, from (2.2), we have

$$(n-(n-1)\beta)^N |a_{n+1}^N - a_n^N| = \left| \left(\frac{n-(n-1)\beta}{(n+1)-n\beta}\right)^N p_n^N - p_{n-1}^N \right|.$$

From Lemma 4.3 with $((n-(n-1)\beta)/((n+1)-n\beta))^N =: \xi$, bound in (4.1) for $\beta \in [0,1)$ follows since $\xi \in (0,1)$ whenever $\beta \in (0,1)$. For $\beta = 1$ we have $\xi = 1$. Bounds for $\beta = 1$ are obtained using Lemma 4.1 .

To show the sharpness for $\beta \in [0,1)$, consider the function $\tilde{f}(z)$ given in (2.3). As for $\tilde{f}$, we have

$$|a_{n+1} - a_n| = \frac{2^N}{(n-(n-1)\beta)^N} \left| \frac{(n-(n-1)\beta)^N}{(n+1-n\beta)^N} - 1 \right|,$$

which is same as in (4.1) for $p = 2$. In case of $\beta = 1$, for the function $\tilde{f}(-z)$, we have

$$|a_{n+1} - a_n| = \frac{2^{N+1}}{(n-(n-1)\beta)^N},$$

which coincides with the bounds in (4.1) for odd N and $p = -2$.

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For $N=1$, Theorem 4.1 yields the following bounds:

Corollary 4.2. *If $f \in \mathcal{A}_\beta(p)$ is of the form (1.1), then*

$$|a_{n+1}-a_n| \leq \begin{cases} \dfrac{2(\sigma^n-\mu^n)(\sigma^2+\mu^2-\sigma\mu p)}{(\sigma-\mu)\sigma\mu^{n+1}} + \dfrac{\sigma^n|2\mu-\sigma p|}{\sigma\mu^{n+1}}; & \beta \in [0,1), \\ \dfrac{2\sqrt{2-p}}{\sigma}; & \beta = 1, \end{cases}$$

The class $\mathcal{A}_\beta$ reduces to the class $\mathcal{R}$ for $\beta = 0$. Let us take corresponding class $\mathcal{R}(p) = \{f \in \mathcal{R} : f''(0) = p\}$. Theorem 4.1 gives the following result for the class $\mathcal{R}(p)$ when $\beta = 0$.

Corollary 4.3. *If $f \in \mathcal{R}(p)$ is of the form (1.1), then the following sharp bounds hold:*

$$|a_{n+1}^N - a_n^N| \leq \frac{2(\sigma^n-\mu^n)(2^{N-1}\sigma^2+2^{N-1}\mu^2-\sigma\mu p^N)}{(\sigma-\mu)\sigma\mu^{n+1}} + \frac{\sigma^n|2^N\mu-\sigma p^N|}{\sigma\mu^{n+1}}.$$

5 Growth Theorem and Bohr Phenomenon

Theorem 5.1. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then for $|z| \leq r$, the following hold:*

(i) $-\dfrac{\tilde{f}(-r)}{r} \leq \operatorname{Re}\left(\dfrac{f(z)}{z}\right) \leq \dfrac{\tilde{f}(r)}{r}$,

(ii) $-\tilde{f}(-r) \leq |f(z)| \leq \tilde{f}(r)$,

where $\tilde{f}(z)$ is given by (2.3). All these estimations are sharp.

Proof. (i) Let $f \in \mathcal{A}_\beta$. Consider $p(z) = f(z)/z$, then we have

$$\operatorname{Re}(p(z) + (1-\beta)zp'(z)) > 0.$$

It can be viewed as $p(z) + (1-\beta)zp'(z) \prec (1+z)/(1-z)$. Further, by Hallenbeck and Rusheweyeh [33, Theorem 3.1b], it follows that

$$p(z) \prec q(z) \prec \frac{1+z}{1-z},$$

where $q(z)$ is convex and best dominant, given by

$$\begin{aligned} q(z) &= \frac{1}{(1-\beta)z^{(\frac{1}{1-\beta})}} \int_0^z \left(\frac{1+t}{1-t}\right) t^{(\frac{1}{1-\beta}-1)} dt \\ &= \frac{\tilde{f}(z)}{z}, \end{aligned}$$

where $\tilde{f}(z)$ is defined in (2.3). Since $q(z)$ is convex and all coefficients are real for $\beta \in [0,1]$, therefore image domain of $\mathbb{D}$ under the function $q(z)$ is symmetric with respect to real axis and

$$q(-r) \leq \operatorname{Re}(q(z)) \leq q(r), \quad |z| = r < 1.$$

As $p(z) = f(z)/z \prec q(z)$, so required bound of $\operatorname{Re}(f(z)/z)$ follows. This completes the first part. Sharpness of the bounds follow as $q(z)$ is the best dominant. (ii) From [7, Lemma 4.10], $f \in \mathcal{A}_\beta$ if and only if

$$f(z) = z\int_0^1 p(t^{1-\beta}z)dt, \tag{5.1}$$

where $p \in \mathcal{P}$. Using the well known bound $|p(z)| \leq (1+r)/(1-r)$ of Carathéodory functions, we have

$$|f(z)| \leq r \int_0^1 \frac{1+rt^{1-\beta}}{1-rt^{1-\beta}} dt = \tilde{f}(r),$$

Now, we proceed for the lower bound of $|f(z)|$. After solving the integration in (5.1) for $p(z) = (1+z)/(1-z)$, we get

$$f(z) = z(-1+2H(z)),$$

where

$$H(z) = {}_2F_1\left[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, z\right].$$

Thus for $z = re^{i\theta}$,

$$|f(z)| = |z(-1+2H(z))| \geq \min_{\theta \in [0,2\pi]} g(\theta) \tag{5.2}$$

where

$$g(\theta) = \sqrt{\mathrm{Re}(re^{i\theta}(-1+2H(re^{i\theta})))^2 + \mathrm{Im}(re^{i\theta}(-1+2H(re^{i\theta})))^2},$$

Since for different choices of β in $[0,1)$, $H(z)$ reduces to different functions. For instance, when $\beta = 0$, it becomes $-2\log(1-z)/z$ and for $\beta = 1/2$, it reduces to $-4(z+\log(1-z))/z^2$. By a simple calculation, we find that the function $g(\theta)$ is decreasing from $[0,\pi]$ and increasing from $[\pi, 2\pi]$ for $r \in (0,1)$ and $\beta \in [0,1)$. Hence $g(\theta)$ attains its minimum value at $\theta = \pi$. Thus from (5.2), we get

$$\begin{aligned} |f(z)| &\geq |-r(-1+2H(-r))| \\ &= r(-1+2H(-r)) = -r\tilde{f}(-r), \end{aligned}$$

which completes the proof. Bounds are sharp for the function $\tilde{f}(z)$.

Theorem 5.2. *If $f \in \mathcal{A}_\beta$ is of the form (1.1), then for $m \in \mathbb{N}$*

$$|\omega(z^m)| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

in $|z| \leq r^*$, where r^* is the smallest positive root of

$$r^m + \tilde{f}(r) - r + \tilde{f}(-1) = 0. \tag{5.3}$$

The radius r^* is sharp.

Proof. Let $f \in \mathcal{A}_\beta$, then by Theorem 5.1, the Euclidean distance between $f(0) = 0$ and the boundary of $f(\mathbb{D})$ satisfies

$$d(0, \partial f(\mathbb{D})) \geq \lim_{r \to 1} |f(z)| = -\tilde{f}(-1).$$

Let $|z| \leq r$. Now using (2.1) with the above inequality, we have

$$\begin{aligned} |\omega(z^m)| + \sum_{n=2}^{\infty} |a_n z^n| &\leq r^m + \sum_{n=2}^{\infty} \left(\frac{2}{n-\beta(n-1)}\right) r^n, \\ &= r^m + \tilde{f}(r) - r \\ &\leq -\tilde{f}(-1) \leq d(0, \partial f(\mathbb{D})). \end{aligned}$$

which is true in $|z| = r \leq r^*$, where r^* is the root of $H(r) = r(r^{m-1} - 1) + \tilde{f}(r) + \tilde{f}(-1)$. Note that, $H(0) = \tilde{f}(-1) < 0$ and $H(1) = \tilde{f}(1) + \tilde{f}(-1) > 0$ for all $\beta \in [0,1]$, therefore by the Intermediate value property for continuous functions there must exist a $r^* \in (0,1)$ such that $H(r^*) = 0$.

Sharpness holds for the functions $\tilde{f}(z)$ and $\omega(z) = z$. Since at $z = r^*$,

$$\begin{aligned}|\omega(z^m)| + \sum_{n=2}^{\infty}|a_n z^n| &= (r^*)^m + \sum_{n=2}^{\infty}\frac{2}{n-(n-1)\beta}(r^*)^n \\ &= (r^*)^m + \tilde{f}(r^*) - r^* = -\tilde{f}(-1).\end{aligned}$$

Hence the radius is sharp.

For $w(z) = z$ and $m = 1$, Theorem 5.2 gives the following Bohr-radius for the class $\mathcal{A}_\beta$.

Corollary 5.3. *If $f \in \mathcal{A}_\beta$, then $|z| + \sum_{n=2}^{\infty}|a_n z^n| \le d(0, \partial f(\mathbb{D}))$ in $|z| \le r_b$, where r_b is root of $\tilde{f}(r) + \tilde{f}(-1) = 0$. The radius r_b is sharp.*

For various values of $\beta \in [0, 1]$, the root r_b is shown in Figure 1 and Table 1.

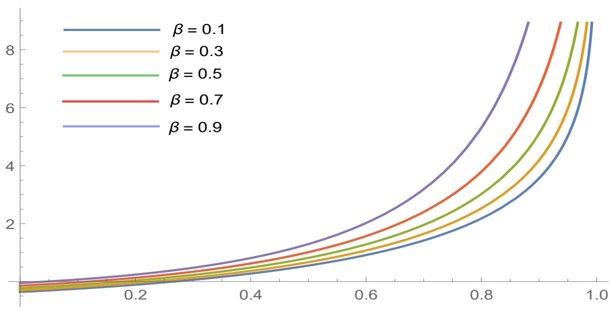

Figure 1

Table 1: Radius r^* for various choices of β

β	0.1	0.2	0.3	0.5	0.7	0.8	0.9
r_b	0.267139	0.24766	0.22655	0.178366	0.119726	0.085113	0.0457777

Theorem 5.4. *If $f \in \mathcal{A}_\beta$, then*

$$|f(z^m)| + \sum_{k=N}^{\infty}|a_k z^k| \le d(0, \partial f(\mathbb{D})) \tag{5.4}$$

hold for $|z| = r \le r_N$, where r_N is the root of the equation

$$\tilde{f}(r^m) + \tilde{f}(r) - \hat{f}(r) + \tilde{f}(-1) = 0,$$

with

$$\hat{f}(r) = \begin{cases} 0 & N = 1, \\ r & N = 2, \\ r + \sum_{n=2}^{N-1}\frac{2}{n-(n-1)\beta}r^n & N \ge 3. \end{cases}$$

The radius is sharp.

Proof. Suppose $f \in \mathcal{A}_\beta$, then from (1.2) and Theorem 5.1, we have

$$|f(z^m)| + \sum_{k=N}^{\infty}|a_k z^k| \le \tilde{f}(r^m) + \sum_{n=N}^{\infty}\frac{2}{n-(n-1)\beta}r^n$$

$$
\begin{aligned}
&= \tilde{f}(r^m) - \hat{f}(r) + \tilde{f}(r) \\
&\leq -\tilde{f}(-1) \\
&\leq d(0, \partial f(\mathbb{D}))
\end{aligned}
$$

holds in $|z| = r_N$, where r_N is the root of

$$G(r) := \tilde{f}(r^m) - \hat{f}(r) + \tilde{f}(r) + \tilde{f}(-1) = 0.$$

Since $G(0) = \tilde{f}(-1) < 0$ and $G(1) = (\tilde{f}(1) - \hat{f}(1)) + (\tilde{f}(1) + \tilde{f}(-1)) > 0$, therefore there exist a $r_N \in (0, 1)$ such that (5.4) holds. Note that, for the function $\tilde{f}(z)$ at $|z| = r_N$,

$$
\begin{aligned}
|f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| &= \tilde{f}((r_N)^m) + \sum_{n=N}^{\infty} \frac{2}{n - (n-1)\beta} (r_N)^n \\
&= -\tilde{f}(-1),
\end{aligned}
$$

which proves the sharpness of radius.

6 Declarations

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Conflict of interest

The authors declare that they have no conflict of interest.

Author Contribution

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Data Availability

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