Linear maps preserving equivalence on $\mathcal{J}$-subspace lattice algebras

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Abstract

The structure of linear maps preserving certain equivalence relation on reflexive algebras with $\mathcal{J}$-subspace lattices is characterized. This result can apply to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras, respectively.

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1 Introduction

The problem of characterizing maps preserving certain equivalence relations has received the attention of many researchers in the last few decades. One of the topics is the study of similarity preserving maps.

Let $\mathcal{A}$ be a unital algebra. Two elements A and B in $\mathcal{A}$ are said to be similar if $A = TBT^{-1}$ for some invertible $T \in \mathcal{A}$. Hiai [2] characterized linear maps Φ defined on the matrix algebra that preserve similarity, which means that if matrices A and B are similar, then $\Phi(A)$ and $\Phi(B)$ are similar as well. Maps preserving similarity on infinite-dimensional spaces have been considered by many authors [1, 4, 5, 7, 10, 15, 16, 18]. For the Banach space case, Lu and Peng [10] proved that if X is an infinite-dimensional complex Banach space and Φ is a similarity preserving linear map on $B(X)$, the algebra of all bounded linear operators on X, then Φ must be of the form either $\Phi(A) = cTAT^{-1} + h(A)I$ or $\Phi(A) = cTA^*T^{-1} + h(A)I$ for some complex number c, some invertible operator T and some similarity-invariant linear functional h. Recently, for the non-prime algebras case, the author and Lu [16] characterized the structure of linear maps preserving similarity on $\mathcal{J}$-subspace lattice algebras.

In this paper we will deal with maps that preserve another equivalence relation. Let $\mathcal{A}$ be an algebra with unit I. Recall that two elements A and B in

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$\mathcal{A}$ are equivalent, denoted by $A \sim B$, if there exist invertible elements $T, S \in \mathcal{A}$ such that $A = TBS$. Obviously, equivalence is a weaker relation than similarity. A map Φ from $\mathcal{A}$ into another algebra is said to be equivalence preserving if $\Phi(A) \sim \Phi(B)$ whenever $A \sim B$; Φ is said to be equivalence preserving in both directions if $A \sim B$ if and only if $\Phi(A) \sim \Phi(B)$. In [3], linear maps on the algebra of all $n \times n$ matrices preserving equivalence were characterized. For the infinite-dimensional case, Petek and Radić [12] proved that if X is an infinite-dimensional reflexive complex Banach space, then linear bijections $\Phi : B(X) \to B(X)$ preserve equivalence if and only if there exist bounded invertible linear operators T, S such that either $\Phi(A) = TAS$ for all A, or $\Phi(A) = TA^*S$ for all A, where A^* denotes the adjoint of A. Later, in the paper [13], they studied the nonlinear map case. Recently, Radić [17] considered linear maps on $B(X)$ preserving another type of equivalence, and then refined the corresponding result stated in [12].

Since $B(X)$ is prime, more generally, one may ask what is the structure of linear maps preserving equivalence on non-prime algebras. The purpose of this paper is to give the structure of linear maps preserving equivalence on reflexive algebras with $\mathcal{J}$-subspace lattices. Note that our approach is quite different from that of [12].

Let us introduce some notations used in this paper. Throughout, X will be a Banach space over the real or complex field $\mathbb{F}$. By X^* we denote the topological dual of X. A family $\mathcal{L}$ of closed subspaces of X is called a subspace lattice on X if it contains (0) and X, and is closed under the operations closed linear span $\vee$ and intersection $\wedge$ in the sense that $\vee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\wedge_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma : \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. Given a subspace lattice $\mathcal{L}$ on X, the associated subspace lattice algebra Alg$\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant, that is,

$$\mathrm{Alg}\mathcal{L} = \{A \in B(X) : Ax \in L \text{ for every } x \in L \text{ and for every } L \in \mathcal{L}\}.$$

Given a subspace lattice $\mathcal{L}$ of X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq (0) \text{ and } K_- \neq X\},$$

where $K_- = \vee\{L \in \mathcal{L} : L \not\supseteq K\}$. Call $\mathcal{L}$ a $\mathcal{J}$-subspace lattice (simply, JSL) if

(1) $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$;

(2) $\wedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = (0)$;

(3) $K \vee K_- = X$, for every $K \in \mathcal{J}(\mathcal{L})$;

(4) $K \wedge K_- = (0)$, for every $K \in \mathcal{J}(\mathcal{L})$.

Note that if $\mathcal{L} = \{(0), X\}$, then $\mathcal{J}(\mathcal{L}) = \{X\}$ and Alg$\mathcal{L} = B(X)$. An example of a $\mathcal{J}$-subspace lattice is any pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, X\}$. Here K, L and M are subspaces of X satisfying $K \vee L = X$, $K \wedge M = (0)$ and $L \subset M$. In this case, $K_- = M$, $L_- = K$ and $\mathcal{J}(\mathcal{P}) = \{K, L\}$. For further discussion of pentagon subspace lattice see [6]. Another important element of the class of $\mathcal{J}$-subspace lattices is the Boolean subspace lattice [9].

For $L \in \mathcal{L}$, $L^\perp$ denotes the annihilator of L, that is, $L^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. For nonzero vectors $x \in X$ and $f \in X^*$, we define the rank-one operator $x \otimes f$ by $y \mapsto f(y)x$ for $y \in X$.

We close this section by summarizing some lemmas on JSL algebras, which will be used to prove our main result.

Lemma 1.1. *([8]) Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then $x \otimes f \in$ Alg$\mathcal{L}$ if and only if there exists a unique element $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$, where $K_-^\perp$ means $(K_-)^\perp$.*

Lemma 1.2. *([14, Lemma 1.2]) Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X.*

(1) *For $E, F \in \mathcal{J}(\mathcal{L})$, $E \neq F$ implies that $F \subseteq E_-$.*

(2) *For $E, F \in \mathcal{J}(\mathcal{L})$, $E \neq F$ implies that $E \wedge F = (0)$.*

Remark 1.3. Suppose $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space X. Let $E, F \in \mathcal{J}(\mathcal{L})$ with $E \neq F$. Take nonzero vectors $x \in E$, $y \in F$, $f \in E_-^\perp$, $g \in F_-^\perp$. By Lemma 1.1, $x \otimes f, y \otimes g \in$ Alg$\mathcal{L}$. For any $A \in$ Alg$\mathcal{L}$, by Lemma 1.2, we have $x \otimes f A y \otimes g = 0$. However, both $x \otimes f$ and $y \otimes g$ are not zero. So, Alg$\mathcal{L}$ is not prime.

2 Preliminaries

Throughout this section, $\mathcal{L}$ will always denote a $\mathcal{J}$-subspace lattice on a real or complex Banach space X. For $K \in \mathcal{J}(\mathcal{L})$, $\mathcal{F}_1(K)$ stands for the set of all rank-one operators $x \otimes f$ with $x \in K$ and $f \in K_-^\perp$. By Lemma 1.1, $\varnothing \neq \mathcal{F}_1(K) \subseteq$ Alg$\mathcal{L}$ for each $K \in \mathcal{J}(\mathcal{L})$. Observe that $A|_{K_-} = 0$ for every $A \in \mathcal{F}_1(K)$, where $A|_{K_-}$ denotes the restriction of A to K_-.

We begin with an easy and useful lemma.

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Lemma 2.1. *Let $K \in \mathcal{J}(\mathcal{L})$ and $x \otimes f \in \mathcal{F}_1(K)$. Then $I + x \otimes f$ is invertible in* $\mathrm{Alg}\mathcal{L}$ *if and only if $f(x) \neq -1$.*

Proof. If $f(x) = -1$, then we have $(I + x \otimes f)x = 0$. So, $I + x \otimes f$ is not invertible. Now assume that $f(x) \neq -1$. Then

$$(I + x \otimes f)\left(I - \frac{1}{1 + f(x)} x \otimes f\right) = I,$$

which implies that $I + x \otimes f$ is invertible in $\mathrm{Alg}\mathcal{L}$. □

By the above lemma, we know that for every rank-one operator $x \otimes f \in \mathrm{Alg}\mathcal{L}$, $I + x \otimes f$ is invertible in $\mathrm{Alg}\mathcal{L}$ if and only if $I + x \otimes f$ is invertible in $B(X)$. It is easy to see that all rank-one operators in $B(X)$ are mutually equivalent. Next we give a necessary and sufficient condition for two rank-one operators in JSL algebras to be equivalent.

Proposition 2.2. *Let $K_1, K_2 \in \mathcal{J}(\mathcal{L})$. Suppose $R_1 \in \mathcal{F}_1(K_1)$ and $R_2 \in \mathcal{F}_1(K_2)$. Then $R_1 \sim R_2$ if and only if $K_1 = K_2$.*

Proof. Let $R_1 = x \otimes f$ and $R_2 = y \otimes g$, where $x \in K_1$, $y \in K_2$, $f \in K_{1-}^{\perp}$, $g \in K_{2-}^{\perp}$ are nonzero. First show the necessity. Assume that there exist invertible operators $T, S \in \mathrm{Alg}\mathcal{L}$ such that $x \otimes f = Ty \otimes gS$. So, x and Ty are linearly dependent. Since Ty is a nonzero vector in K_2, it follows from Lemma 1.2 that $K_1 = K_2$. Now we show the sufficiency. For this, let $K = K_1 = K_2$. We first prove two claims.

Claim 1. If $0 \neq x, y \in K$ and $0 \neq f \in K_{-}^{\perp}$, then $x \otimes f \sim y \otimes f$.

If x and y are linearly dependent, then we can write $y = \alpha x$ for some nonzero $\alpha \in \mathbb{F}$. Let $T = \alpha I$ and $S = I$. Then T, S are invertible in $\mathrm{Alg}\mathcal{L}$ and $Tx \otimes fS = y \otimes f$. So, $x \otimes f \sim y \otimes f$. Now assume that x and y are linearly independent. Take $h \in K_{-}^{\perp}$ such that $h(x) = 1$ and $h(y) = 1$. Let $T = I - (x - y) \otimes h$ and $S = I$. Then T is invertible in $\mathrm{Alg}\mathcal{L}$ by Lemma 2.1 and $Tx \otimes fS = y \otimes f$. So, $x \otimes f \sim y \otimes f$.

Claim 2. If $0 \neq x \in K$ and $0 \neq f, g \in K_{-}^{\perp}$, then $x \otimes f \sim x \otimes g$.

If f and g are linearly dependent, then we are done by Claim 1. Now assume that f and g are linearly independent. Then we can find $z \in K$ such that $f(z) = g(z) = 1$. Let $T = I$ and $S = I - z \otimes (f - g)$. Then, S is invertible in $\mathrm{Alg}\mathcal{L}$ by Lemma 2.1 and $Tx \otimes fS = x \otimes g$. So, $x \otimes f \sim x \otimes g$.

Now, by Claim 1, we get $x \otimes f \sim y \otimes f$. By Claim 2, we get $y \otimes f \sim y \otimes g$. Hence, by the transitivity, we arrive at $R_1 \sim R_2$, completing the proof. □

The following lemma gives a characterization of rank-one operators in JSL algebras involving equivalence relation, which in fact plays a key role in this paper.

Lemma 2.3. *Let R be a nonzero operator in* $\mathrm{Alg}\mathcal{L}$*. Then the following are equivalent.*

(1) *R is of rank one.*

(2) *$R \sim 2R$ and for every $A \sim R$ with $R \neq A \in \mathrm{Alg}\mathcal{L}$, $A + R \sim R$ implies that $A - R \sim R$.*

Proof. (1) $\Rightarrow$ (2). It is a direct consequence of Proposition 2.2.

(2) $\Rightarrow$ (1). Since $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$ and $R \neq 0$, there exist some $K \in \mathcal{J}(\mathcal{L})$ and some $x \in K$ such that $Rx \neq 0$. Take $f \in K_-^\perp$ such that $f(x) = 1$. Then, by Lemma 2.1, $I + 2x \otimes f$ and $I + x \otimes f$ are both invertible in $\mathrm{Alg}\mathcal{L}$. Compute

$$R(I + 2x \otimes f) = R + 2Rx \otimes f.$$

Let $A = R + 2Rx \otimes f$. Then $A \neq R$ and $A \sim R$ by the above equation. Moreover, since

$$2R(I + x \otimes f) = 2R + 2Rx \otimes f,$$

we have $A + R \sim 2R \sim R$. It follows that $R \sim A - R = 2Rx \otimes f$ by the assumption. So, R is of rank one. □

3 Main result

In this section, we will give the structure of linear maps preserving equivalence in both directions on JSL algebras. For a $\mathcal{J}$-subspace lattice $\mathcal{L}$ on a Banach space X, we denote by $\mathcal{J}_2(\mathcal{L})$ the set $\{K \in \mathcal{J}(\mathcal{L}) : \dim K \geq 2\}$. Our main result reads as follows.

Theorem 3.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be $\mathcal{J}$-subspace lattices on real or complex Banach spaces X_1 and X_2, respectively. Suppose $\Phi : \mathrm{Alg}\mathcal{L}_1 \to \mathrm{Alg}\mathcal{L}_2$ is a surjective linear map preserving equivalence in both directions.*

(1) *There exists a bijection $K \mapsto \hat{K}$ from $\mathcal{J}(\mathcal{L}_1)$ onto $\mathcal{J}(\mathcal{L}_2)$.*

(2) *Assume that $\Phi(U) = I$ for some $U \in \mathrm{Alg}\mathcal{L}_1$. Then for each $K \in \mathcal{J}_2(\mathcal{L}_1)$, one of the following holds.*

(a) *U_K is invertible and there exists a bijective continuous linear map $T_K : K \to \hat{K}$ such that*

$$\Phi(A)y = T_K A U_K^{-1} T_K^{-1} y$$

for all $A \in \mathrm{Alg}\mathcal{L}_1$ and all $y \in \hat{K}$, where U_K denotes the operator $U|_K : K \to K$.

(b) *$U^*_{K_-^\perp}$ is invertible and there exists a bijective continuous linear map $T_K : K_-^\perp \to \hat{K}$ such that*

$$\Phi(A)y = T_K A^* (U^*_{K_-^\perp})^{-1} T_K^{-1} y$$

for all $A \in \mathrm{Alg}\mathcal{L}_1$ and all $y \in \hat{K}$, where $U^*_{K_-^\perp}$ denotes the operator $U^*|_{K_-^\perp} : K_-^\perp \to K_-^\perp$.

(3) *For each $L \in \mathcal{J}(\mathcal{L}_2)$ with $\dim L = 1$, there exists a linear functional h_L on $\mathrm{Alg}\mathcal{L}_1$ such that*

$$\Phi(A)y = h_L(A)y$$

for all $A \in \mathrm{Alg}\mathcal{L}_1$ and all $y \in L$.

To prove Theorem 3.1, we need several lemmas. In the following, let the map Φ satisfy the hypotheses of Theorem 3.1.

Lemma 3.2. *Φ is injective.*

Proof. Let $\Phi(A) = \Phi(B)$ for some $A, B \in \mathrm{Alg}\mathcal{L}_1$. Then we have $\Phi(A - B) = \Phi(A) - \Phi(B) = 0$. This together with the fact that $\Phi(0) = 0$ gives us $A - B \sim 0$. Hence, $A = B$. □

Lemma 3.3. *Φ preserves rank-one operators in both directions.*

Proof. Let A be of rank one. Then by Proposition 2.2, $A \sim 2A$. It follows that $\Phi(A) \sim 2\Phi(A)$. Suppose $\Phi(B) \sim \Phi(A)$ with $\Phi(B) \neq \Phi(A)$ such that $\Phi(B) + \Phi(A) \sim \Phi(A)$. Then $B \sim A$, $B \neq A$ and $B + A \sim A$. So we can apply Lemma 2.3 to conclude that $B - A \sim A$, which implies that $\Phi(B) - \Phi(A) \sim \Phi(A)$. Applying Lemma 2.3 again, we obtain that $\Phi(A)$ is of rank one. The same discussion implies that if $\Phi(A)$ is of rank one, then A is of rank one. Consequently, Φ preserves rank-one operators in both directions. □

Lemma 3.4. *Let K be in $\mathcal{J}(\mathcal{L}_1)$. Then there is a bijection $K \mapsto \hat{K}$ from $\mathcal{J}(\mathcal{L}_1)$ onto $\mathcal{J}(\mathcal{L}_2)$ such that $\Phi(\mathcal{F}_1(K)) = \mathcal{F}_1(\hat{K})$.*

Proof. Let K be in $\mathcal{J}(\mathcal{L}_1)$. Fix an operator $R_0 \in \mathcal{F}_1(K)$. By Lemma 3.3, there exists a rank-one operator $W_0 \in \mathrm{Alg}\mathcal{L}_2$ such that $\Phi(R_0) = W_0$. It follows from Lemma 1.1 that there exists a unique element $\hat{K} \in \mathcal{J}(\mathcal{L}_2)$ such that $W_0 \in \mathcal{F}_1(\hat{K})$. For any $R \in \mathcal{F}_1(K)$, by Proposition 2.2, we have $R \sim R_0$. Then $\Phi(R) \sim \Phi(R_0)$. This together with Proposition 2.2 gives $\Phi(R) \in \mathcal{F}_1(\hat{K})$. Thus, the map $K \mapsto \hat{K}$ is well defined and $\Phi(\mathcal{F}_1(K)) \subseteq \mathcal{F}_1(\hat{K})$. Since Φ^{-1} has the same property as Φ, one can get that $\mathcal{F}_1(\hat{K}) \subseteq \Phi(\mathcal{F}_1(K))$. So, $\Phi(\mathcal{F}_1(K)) = \mathcal{F}_1(\hat{K})$.

Next we show that the map $K \mapsto \hat{K}$ is injective. Let $\hat{K}_1 = \hat{K}_2$, $K_1, K_2 \in \mathcal{J}(\mathcal{L}_1)$. Take $R_1 \in \mathcal{F}_1(K_1)$ and $R_2 \in \mathcal{F}_1(K_2)$. Then we can obtain that $\Phi(R_1) \in \mathcal{F}_1(\hat{K}_1)$ and $\Phi(R_2) \in \mathcal{F}_1(\hat{K}_2)$. Since $\hat{K}_1 = \hat{K}_2$, we have $\Phi(R_1) \sim \Phi(R_2)$ by Proposition 2.2. It follows that $R_1 \sim R_2$. Applying Proposition 2.2 again, we get $K_1 = K_2$.

Finally, we prove that the map $K \mapsto \hat{K}$ is surjective. Let L be an arbitrary element in $\mathcal{J}(\mathcal{L}_2)$. Take $W \in \mathcal{F}_1(L)$. By Lemma 3.3, there exists a rank-one operator $R \in \mathrm{Alg}\mathcal{L}_1$ such that $\Phi(R) = W$. It follows from Lemma 1.1 that there exists a unique element $K \in \mathcal{J}(\mathcal{L}_1)$ such that $R \in \mathcal{F}_1(K)$. By the definition of the map, we have $W \in \mathcal{F}_1(\hat{K})$. This together with Lemma 1.2 implies that $L = \hat{K}$. □

Proposition 3.5. *Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then one of the following holds.*

(1) *There exist bijective linear maps $T_K : K \to \hat{K}$ and $S_K : K_-^\perp \to \hat{K}_-^\perp$ such that*

$$\Phi(x \otimes f) = T_K x \otimes S_K f$$

for every $x \otimes f \in \mathcal{F}_1(K)$.

(2) *There exist bijective linear maps $T_K : K_-^\perp \to \hat{K}$ and $S_K : K \to \hat{K}_-^\perp$such that*

$$\Phi(x \otimes f) = T_K f \otimes S_K x$$

for every $x \otimes f \in \mathcal{F}_1(K)$.

Proof. The proof is similar to the proof of Theorem 3.1 and 3.3 in [11]. □

Since Φ is surjective, we can assume that $\Phi(U) = I$ for some $U \in \mathrm{Alg}\mathcal{L}_1$. In the following, we always assume that the first case in Proposition 3.5 holds. The proof in the second case is similar. Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then there exist bijective linear maps $T_K : K \to \hat{K}$ and $S_K : K_-^\perp \to \hat{K}_-^\perp$ such that

$$\Phi(x \otimes f) = T_K x \otimes S_K f \tag{3.1}$$

for every $x \otimes f \in \mathcal{F}_1(K)$.

Lemma 3.6. *Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then for every $x \in K$ and $f \in K_-^{\perp}$, we have*

$$S_K f(T_K U x) = f(x) \tag{3.2}$$

and

$$S_K U^* f(T_K x) = f(x). \tag{3.3}$$

Proof. Let $x_0 \in K$ and $f_0 \in K_-^{\perp}$ be such that $f_0(x_0) = 0$. For every $\lambda \in \mathbb{F}$, $I + \lambda x_0 \otimes f_0$ is invertible by Lemma 2.1. Then

$$U \sim U(I + \lambda x_0 \otimes f_0) = U + \lambda U x_0 \otimes f_0.$$

This together with Eq. (3.1) gives

$$I \sim I + \lambda T_K U x_0 \otimes S_K f_0.$$

So, $I + \lambda T_K U x_0 \otimes S_K f_0$ is invertible. Applying Lemma 2.1 again, we can obtain that $\lambda S_K f_0(T_K U x_0) \neq -1$. As λ is an arbitrary scalar, we have

$$S_K f_0(T_K U x_0) = 0 \tag{3.4}$$

for every $x_0 \in K$ and $f_0 \in K_-^{\perp}$ with $f_0(x_0) = 0$.

Now we will prove that there exists a scalar $c \in \{0, 1\}$ such that

$$S_K f(T_K U x) = c f(x) \tag{3.5}$$

for every $x \in K$ and $f \in K_-^{\perp}$. Fix $x_1 \in K$ and $f_1 \in K_-^{\perp}$ with $f_1(x_1) = 1$ and set $c = S_K f_1(T_K U x_1)$. Choose any $\lambda \in \mathbb{F} \setminus \{-1\}$. Then by Lemma 2.1, $I + \lambda x_1 \otimes f_1$ is invertible. So,

$$U \sim U(I + \lambda x_1 \otimes f_1) = U + \lambda U x_1 \otimes f_1.$$

From this and Eq. (3.1) we get that

$$I \sim I + \lambda T_K U x_1 \otimes S_K f_1,$$

which implies that $I + \lambda T_K U x_1 \otimes S_K f_1$ is invertible. Applying Lemma 2.1, for any $\lambda \in \mathbb{F} \setminus \{-1\}$, we have $\lambda c \neq -1$. This further yields that $c = 0$ or $c = 1$.

We claim that for every $z \in K$ and $h \in K_-^{\perp}$ with $f_1(x_1) = h(z) = 1$ and $f_1(z) = h(x_1) = 0$, $S_K h(T_K U z) = c$. Actually, since $f_1(z) = h(x_1) = 0$, by Eq. (3.4), we have $S_K f_1(T_K U z) = 0 = S_K h(T_K U x_1)$. This together with Eq. (3.4) gives us

$$\begin{aligned} 0 &= S_K(f_1 + h)(T_K U(x_1 - z)) \\ &= S_K f_1(T_K U x_1) - S_K f_1(T_K U z) + S_K h(T_K U x_1) - S_K h(T_K U z) \\ &= S_K f_1(T_K U x_1) - S_K h(T_K U z), \end{aligned}$$

which implies that $S_K h(T_K U z) = c$.

Now we distinguish two cases according to the dimension of K.

Case 1: $2 \leq \dim K < \infty$.

Assume that $\dim K = n$, where $2 \leq n < \infty$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of K and $\{f_1, f_2, \ldots, f_n\}$ be a basis of $K_-^\perp$ satisfying $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. For any $x \in K$ and any $f \in K_-^\perp$, write $x = \sum_{i=1}^n \alpha_i x_i$ and $f = \sum_{j=1}^n \beta_j f_j$ for some $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{F}$. Then $f(x) = \sum_{i=1}^n \alpha_i \beta_i$. By the above claim and Eq. (3.4), we see that

$$\begin{aligned} S_K f(T_K U x) &= S_K \left(\sum_{j=1}^n \beta_j f_j \right) \left(T_K U \left(\sum_{i=1}^n \alpha_i x_i \right) \right) \\ &= \sum_{i=1}^n \alpha_i \beta_i S_K f_i(T_K U x_i) \\ &= c \sum_{i=1}^n \alpha_i \beta_i = c f(x). \end{aligned}$$

Case 2: K is infinite-dimensional.

By the linearity of T_K and S_K, it suffices to show that $S_K f(T_K U x) = c$ for every $x \in K$ and $f \in K_-^\perp$ with $f(x) = 1$. Since K is infinite-dimensional, we can choose a vector $y \in K$ such that $f(y) = f_1(y) = 0$ and $y \notin \operatorname{span}\{x, x_1\}$ as follows: take linearly independent vectors $y_1, y_2 \in K$ such that $f(y_1) = f(y_2) = 0$ and $y_1, y_2 \notin \operatorname{span}\{x, x_1\}$. We firstly assume that $f_1(y_1) = 0$ or $f_1(y_2) = 0$. Set $y = y_i$ if $f_1(y_i) = 0$, for $i = 1, 2$. Then the vector $y \in K$ is as required. Now assume that $f_1(y_1) \neq 0 \neq f_1(y_2)$. Set $y = f_1(y_2) y_1 - f_1(y_1) y_2$. Then $f(y) = f_1(y) = 0$, as desired. Now we can find a functional $g \in K_-^\perp$ such that $g(y) = 1$ and $g(x) = g(x_1) = 0$. By the above claim, we have $S_K f_1(T_K U x_1) = S_K g(T_K U y)$ and $S_K f(T_K U x) = S_K g(T_K U y)$, which yields that $S_K f(T_K U x) = c$.

Finally, we will show that $c = 1$. For this, assume on the contrary that $c = 0$. Then $S_K f(T_K U x) = 0$ for all $x \in K$ and $f \in K_-^\perp$. By the surjectivity of S_K and the injectivity of T_K, we have

$$Ux = 0 \tag{3.6}$$

for all $x \in K$. Take $y_0 \in \hat{K}$ and $g_0 \in \hat{K}_-^\perp$ such that $g_0(y_0) = 1$. By Lemma 3.4, there exists an operator $x_0 \otimes f_0 \in \mathcal{F}_1(K)$ such that $\Phi(x_0 \otimes f_0) = y_0 \otimes g_0$. Note that all invertible operators in $\operatorname{Alg}\mathcal{L}_2$ are mutually equivalent, and that $I + y_0 \otimes g_0$ is invertible by Lemma 2.1. It follows that

$$I \sim I + y_0 \otimes g_0.$$

Then

$$U \sim U + x_0 \otimes f_0.$$

From this, we can get

$$TUS = U + x_0 \otimes f_0$$

for some invertible operators $T, S \in \mathrm{Alg}\mathcal{L}_1$. Take $z_0 \in K$ such that $f_0(z_0) = 1$. Applying the above equation to z_0, we can get

$$TUSz_0 = Uz_0 + x_0.$$

Note that $Sz_0 \in K$. By Eq. (3.6), $x_0 = 0$, a contraction.

If we started the proof of this Lemma by $U \sim (I + \lambda x \otimes f)U$ instead of $U \sim U(I + \lambda x \otimes f)$ in the proof of Eq. (3.4) and then continuing the proof in the same way, we would get Eq. (3.3). □

Lemma 3.7. *Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then T_K and S_K are continuous.*

Proof. First we show the continuity of the operator $T_K U_K$. By the closed graph theorem, it suffices to prove that $T_K U_K$ is a closed operator. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K such that $x_n \to x$ and $T_K U_K x_n \to y$ $(n \to \infty)$, where $x \in K$ and $y \in \hat{K}$. Take any $f \in K_-^{\perp}$. Then $S_K f(T_K U_K(x_n - x)) \to S_K f(y - T_K U_K x)$. On the other hand, by Eq. (3.2), we have $S_K f(T_K U_K(x_n - x)) = f(x_n - x) \to 0$. Hence, $S_K f(y - T_K U_K x) = 0$. Since S_K is surjective, we can get that $g(y - T_K U_K x) = 0$ for all $g \in \hat{K}_-^{\perp}$. Note that $\hat{K} \wedge \hat{K}_- = (0)$ and $y - T_K U_K x \in \hat{K}$. So, $y = T_K U_K x$.

Applying Eq. (3.2) and the bijectivity of S_K, we have $S_K^{-1} g(x) = g(T_K U_K x)$ for every $x \in K$ and every $g \in \hat{K}_-^{\perp}$. Then

$$|S_K^{-1} g(x)| = |g(T_K U_K x)| \leq \|g\| \|T_K U_K\| \|x\|$$

for every $x \in K$. Since $K \wedge K_- = (0)$ and $K \vee K_- = X$, we can regard $K_-^{\perp}$ as the dual space K^* of K. Hence,

$$\|S_K^{-1} g\| \leq \|T_K U_K\| \|g\|$$

for every $g \in \hat{K}_-^{\perp}$. So, $\|S_K^{-1}\| \leq \|T_K U_K\|$, and hence, S_K^{-1} as well as S_K is continuous. By a similar argument as above, we can obtain that T_K is continuous. The proof is complete. □

Lemma 3.8. *Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then the operator U_K is invertible.*

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Proof. First we show that U_K has dense range. Otherwise, there exists a nonzero functional $f \in K_-^\perp$ such that $f(Ux) = 0$ for all $x \in K$, that is, $U^*f(x) = 0$ for all $x \in K$. Note that $U^*f \in K_-^\perp$ and $K \vee K_- = X$. So, $U^*f = 0$ and hence, $S_K U^* f = 0$. By Eq. (3.3), we have $f(x) = 0$ for all $x \in K$. This together with the fact that $K \vee K_- = X$ gives $f = 0$, a contraction.

Next we show that U_K is bounded below. Let x be any nonzero vector in K. Since $K \wedge K_- = (0)$, we can find $f_x \in K_-^\perp$ with $\|f_x\| = 1$ such that $f_x(x) = \|x\|$. Note that S_K and T_K are continuous by Lemma 3.7. It follows from Eq. (3.2) that

$$\|x\| = |f_x(x)| = |S_K f_x(T_K U_K x)| \leq \|S_K f_x\| \cdot \|T_K U_K x\| \leq \|S_K\| \cdot \|T_K\| \cdot \|U_K x\|.$$

As x is arbitrary, the operator U_K is bounded below. The proof is complete. □

Lemma 3.9. *Let $K \in \mathcal{J}_2(\mathcal{L}_1)$. Then for every $y \in \hat{K}$ and $f \in K_-^\perp$, we have $S_K f(y) = f(U_K^{-1} T_K^{-1} y)$. Moreover, $\Phi(x \otimes f)y = T_K x \otimes f U_K^{-1} T_K^{-1} y$ for every $x \otimes f \in \mathcal{F}_1(K)$ and $y \in \hat{K}$.*

Proof. By Lemma 3.8 and Eq. (3.2), we have $S_K f(T_K x) = f(U_K^{-1} x)$ for every $x \in K$ and $f \in K_-^\perp$. Noticing the bijectivity of T_K, we can change the above equation into $S_K f(y) = f(U_K^{-1} T_K^{-1} y)$ for every $y \in \hat{K}$ and $f \in K_-^\perp$. From this and Eq. (3.1), we conclude that $\Phi(x \otimes f)y = T_K x \otimes S_K f y = T_K x \otimes f U_K^{-1} T_K^{-1} y$ for every $x \otimes f \in \mathcal{F}_1(K)$ and $y \in \hat{K}$. The proof is complete. □

The proof of Theorem 3.1. (1) follows from Lemma 3.4 and (3) is obvious. To show (2), let K be in $\mathcal{J}_2(\mathcal{L}_1)$. By Lemma 3.8, U_K is invertible. We will show that

$$\Phi(A)y = T_K A U_K^{-1} T_K^{-1} y$$

for all $A \in \mathrm{Alg}\mathcal{L}_1$ and all $y \in \hat{K}$.

To this end, let $A \in \mathrm{Alg}\mathcal{L}_1$ and set $B = \Phi(A)$. We can assume that $A|_K \neq 0$ and B is invertible in $\mathrm{Alg}\mathcal{L}_2$. Actually, if $A|_K = 0$ or B is non-invertible, then we can take a nonzero scalar $\lambda \in \mathbb{F}$ such that $(A + \lambda U)|_K \neq 0$ and $B + \lambda I$ is invertible in $\mathrm{Alg}\mathcal{L}_2$. In this case, we may replace A by $A + \lambda U$.

Choose any nonzero vectors $x \in K$ and $f \in K_-^\perp$ such that $f(x) = 0$. Let $\lambda \in \mathbb{F}$ be arbitrary. By Lemma 2.1, $I + \lambda x \otimes f$ is invertible. Then we have

$$A \sim A(I + \lambda x \otimes f) = A + \lambda A x \otimes f.$$

This together with Eq. (3.1) gives us

$$B \sim B + \lambda T_K A x \otimes S_K f = B(I + \lambda B^{-1} T_K A x \otimes S_K f).$$

Since B is invertible in $\mathrm{Alg}\mathcal{L}_2$, $I + \lambda B^{-1}T_K Ax \otimes S_K f$ is invertible. Applying Lemma 2.1, we have $\lambda S_K f(B^{-1}T_K Ax) \neq -1$. Note that λ is an arbitrary scalar. Hence, $S_K f(B^{-1}T_K Ax) = 0$, which, together with Lemma 3.9, implies that

$$f(U_K^{-1}T_K^{-1}B^{-1}T_K Ax) = 0$$

for every $x \in K$ and $f \in K_-^\perp$. As $K \wedge K_- = (0)$, there exists a scalar $\mu \in \mathbb{F}$ such that

$$U_K^{-1}T_K^{-1}B^{-1}T_K Ax = \mu x \tag{3.7}$$

for all $x \in K$.

Now take $x_1 \in K$ and $f_1 \in K_-^\perp$ such that $f_1(x_1) = 1$. Then, by Lemma 2.1, $I + \lambda x_1 \otimes f_1$ is invertible for every $\lambda \in \mathbb{F} \setminus \{-1\}$. So, for every $\lambda \in \mathbb{F} \setminus \{-1\}$, we have

$$A \sim A(I + \lambda x_1 \otimes f_1) = A + \lambda A x_1 \otimes f_1.$$

It follows from Eq. (3.1) that

$$B \sim B + \lambda T_K A x_1 \otimes S_K f_1 = B(I + \lambda B^{-1}T_K A x_1 \otimes S_K f_1).$$

Since B is invertible in $\mathrm{Alg}\mathcal{L}_2$, $I + \lambda B^{-1}T_K A x_1 \otimes S_K f_1$ is invertible for every $\lambda \in \mathbb{F} \setminus \{-1\}$. So, $\lambda S_K f_1(B^{-1}T_K A x_1) \neq -1$ for every $\lambda \in \mathbb{F} \setminus \{-1\}$ by Lemma 2.1. This implies that either $S_K f_1(B^{-1}T_K A x_1) = 0$ or $S_K f_1(B^{-1}T_K A x_1) = 1$. Therefore, by Lemma 3.9, we have either

$$f_1(U_K^{-1}T_K^{-1}B^{-1}T_K A x_1) = 0 \tag{3.8}$$

or

$$f_1(U_K^{-1}T_K^{-1}B^{-1}T_K A x_1) = 1. \tag{3.9}$$

First assume that Eq. (3.8) holds. Then, by Eq. (3.7), we have

$$\mu = \mu f_1(x_1) = f_1(\mu x_1) = f_1(U_K^{-1}T_K^{-1}B^{-1}T_K A x_1) = 0.$$

So, $U_K^{-1}T_K^{-1}B^{-1}T_K Ax = 0$ for all $x \in K$, which further yields that $A|_K = 0$, a contraction. Now assume that Eq. (3.9) holds. Then, by Eq. (3.7), we have

$$\mu = \mu f_1(x_1) = f_1(\mu x_1) = f_1(U_K^{-1}T_K^{-1}B^{-1}T_K A x_1) = 1.$$

So,

$$U_K^{-1}T_K^{-1}B^{-1}T_K Ax = x$$

for all $x \in K$. Equivalently,

$$T_K A U_K^{-1} T_K^{-1} y = By$$

for all $y \in \hat{K}$. The proof is complete. □

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