Uniform finite-time stability of non-linear time-varying parameter-dependent systems

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Abstract

In this paper, we address uniform finite-time stability of time-varying parameter-dependent systems. Specifically, we provide a new Lyapunov and converse Lyapunov conditions for uniform finite-time stability of time-varying system. Furthermore, we show that uniform finite-time stability leads to uniqueness of solutions in forward time. In addition, we establish necessary and sufficient conditions for continuity of the settling-time function of a nonlinear time-varying system. Finally, we give an application of our result for perturbed system.

keywords: Nonlinear time-varying system, parameter-dependent system, Lyapunov stability, Uniform finite-time-stability

1 Introduction

Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Since finite time convergence implies a non uniqueness of system solutions in backward time, these systems have non-Lipschitz dynamics. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [6]. In addition, it is shown in [4][Theorem 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov [13]. The notions of uniform asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to a Lyapunov stable equilibrium state over an infinite horizon: [22] investigate the asymptotic stability of the zero solution and boundedness of all solutions of a certain third order nonlinear ordinary vector differential equation and in [14], the author presents a practical method to solve the problem

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of global output feedback tracking trajectories for a class of EulerLagrange systems which globally exponentially stabilize trajectories.

A converse Lyapunov theorem will be expecting for theoretical and practicalusage. In fact, converse Lyapunov theorems always attract much attention within stability theory. [11] threw first light on this question for deterministic systems.

In [10], a converse theorem for uniform asymptotic stability is established, and confirms that if the origin is uniformly asymptotically stable, then there is a Lyapunov function that meets some conditions. Motivated by robust control analysis and design, the authors in [12] establish a smooth converse Lyapunov theorem for uniform global asymptotic robust stability. The author in [21], give a Lyapunov characterization of a concept of, non-uniform in time, global exponential robust stability of the origin. In [9], a converse Lyapunov theorem of a concept of global asymptotic robust non-uniform stability of the origin is shown. The authors in [1] and [5] studied this problem when the origin is not necessary an equilibrium point. When the perturbed term is small, then the trajectory will be ultimately bound and tends to the origin when the ultimate bound approaches to zero. In [2], the author present a converse Lyapunov theorem for the notion of uniform practical stability for nonlinear time varying systems in presence of small perturbation.

The key work of converse Lyapunov theorems is to construct auxiliary functions that satisfy the conditions of the respective theorems. Usually some K-class or KL-class functions from stability definition play an important role in constructing Lyapunov functions. With regard to finite-time stability, the settling-time function will take place of K-class or KL-class functions.

The existence of a Hölder continuous Lyapunov function assumes importance in literature where we investigate the sensitivity of stability properties to perturbations of systems with a finite-time-stable equilibrium under the assumption of the existence of a Lipschitz continuous Lyapunov function. For the sake of completeness, it should be noticed that a more recent notion of finite-time stability, which is strictly related to Lyapunov asymptotic stability, has been given in [3]-[16] for continuous autonomous systems and in [18]-[7] for nonlinear time varying dynamical systems. This different concept of finite-time stability requires convergence of system trajectories to an equilibrium state in finite-time. Several finite-time stabilization results have been obtained by combining the finite-time stability results [17]-[8]-[19].

In this paper, we extend the results of [3], [17] and [7] to address uniform finite-time stability for time-varying systems. In addition, we establish a new necessary and sufficient conditions for continuity of the settling-time function, that is, the time at which a system trajectory reaches an equilibrium state. Then, we give an application of perturbed system to show the applicability of the result.

The rest of this paper is organized as follows. In Section 2, we give the basic concepts about uniform-finite-time stability. In Section 3, we provide a new Lyapunov and converse Lyapunov results for uniform finite-time stability of time-varying parameter dependent systems in terms of scalar differential inequalities, Such that the uniform stability in finite time of this system can not be attained by the theorems given by [3], [17] and [7]. In section 4, we present an application of our results for perturbed system and we prove it with a numerical example.

2 Preliminaries results

In this section, we introduce notation and definitions, and present some key results needed for developing the main results of this paper. We consider the system:

$$\dot{x} = f^{\xi}(t, x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$, ξ and $f^{\xi} : [0, +\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is such that $f^{\xi}(.,.)$ is jointly continuous in t and x, and for every $t \in [0, +\infty)$, $f^{\xi}(t, 0) = 0$.

Definition 2.1. 1) A scalar continuous function $\alpha(r)$, defined for $r \in [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_{\infty}$ If more $\alpha(r) \to \infty$ as $r \to \infty$.

2) A scalar continuous function $\beta(r, s)$, defined for $r \in [0, +\infty)$ and $s \in [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

3) A scalar function $\gamma(r)$, defined for $r \in [0, +\infty)$ is said to belong to class $\mathcal{KI}$, if it is a class $\mathcal{K}$-function, locally Lipschitz continuous in some neighborhood outside the origin and satisfies: there exists $\epsilon > 0$ such that

$$\int_0^{\epsilon} \frac{dz}{\gamma(z)} < +\infty.$$

The Lie derivative of $V_{\xi} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ along $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathcal{L}_f V_{\xi} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}, \quad \mathcal{L}_f V_{\xi} = \frac{\partial V_{\xi}}{\partial t} + \frac{\partial V_{\xi}}{\partial x} f(t, x).$$

The next result presents the classical comparison principle for nonlinear time-varying dynamical systems.

Theorem 2.1. *[10] Consider the nonlinear dynamical system (2.1) with $n = 1$ and let $x(t), t \geq t_0$, be the solution to (2.1) with $x(t_0) = x_0$. Assume that there exists a continuously differentiable function $V_{\xi} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that*

$$\mathcal{L}_f V_{\xi} \leq w_{\xi}(t, V_{\xi}(t, x)), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{2.2}$$

where $w_{\xi}(t, .) : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ for all $t \in [t_0, +\infty)$, and $w_{\xi}(., y) : \mathbb{R} \to \mathbb{R}$ is continuous on $[t_0, +\infty)$ for all $y \in \mathbb{R}$, and

$$\dot{z}(t) = w_{\xi}(t, z(t)), \quad z_{\xi 0}(t_0) = z_{\xi}, \ t \in \mathcal{I}_{z_0, t_0}, \tag{2.3}$$

has a unique solution $z_{\xi}(t)$, $t \in \mathcal{I}_{z_{\xi 0}, t_0}$, where $\mathcal{I}_{z_0, t_0}$ designates the maximal interval of existence of a solution $z_{\xi}(t)$ of (2.3)with $z_{\xi}(t_0) = z_0$.
If $[t_0, t_0 + \tau] \subset \mathcal{I}_{x_0, t_0} \cap \mathcal{I}_{z_{\xi 0}, t_0}$ and $V_{\xi}(t_0, x_0) \leq z_{\xi 0}$, $z_0 \in \mathbb{R}$, then

$$V_{\xi}(t, x(t)) \leq z_{\xi}(t), \ \forall t \in [t_0, t_0 + \tau].$$

Now, we develop the notion of finite-time stability for time-varying nonlinear dynamical systems. The following definition generalizes Definition of [3] to time-varying systems. Without loss of generality, let $\mathcal{D} \subset \mathbb{R}^n$ an open set containing 0. We assume that for each $x \in \mathcal{D}$ and $t_0 \in \mathbb{R}_+ \mathcal{I}_{x_0,t_0}$ is chosen to be the the maximal interval of existence of a solution $x(t)$ of (2.1) with $x(t_0) = x_0$.
In this case, we denote the trajectory or solution curve of (2.1) on $\mathcal{I}_{x_0,t_0}$ satisfying the consistency property $\varphi(t_0, t_0, x_0) = x_0$ and the semi-group property $\varphi(t_2, t_1, \phi(t_1, t_0, x_0)) = \varphi(t_2, t_0, x_0)$ for every $x_0 \in \mathcal{D}$, $t_0 \in \mathbb{R}_+$, and $t_1 \leq t_2 \in \mathcal{I}_{x_0,t_0}$ by $\varphi(., t_0, x_0)$.

Definition 2.2. [7]
Consider the nonlinear dynamical system (2.1). The zero solution $x(t) = 0$ to (2.1) is finite-time stable, if there exist an open neighborhood $\mathcal{N}$ of the origin and a function $T_\xi : \mathbb{R}_+ \times \mathcal{N} \setminus \{0\} \to \mathbb{R}_+$, called the settling-time function, such that the following statements hold:

1) Finite-time convergence: if for every $t_0 \in [0, +\infty)$, and $x_0 \in \mathcal{N} \setminus \{0\}$, $\varphi_\xi(t, t_0, x_0)$ is defined on $[t_0, T_\xi(t_0, x_0))$, $\varphi_\xi(t, t_0, x_0) \in \mathcal{N} \setminus \{0\}$ for all $t \in [t_0, T_\xi(t_0, x_0))$, and

$$\lim_{t \to T_\xi(t_0,x_0)} \varphi_\xi(t, t_0, x_0) = 0.$$

ii) Lyapunov stability : if for all $\eta > 0$, there exists $\delta = \delta(t_0, \eta) > 0$, such that $B_\delta(0) \subset \mathcal{N}$, and for every $x_0 \in B_\delta(0) \setminus \{0\}$, $\varphi_\xi(t, t_0, x_0) \in B_\eta(0), \quad \forall t \in [t_0, T_\xi(t_0, x_0))$.

Finally, the zero solution $x(t) = 0$ to (2.1) is globally finite-time stable if it is finite-time stable with $\mathcal{N} = \mathbb{R}^n$.

Remark 2.2.
Note that the definition of uniform finite-time stability differs from that of finite time stability in that it requires Lyapunov stability to be uniform with respect to the initial time. Since the classical definition of uniform asymptotic stability requires uniform Lyapunov stability as well as uniform attractivity with respect to the initial time, a more mainstream definition for uniform finite-time stability would involve uniform Lyapunov stability with uniform finite-time convergence.
We show that if the zero solution $x(t) = 0$ to (2.1) is finite-time stable, then (2.1) possesses a unique solution $\varphi_\xi(., t_0, x_0)$ for every initial condition in an open neighborhood of the origin, including the origin, and $\varphi_\xi(t, t_0, x_0) = 0$ for all $t \geq T_\xi(t_0, x_0),\ t_0 \in [0, +\infty),\ x_0 \in \mathcal{N}$, where $T_\xi(t_0, 0) = t_0$.

Definition 2.3. [7]
The zero solution $x(t) = 0$ to the system (2.1) is said to be Uniformly finite-time stable (**UFTS**) if: there exist an open neighborhood $\mathcal{N}$ of the origin and a function $T_\xi : \mathbb{R}_+ \times \mathcal{N} \setminus \{0\} \to \mathbb{R}_+$, called the settling-time function, such that the following statements hold:

1) Finite-time convergence: if for every $t_0 \in [0, +\infty)$, and $x_0 \in \mathcal{N} \setminus \{0\}$, $\varphi_\xi(t, t_0, x_0)$ is defined on $[t_0, T_\xi(t_0, x_0))$, $\varphi_\xi(t, t_0, x_0) \in \mathcal{N} \setminus \{0\}$ for all $t \in [t_0, T_\xi(t_0, x_0))$, and

$$\lim_{t \to T_\xi(t_0,x_0)} \varphi_\xi(t, t_0, x_0) = 0.$$

ii) Uniform Lyapunov stability : if for all $\eta > 0$, there exists $\delta = \delta(\eta) > 0$, such that $B_\delta(0) \subset \mathcal{N}$, and for every $x_0 \in B_\delta(0) \setminus \{0\}$, $\varphi_\xi(t, t_0, x_0) \in B_\eta(0), \quad \forall t \in [t_0, T_\xi(t_0, x_0))$ and every $t_0 \in [0, +\infty)$

Finally, the zero solution $x(t) = 0$ to (2.1) is globally uniformly finite-time stable if it is uniformly finite-time stable with $\mathcal{N} = \mathbb{R}^n$.

The set of the solutions $\phi_\xi(t, t_0, x_0)$ corresponding to the common initial condition x_0 we will denote as $\mathcal{S}(x_0)$, let $S = \bigcup_{x_0 \in \mathcal{N}} S(x_0)$ be the set of all possible solutions of (2.1) starting in $\mathcal{N}$.

Proposition 2.3. *the nonlinear dynamical system (2.1) is uniformly-finite-time stable at the origin if*

1) *there exist a class $\mathcal{K}$ function $\alpha_\xi(.)$ and a positive constant c, independent of t_0, such that*

$$\|\varphi_\xi(t, t_0, x_0)\| \le \alpha_\xi(\|x_0\|), \quad \forall t \ge t_0 \ge 0, \forall x_0 \in S(x_0).$$

2) *there exist a function $T_{\xi 0} : S \to \mathbb{R}_+$ such that for all $x_0 \in \mathcal{N}$, and $\varphi_\xi(t, t_0, x_0) = 0, \ \forall t \ge t_0 + T_\xi(t, \varphi_\xi(t, t_0, x_0))$.*
$T_{\xi 0}$ is called the settling-time function of the solution $\varphi_\xi(t, t_0, x_0)$.
If $T_\xi(t_0, x_0) = sup_{\varphi_\xi(t,t_0,x_0) \in S(x_0)} T_{\xi 0}(\varphi_\xi(t, t_0, x_0)) < +\infty$, then $T_\xi(.,.)$ is the settling-time function of the system (2.1).

A sufficient condition of finite-time stable can be formulated for (2.1) using the Lyapunov theory. Let us recall the following theorem

Theorem 2.4. *[15]*
Suppose there exists a continuous Lyapunov function $V_\xi : \mathbb{R}^n \to \mathbb{R}_+$
locally Lipschitz, positive definite and radially unbounded such that the following conditions hold:

$$\dot{V}_\xi(t, x) \le -\gamma(V_\xi(t, x)),$$

where γ_ξ is $\mathcal{KI} - function$.
Then the origin is finite-time stable for (2.1), and the settling-time function

$$T_\xi(x) \le \int_0^{V_\xi(t,x)} \frac{dz}{\gamma_\xi(z)},$$

A particular possible choice is $\gamma_\xi(s) = cs^p$, where $c > 0$ and $p \in (0, 1)$.

Theorem 2.5.
There exist real numbers $c > 0$ and $p \in (0, 1)$ and an open neighborhood $D \subset \mathbb{R}^n$ of the origin such that

$$\dot{V}_\xi(x) \le -c(V_\xi(x))^p, \quad \forall x \in D \setminus \{0\} \tag{2.4}$$

Then the origin is a finite-time-stable equilibrium of (2.1). Moreover, the settling-time function T_ξ of the system (2.1) satisfies:

$$T_\xi(x) \le \frac{1}{c(1-p)} V_\xi^{1-p}(x), \ \forall x \in D,$$

and $T_\xi(.)$ is continuous on D .

If in addition $D = \mathbb{R}^n$, V_ξ is proper, and $\dot{V}_\xi$ takes negative values on $\mathbb{R}^n \setminus \{0\}$, then the origin is a globally finite-time-stable equilibrium of (2.1).

3 Lyapunov and converse Lyapunov theory for uniform finite-time stability

We start this section by considering an example of uniform finite-time stable system with a continuous but non Lipschitzian vector field. This system class is different from that given by [7].

Example 1. *Consider the scalar nonlinear time-varying dynamical system*

$$\dot{y}_\xi(t) = -k_\xi(t) sign(y_\xi(t))|y_\xi(t)|^\lambda + \gamma_\xi y_\xi(t), \quad y(t_{\xi 0}) = y_{\xi 0},\ t \geq t_0, \tag{3.1}$$

where $sign(y_\xi) := \frac{y_\xi}{|y_\xi|}$, $y_\xi \neq 0$, $y_{\xi 0} \in \mathbb{R}_+$, $\gamma_\xi > 0, \xi > 0$, $k_\xi(.)$ *is a continuous function on* $\mathbb{R}$ *and* $k_\xi(t) > 0$ *for almost all* $t \in [t_0, +\infty)$, *and* $\lambda \in (0,1)$. *The right-hand side of (3.1) is continuous everywhere and locally Lipschitz continuous everywhere except the origin. Hence, every initial condition* $y_{\xi 0} \in \mathbb{R}_+ \setminus \{0\}$ *has a unique solution in forward time on a sufficiently small time interval. Let* $\lambda \in (0,1)$.
Let $z_\xi(t) = |y_\xi(t)|^{1-\lambda}$, *the derivative of* $z_\xi(t)$ *along the trajectories of system (3.1) is given by:*

$$\dot{z}_\xi(t) = \gamma_\xi z_\xi(t) - (1-\lambda) k_\xi(t), \tag{3.2}$$

and the solution to (3.2) is given by:

$$\begin{aligned} z_\xi(t, t_0, z_{\xi 0}) = \quad & sign(y_{\xi 0})[z_{\xi 0} e^{-\gamma_\xi (t-t_0)} - (1-\lambda) e^{-\gamma_\xi t} \\ & \left(\int_{t_0}^{t} k_\xi(\tau) e^{\gamma_\xi \tau}\, d\tau \right)], \end{aligned} \tag{3.3}$$

$\forall t_0 \leq t,\ z_{\xi 0} = z_\xi(t_0) \neq 0$.
In this case, for every $z_{\xi 0} \in \mathbb{R}$, *since* $k_\xi(.)$ *is continuous on* $\mathbb{R}$ *and* $k_\xi(t) > 0$ *for almost all* $t \in [t_0, +\infty)$, *there exist* $T_{1\xi} := T_{1\xi}(t_0, z_{\xi 0}) \geq t_0$ *such that*

$$\int_{t_0}^{T_{1\xi}} k_\xi(\tau) e^{\gamma_\xi \tau}\, d\tau = \frac{z_{\xi 0}}{1-\lambda} e^{\gamma_\xi t_0}. \tag{3.4}$$

Note that if $K_\xi : [t_0, +\infty) \to \mathbb{R}$ *is a continuously differentiable function such that*

$$\int k_\xi(\tau) e^{\gamma_\xi \tau}\, d\tau = K_\xi(\tau).$$

i.e. $\dot{K}_\xi(t) = k_\xi(t) e^{\gamma_\xi t} > 0$, *for all* $t \in [t_0, T_{1\xi}]$, *then* $K_\xi(.)$ *is strictly increasing function and its inverse function* $K_\xi^{-1}(.)$ *exists.*
From (3.4)we are:

$$[K_\xi(T_{\xi 1}) - K_\xi(t_0)] = \frac{z_{\xi 0}}{1-\lambda} e^{\gamma_\xi t_0},$$

which implies that

$$T_{1\xi}(t_0, z_{\xi 0}) = K_\xi^{-1} \left(K_\xi(t_0) + \frac{z_{\xi 0}}{1-\lambda} e^{\gamma_\xi t_0} \right).$$

Hence

$$T_\xi(t_0, y_0) = K_\xi^{-1} \left(K_\xi(t_0) + \frac{|y_{\xi 0}|^{1-\lambda}}{1-\lambda} e^{\gamma_\xi t_0} \right) \tag{3.5}$$

Note that $T_\xi(t_0, y_{\xi 0})$ is the setting-time function of a finite time stable system (3.1) and $T_\xi(.,.)$ is unique since $k_\xi(t) > 0$ for almost all $t \in [t_0, +\infty)$. Thus the zero solution $y_\xi = 0$ to equation (3.1) is globally uniformly finite-time stable. ▲

Next, we present necessary and sufficient conditions for finite-time stability using a Lyapunov function involving a vector differentiable inequality. For the following result, we define:

$$\dot{V}_\xi = \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f^\xi(t, x)$$

for a given continuously differentiable function $V_\xi : [0, +\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$.

Theorem 3.1. *Consider the nonlinear dynamical system (2.1). Then the following statements hold:*

i) *If there exist a continuously differentiable function $V_\xi : [0, +\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$, a class $\mathcal{K}$ function $\alpha_\xi(.)$ and $\beta_\xi(.)$, a function $k_\xi : [0, +\infty) \longrightarrow \mathbb{R}_+$ is continuously differentiable, such that $k_\xi(t) > 0$ for almost all $t \in [0, +\infty)$, a real number $\lambda \in (0, 1)$, $\gamma_\xi > 0$ and an open neighborhood $M \subset \mathcal{N}$, of the origin such that, $\forall x \in M$, $\forall t \in [0, +\infty)$,*

$$V_\xi(t, 0) = 0, \ t \in [0, +\infty) \tag{3.6}$$

$$e^{-\gamma_\xi t}\alpha_\xi(\|x\|) \le V_\xi^{1-\lambda}(t, x) \le e^{-\gamma_\xi t}\beta_\xi(\|x\|), \tag{3.7}$$

$$\dot{V}_\xi(t, x) \le -k_\xi(t)(V_\xi(t, x))^\lambda + \gamma_\xi V_\xi(t, x). \tag{3.8}$$

Then the system (2.1) *is uniformly finite-time stable.*

ii) *If $\mathcal{N} = \mathbb{R}^n$ and there exist a class $\mathcal{K}_\infty$ function $\alpha_\xi(.)$ and $\beta_\xi(.)$, a function $k_\xi : [0, +\infty) \longrightarrow \mathbb{R}_+$ such that $k_\xi(t) > 0$ for almost all $t \in [0, +\infty)$, and an open neighborhood $M \subset \mathcal{N}$ of the origin such that* (3.6)-(3.8) *hold, then the system* (2.1) *is globally uniformly finite-time stable.*

Proof. **i)** Let $t_0 \in [0, +\infty)$, let $\eta > 0$ be such that $B_\eta(0) = \{x \in \mathcal{N}, \ \|x\| < \eta\}$, define $\nu := \alpha(\eta)$, and define $D_\nu := \{x \in B_\eta(0) \ \setminus \ V_\xi^{1-\lambda}(t_0, x)e^{\gamma t_0} < \nu\}$. Since $V_\xi^{1-\lambda}(t_0, .)$ is continuous and $V_\xi(t_0, 0) = 0$, it follows that D_ν is nonempty and there exists $\delta = \delta(\eta, t_0) > 0$ such that $V_\xi^{1-\lambda}(t_0, x)e^{\gamma t_0} < \nu$, $\forall \ x \in B_\delta(0)$. Hence, $B_\delta(0) \subset D_\nu$.
Furthermore, it follows from (3.8) and Theorem 2.1 that

$$V_\xi(t, x(t)) \le y_\xi(t, t_0, V_\xi(t_0, x_0)), \ \forall x_0 \in B_\delta(0), \ t \in [t_0, +\infty),$$

where $y_\xi(.,.,.)$ is given by (3.1) with $y_\xi(t) = V_\xi(t, x(t))$.
Now, it follows from (3.1) and (3.7) that for every $x_0 \in B_\delta(0) \subset D_\eta$,

$$\begin{aligned} e^{-\gamma_\xi t}\alpha_\xi(\|x\|) &\le V_\xi^{1-\lambda}(t, x) \\ &\le V_\xi^{1-\lambda}(t_0, x_0)e^{-\gamma_\xi(t-t_0)} - \\ &\quad (1-\lambda)e^{-\gamma_\xi t}\left(\int_{t_0}^t k_\xi(\tau)e^{\gamma_\xi \tau}\, d\tau\right) \end{aligned}$$

which implies that

$$
\begin{aligned}
\alpha_\xi(\|x\|) &\leq V_\xi^{1-\lambda}(t,x)e^{\gamma_\xi t}\\
&\leq V_\xi^{1-\lambda}(t_0,x_0)e^{\gamma_\xi t_0}-\\
&\quad (1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma_\xi\tau}\,d\tau\right)\\
&< \alpha_\xi(\eta)-(1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma\tau}\,d\tau\right)
\end{aligned}
$$

that $\|x(t)\| \leq \eta$ for every $x_0 \in B_\delta(0)$, $\forall t \geq T_{1\xi}$, where $T_{1\xi}$ is given by (3.5). Which implies the finite-time stability of the zero solution $x(t) = 0$ to (2.1) and the finite-time convergence of the trajectory of (2.1), $\forall t_0 \in [0,+\infty)$, and $x_0 \in B_\delta(0)$.

Let $\eta > 0$ and let $\delta = \delta(\eta) > 0$, be such that $\beta_\xi(\delta) = \alpha_\xi(\eta)$. Hence, it follows from (4.5) that, for all $t_0 \in [0,+\infty)$ and $x_0 \in B_\delta(0)$, it follows from (3.8) and Theorem (2.1) that

$$
V_\xi(t,x(t)) \leq y_\xi(t,t_0,V_\xi(t_0,x_0)),\ \forall x_0 \in B_\delta(0), t \in [t_0,+\infty),
$$

where $y_\xi(.,.,.)$ is given by (3.1) with $y(t) = V_\xi(t,x(t))$

$$
\begin{aligned}
e^{-\gamma_\xi t}\alpha_\xi(\|x\|) &\leq V_\xi^{1-\lambda}(t,x)\\
&\leq V_\xi^{1-\lambda}(t_0,x_0)e^{\gamma_\xi(t-t_0)}-\\
&\quad (1-\lambda)e^{\gamma_\xi t}\left(\int_{t_0}^{t} k_\xi(\tau)e^{-\gamma_\xi\tau}\,d\tau\right)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\alpha_\xi(\|x\|) &\leq V_\xi^{1-\lambda}(t,x)e^{\gamma_\xi t}\\
&\leq V_\xi^{1-\lambda}(t_0,x_0)e^{\gamma_\xi t_0}-\\
&\quad (1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma_\xi\tau}\,d\tau\right)\\
&< \beta_\xi(\|x_0\|)-(1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma_\xi\tau}\,d\tau\right)\\
&< \alpha_\xi(\eta)
\end{aligned}
$$

and hence,

$$
\|x(t)\| \leq \eta,\ \forall t \geq T_{1\xi}.
$$

Which implies uniform finite-time stability of the zero solution $x(t) = 0$ to (2.1).

ii) Let $\alpha_\xi(.)$ and $\beta_\xi(.)$ are class $\mathcal{K}_\infty$ function let $\delta = \delta(\eta) > 0$ be such that $\|x_0\| \leq \delta$, there exist $\eta > 0$ such that $\beta_\xi(\delta) \leq \alpha_\xi(\eta)$. Hence, it follows from (4.5) that, for all $t_0 \in [0,+\infty)$ and $x_0 \in B_\delta(0)$,

$$
\begin{aligned}
\alpha_\xi(\|x\|) &\leq V_\xi^{1-\lambda}(t_0,x_0)e^{\gamma_\xi t_0}-(1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma_\xi\tau}\,d\tau\right)\\
&\leq \alpha(\eta)-(1-\lambda)\left(\int_{t_0}^{t} k_\xi(\tau)e^{\gamma_\xi\tau}\,d\tau\right),\ \forall t \geq t_0
\end{aligned}
$$

and hence,

$$
\|x(t)\| \leq \eta,\ \forall t \geq t_0,\ \forall t_0 \in [0,+\infty).
$$

Finite-time convergence follows as in the proof of i) implying global uniform finite-time stability of the system (2.1).

□

Remark 3.2. It is necessary to indicate that the exponential term that we have shown it in the expression of V_ξ is the one that allowed us to reach stability in finite time for a class of system larger than that given in the literature. This example shows this difference.

Example 2. *We consider the system*

$$\dot{x} = -\xi t^2 \ x^{\frac{1}{3}} + sin(\xi t)x \tag{3.9}$$

with the function

$$f^\xi(t, x(t)) = -\xi t^2 (x(t))^{\frac{1}{3}} + sin(\xi t)x(t)$$

is continuous. Considering a continuously differentiable function $V_\xi : [0, +\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$*, a Lyapunov function associated to system (3.9)*

$$V_\xi(t, x) = \frac{1}{2} e^{\xi t} x^{\frac{3}{2}},$$

which verify $V_\xi(t, 0) = 0, \ \forall t \in [0, +\infty)$.
It is easy to compute $\dot{V}_\xi(t, x)$ *along the solutions of the system (3.9):*

$$\begin{aligned}
\dot{V}_\xi(t, x) \ &= \ \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f^\xi(t, x) \\
&= \ -\frac{3}{4}\xi t^2 e^{\xi t} x^{\frac{5}{6}} + (\xi + \frac{3}{2} sin(\xi t))\frac{1}{2} e^{\xi t} x^{\frac{3}{2}} \\
&\leq \ -\frac{3}{4}\xi t^2 e^{\xi t} x^{\frac{5}{6}} + (\xi + \frac{3}{2})\frac{1}{2} e^{\xi t} x^{\frac{3}{2}}
\end{aligned}$$

Then

$$\dot{V}_\xi(t, x) \leq -k_\xi(t)(V_\xi(t, x))^\lambda + \gamma_\xi V_\xi(t, x)$$

with $k_\xi(t) := \dfrac{3}{2^{\frac{13}{9}}} \xi t^2 e^{\frac{4}{9}\xi t}$, $\lambda := \dfrac{5}{9}$ *and* $\gamma_\xi := \xi + \frac{3}{2}$.
Let $v_\xi(t) = (V_\xi(t, x(t)))^{\frac{4}{9}}$*, then we have*

$$\dot{v}_\xi(t) \leq \left(\xi + \frac{3}{2}\right) v_\xi(t) - \frac{4}{9}\frac{3}{2^{\frac{13}{9}}} \xi t^2 e^{\frac{4}{9}\xi t}.$$

Thus,

$$v_\xi(t) \leq v_\xi(t_0) e^{-\left(\xi + \frac{3}{2}\right)(t - t_0)} - \frac{4}{9} e^{-\left(\xi + \frac{3}{2}\right)t} \left(\int_{t_0}^t \frac{3}{2^{\frac{13}{9}}} \xi \tau^2 e^{\frac{4}{9}\xi \tau} e^{\left(\xi + \frac{3}{2}\right)\tau} \, d\tau \right)$$

$$v_\xi(t) \leq v_\xi(t_0) e^{-\gamma_\xi (t - t_0)} - (1 - \lambda) e^{-\gamma_\xi t} \left(\int_{t_0}^t k(\tau) e^{\gamma_\xi \tau} \, d\tau \right)$$

$$v_\xi(t) \leq v_\xi(t_0) e^{-\left(\xi + \frac{3}{2}\right)(t - t_0)} - \frac{4}{9} e^{\frac{4}{9}\xi t} \left((t - t_0)(t + t_0 - 2)\right)$$

with $v_\xi(t) = \dfrac{1}{2} e^{\frac{4}{9}\xi t} x^{\frac{2}{3}}$ *and* $v_\xi(t_0) = \dfrac{1}{2^{\frac{4}{9}}} e^{\frac{4}{9}\xi t_0} x_0^{\frac{2}{3}}$*, implies that*

$$x^{\frac{2}{3}} \leq {x_0}^{\frac{2}{3}} e^{-\frac{4}{9}\xi (t - t_0)} e^{-(\xi + \frac{3}{2})(t - t_0)} - 2^{\frac{4}{9}} \frac{4}{9} \ \left((t - t_0)(t + t_0 - 2)\right)$$

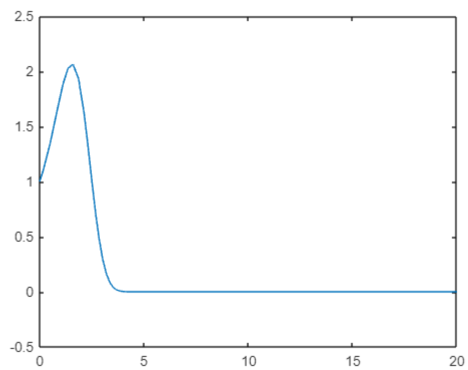
$$x^{\frac{2}{3}} \leq {x_0}^{\frac{2}{3}} e^{-(\frac{13}{9}\xi + \frac{3}{2})(t - t_0)} - \frac{2^{\frac{22}{9}}}{9} \ \left((t - t_0)(t + t_0 - 2)\right)$$

$$(x(t))^2 \leq \left(x(t_0)^{\frac{2}{3}} e^{-(\frac{13}{9}\xi+\frac{3}{2})(t-t_0)} - \frac{2^{\frac{22}{9}}}{9} \ ((t-t_0)(t+t_0-2)) \right)^3 .$$

Hence, by Theorem3.1, we prove the uniform finite-time stability which indicate that the main result of this section may be valid under the weaker assumption of finite-time stability with a continuous settling-time function $T_\xi(t_0, x_0)$ and t is the smallest real t chooses such as

$$\left(x(t_0)^{\frac{2}{3}} e^{-(\frac{13}{9}\xi+\frac{3}{2})(t-t_0)} - \frac{2^{\frac{22}{9}}}{9} \ ((t-t_0)(t+t_0-2)) \right)$$

is strictly positive.
Fig.1. shows the state trajectory of the system (3.9) for initial condition $x_0 = 1$, $t_0 = 0$ and $t \in [0, 20]$.



Lyapunov and converse Lyapunov results for finite-time stability naturally involve finite-time scalar differential inequalities. The regularity properties of a Lyapunov function satisfying such an inequality strongly depend on the regularity properties of the settling-time function.

Theorem 3.3. *Let $\lambda \in (0,1)$, $\gamma_\xi > 0$ and let $\mathcal{N}$ an open neighborhood of the origin, we assume that there exists a class $\mathcal{K}-$function $\mu_\xi : [0,r] \longrightarrow \mathbb{R}_+$, where $r > 0$, such that $B_r(0) \subset \mathcal{N}$ and*

$$\|f^\xi(t,x)\| \leq e^{-\gamma_\xi \lambda t} \mu_\xi(\|x\|), \ \forall t \in [0,+\infty), \ \forall x \in B_r(0).$$

If the system (2.1) is uniformly finite-time stable, and the settling-time function $T_\xi(.,.)$ is continuous at $(t,0)$, $t \geq t_0$, then there exist a class $\mathcal{K}-$function $\alpha_\xi(.)$, a positive function $k_\xi(t) > 0$ a continuously differentiable function, $V_\xi : [0,+\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$ and a neighborhood $M \subset \mathcal{N}$ of the origin such that that $V_\xi(t,x)$ is defined for all $(t,x) \in [0,+\infty) \times M$

$$V_\xi(t,0) = 0, \ \ t \in [0,+\infty) \tag{3.10}$$

$$e^{-\gamma_\xi t} \alpha_\xi(\|x\|) \leq V_\xi^{1-\lambda}(t,x), \ \forall t \in [0,+\infty), \tag{3.11}$$

$$\dot{V}_\xi(t,x) \leq -k_\xi(t)(V_\xi(t,x))^\lambda + \gamma_\xi V_\xi(t,x), \ \forall t \in [0,+\infty), \tag{3.12}$$

proof:
First, the settling-time function $T_\xi : [0,+\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$ is continuous. Now, consider the Lyapunov function candidate $V_\xi : [0,+\infty) \times \mathcal{N} \longrightarrow \mathbb{R}_+$ given by

$$V_\xi(t,x) = [T_\xi(t,x) - t]^{\dfrac{1}{1-\lambda}} e^{\gamma_\xi t}.$$

Note that

$$V_\xi(t,0) = [T_\xi(t,0) - t]^{\dfrac{1}{1-\lambda}} e^{\gamma_\xi t} = [t-t]^{\dfrac{1}{1-\lambda}} e^{\gamma_\xi t} = 0,$$

for all $t \in [0,+\infty)$, which proof (3.10).

Next, since the system (2.1) is uniformly finite-time stable, then the solution of the system (2.1) converge to B_η. Hence $\forall \eta > 0$, there exists $\delta = \delta(\eta) > 0$ such that $\forall x \in B_\delta(0)$, $\|\phi(\tau,t,x)\| < \eta$, $\forall \tau \geq t \geq 0$,
Now since

$$\phi(\tau,t,x) = x + \int_t^\tau f^\xi(s,x(s))\,ds, \quad \forall \tau \geq t,$$

it follows that, with $\tau = T_\xi(t,x)$,

$$\begin{array}{rcl} \|x\| & \leq & \|\displaystyle\int_t^{T_\xi(t,x)} f^\xi(s,x(s))\,ds\| \\ & \leq & \displaystyle\int_t^{T_\xi(t,x)} \|f^\xi(s,x(s))\|\,ds \\ & \leq & \displaystyle\int_t^{T_\xi(t,x)} e^{-\gamma_\xi \lambda s} \mu_\xi(\|x(s)\|)\,ds \\ & \leq & e^{-\gamma_\xi \lambda t} \displaystyle\int_t^{T_\xi(t,x)} \mu_\xi(\eta)\,ds \\ & \leq & e^{-\gamma_\xi \lambda t} \mu_\xi(\eta)[T_\xi(t,x) - t], \ \forall x \in B_\delta(0). \end{array}$$

Let $M = B_\delta(0) \subset \mathcal{N}$, and note that $\forall x \in M, \forall t \in [0,+\infty)$:

$$\begin{array}{rcl} V_\xi(t,x)^{1-\lambda} & = & [T_\xi(t,x) - t] e^{\gamma_\xi (1-\lambda) t} \\ & \geq & \left(\dfrac{\|x\|}{\mu_\xi(\eta)}\right) e^{\gamma_\xi t} \\ & \geq & e^{\gamma_\xi t} \alpha_\xi(\|x\|), \end{array}$$

where $\alpha_\xi(\|x\|) = \left(\dfrac{\|x\|}{\mu_\xi(\eta)}\right)$, $x \in M$ is a class $\mathcal{K}$ function, this proves (3.11).
Finally, consider the Lyapunov derivative $\dot{V}_\xi(t,x(t))$ for some trajectory $x(t)$ starting at $t_0 \in [0,+\infty)$ and $x_0 \in M$. In this case, note that $T_\xi(.,.)$ is continuous

$$\begin{array}{rcl} \dot{T}_\xi(t,x) & = & \displaystyle\lim_{s \to t} \frac{T_\xi(s,x(s)) - T_\xi(t,x(t))}{s-t} \\ & = & \displaystyle\lim_{s \to t} \frac{T_\xi(t_0,x(t_0)) - T_\xi(t_0,x(t_0))}{s-t} = 0 \end{array}$$

Hence, it follows that

$$\begin{aligned}\dot{V}_\xi(t,x(t)) &= (\frac{1}{1-\lambda}[T_\xi(t,x(t))-t]^{\frac{\lambda}{1-\lambda}}[\dot{T}_\xi(t,x(t))-1]+\gamma_\xi[T_\xi(t,x)-t]^{\frac{1}{1-\lambda}})e^{\gamma_\xi t}\\ &= -\frac{1}{1-\lambda}e^{(1-\lambda)\gamma_\xi t}(V_\xi(t,x))^\lambda+\gamma_\xi V_\xi(t,x).\end{aligned}$$

which proves (3.12) with $k_\xi(t)=\dfrac{1}{1-\lambda}\xi^{1-\lambda}e^{(1-\lambda)\gamma_\xi t}$. □

Remark 3.4. It is clear that theorem3.3 is a converse result respect to theorem3.1.

4 Application of stability of perturbed system

Now we consider the following perturbed system

$$\dot{x}=f^\xi(t,x)+g^\xi(t,x) \tag{4.1}$$

where $t\geq 0$, $x\in\mathbb{R}^n$, f is assumed to be locally Lipschitz in the state and piecewise continuous in the time.
$(\mathcal{A}_1)$ Suppose that the nominal system

$$\dot{x}=f^\xi(t,x) \tag{4.2}$$

is uniformly Fnite-time stable with

$$\|f^\xi(t,x)\|\leq e^{-\gamma_\xi\lambda t}\mu_\xi(\|x\|),\ \forall t\in[0,+\infty),\ \forall x\in B_r(0).$$

$(\mathcal{A}_2)$ We suppose also that there exist a strictly positive function σ such that

$$\|\frac{\partial V_\xi}{\partial x}g^\xi(t,x)\|\leq\sigma(\xi)V_\xi(t,x) \tag{4.3}$$

Corollary 4.1. *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$*, the perturbed system* (4.1) *is globally uniformly finite-time stable.*

Proof. Under assumptions $(\mathcal{A}_1)$ and using the Theorem (3.3), there exist a Lyapunov function V_ξ which verifies:

$$V_\xi(t,0)=0,\ t\in[0,+\infty) \tag{4.4}$$

$$e^{-\gamma_\xi t}\alpha_\xi(\|x\|)\leq V_\xi^{1-\lambda}(t,x)\leq e^{-\gamma_\xi t}\beta_\xi(\|x\|), \tag{4.5}$$

$$\dot{V}_\xi(t,x)\leq -k_\xi(t)(V_\xi(t,x))^\lambda+\gamma_\xi V_\xi(t,x). \tag{4.6}$$

The derivative of V_ξ along the trajectories of system (4.1) is given by

$$\dot{V}_\xi=\frac{\partial V_\xi}{\partial t}+\frac{\partial V_\xi}{\partial x}f^\xi(t,x)+\frac{\partial V_\xi}{\partial x}g^\xi(t,x)$$

$$\begin{array}{rcl}\dot V_\xi(t,x(t)) & \leq & -k_\xi(t)(V_\xi(t,x))^\lambda+\gamma_\xi V_\xi(t,x)+\sigma(\xi)V_\xi(t,x)\\ & \leq & -k_\xi(t)(V_\xi(t,x))^\lambda+(\gamma_\xi+\sigma(\xi))V_\xi(t,x)\end{array}$$

Then the system (4.1) is uniformly finite-time stable. □

Example 3. *We consider the system*

$$\dot x=-\xi t^2\ x^{\frac{1}{3}}+sin(\xi t)x+\xi x \tag{4.7}$$

with the functions

$$f^\xi(t,x(t))=-\xi t^2(x(t))^{\frac{1}{3}}+sin(\xi t)x(t)$$

and

$$g^\xi(t,x(t))=\xi x(t)$$

are continuous. Considering a continuously differentiable function $V_\xi:[0,+\infty)\times\mathcal{N}\longrightarrow\mathbb{R}_+$, *a Lyapunov function associated to system (4.7)*

$$V_\xi(t,x)=\frac{1}{2}e^{\xi t}x^{\frac{3}{2}},$$

which verify $V_\xi(t,0)=0,\ \forall t\in[0,+\infty)$.
It is easy to compute $\dot V_\xi(t,x)$ *along the solutions of the system (4.7):*

$$\begin{array}{rcl}\dot V_\xi(t,x) & = & \dfrac{\partial V_\xi}{\partial t}+\dfrac{\partial V_\xi}{\partial x}f^\xi(t,x)+\dfrac{\partial V_\xi}{\partial x}g^\xi(t,x)\\ & = & -\dfrac{3}{4}\xi t^2e^{\xi t}x^{\frac{5}{6}}+(\xi+\dfrac{3}{2}sin(\xi t))\dfrac{1}{2}e^{\xi t}x^{\frac{3}{2}}+\dfrac{3}{4}\xi e^{\xi t}x^{\frac{3}{2}}\\ & \leq & -\dfrac{3}{4}\xi t^2e^{\xi t}x^{\frac{5}{6}}+(\xi+\dfrac{3}{2})\dfrac{1}{2}e^{\xi t}x^{\frac{3}{2}}+\dfrac{3}{4}\xi e^{\xi t}x^{\frac{3}{2}}\end{array}$$

Then

$$\dot V_\xi(t,x)\leq -k_\xi(t)(V_\xi(t,x))^\lambda+(\gamma_\xi+\sigma(\xi))V_\xi(t,x)$$

with $k_\xi(t):=\dfrac{3}{2^{\frac{13}{9}}}\xi t^2e^{\dfrac{4}{9}\xi t}$, $\lambda:=\dfrac{5}{9}$, $\gamma_\xi:=\xi+\frac{3}{2}$ *and* $\sigma(\xi):=\frac{3}{2}\xi$ *Let* $v_\xi(t)=(V_\xi(t,x(t)))^{\frac{4}{9}}$, *then we have*

$$\dot v_\xi(t)\leq\left(\frac{5}{2}\xi+\frac{3}{2}\right)v_\xi(t)-\frac{4}{9}\frac{3}{2^{\frac{13}{9}}}\xi t^2e^{\frac{4}{9}\xi t}.$$

Thus,

$$v_\xi(t)\leq v_\xi(t_0)e^{-\left(\frac{5}{2}\xi+\frac{3}{2}\right)(t-t_0)}-\frac{4}{9}e^{-\left(\frac{5}{2}\xi+\frac{3}{2}\right)t}\left(\int_{t_0}^t\frac{3}{2^{\frac{13}{9}}}\xi\tau^2e^{\frac{4}{9}\xi\tau}e^{\left(\frac{5}{2}\xi+\frac{3}{2}\right)\tau}\,d\tau\right)$$

$$v_\xi(t)\leq v_\xi(t_0)e^{-\left(\frac{5}{2}\xi+\frac{3}{2}\right)(t-t_0)}-\frac{4}{9}e^{\frac{4}{9}\xi t}\left((t-t_0)(t+t_0-2)\right)$$

with $v_\xi(t)=\dfrac{1}{2}e^{\frac{4}{9}\xi t}x^{\frac{2}{3}}$ *and* $v_\xi(t_0)=\dfrac{1}{2^{\frac{4}{9}}}e^{\frac{4}{9}\xi t_0}x_0^{\frac{2}{3}}$, *implies that*

$$x^{\frac{2}{3}}\leq {x_0}^{\frac{2}{3}}e^{-(\frac{53}{18}\xi+\frac{3}{2})(t-t_0)}-\frac{2^{\frac{22}{9}}}{9}\ \left((t-t_0)(t+t_0-2)\right)$$

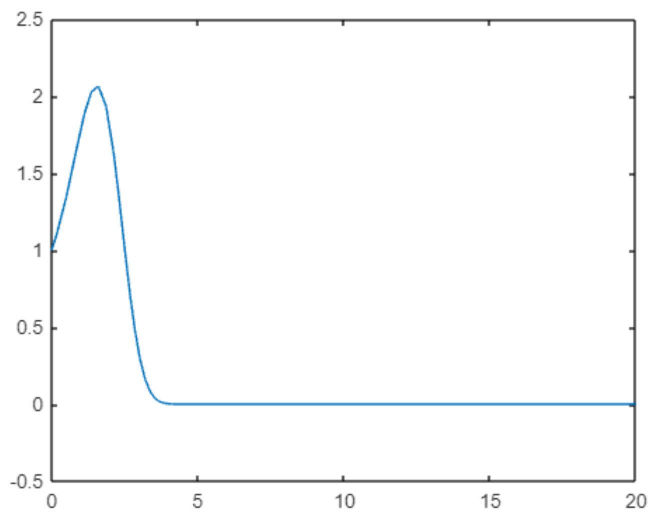
Thus,

$$(x(t))^2 \leq \left(x(t_0)^{\frac{2}{3}} e^{-(\frac{53}{18}\xi + \frac{3}{2})(t-t_0)} - \frac{2^{\frac{22}{9}}}{9} \ ((t-t_0)(t+t_0-2)) \right)^3 .$$

Hence, by Theorem 3.1, we prove the uniform finite-time stability of the perturbed system which is the goal of this section with the continuous settling-time function $T_\xi(t_0, x_0)$ *and* t *is the smallest real* t *chooses such as*

$$\left(x(t_0)^{\frac{2}{3}} e^{-(\frac{53}{18}\xi + \frac{3}{2})(t-t_0)} - \frac{2^{\frac{22}{9}}}{9} \ ((t-t_0)(t+t_0-2)) \right)$$

is strictly positive. Fig.2. shows the state trajectory of the system (4.7) for initial condition $x_0 = 1$, $t_0 = 0$ *and* $t \in [0, 20]$.



Conclusion:
This paper extends the notion of finite-time stability for time-varying dynamical systems to uniform finite-time stability of new time-varying dynamical systems with a perturbation. Specifically, Lyapunov and converse Lyapunov results for uniform finite-time stability involving finite-time scalar differential inequalities are established.
The question now is how to study the uniform finite time stability of time varying paramter dependent system with delays.

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