ANALYSIS OF A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS VIA KRASNOSEL'SKIĬ-PRECUP FIXED POINT INDEX THEOREMS IN CONES

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Abstract. We apply Krasnosel'skiĭ-Precup fixed point index theorems in cones recently published by Rodríguez-López [11] to study the existence of multiple positive solutions to a boundary value problem for a system of higher-order fractional differential equations. An illustrative example is also presented.

Keywords: Caputo fractional derivative, system, boundary conditions, Krasnosel'skiĭ-Precup fixed point theorem.
Mathematics Subject Classifications: 34A08; 34A34; 34B15

1. Introduction

The approach of fixed point theory is found to be of great benefit in studying the existence of solutions for initial and boundary value problems. In particular, Krasnosel'skiĭ compression-expansion fixed point index theorem [1, 2] is extensively used to establish the existence of positive solutions for different types of boundary value problems. In [3], Leggett–Williams and Krasnosel'skiĭ fixed point theorems were employed to accomplish the existence of triple positive solutions to higher-order fractional differential equations with integral conditions. A detailed description of the work on boundary value problems for fractional differential equations and systems based on different kinds fixed point theorems can be found in the text [4]. In a recent paper [5], the author studied the existence of positive solutions for a singular system of nonlinear fractional differential equations. Krasnosel'skiĭ-Precup fixed point theorem was successfully employed to investigate the existence, localization and multiplicity of positive solutions for boundary value problems of systems of differential equations in [6]-[10]. In a recent article [11], Rodríguez-López discussed a fixed point index approach to Krasnosel'skiĭ-Precup fixed point theorem and applied it to show the existence of positive solutions for Hammerstein integral equations.

The objective of the present paper is to apply Krasnosel'skiĭ-Precup fixed point index theorems in cones presented in [11] to study the existence of multiple positive solutions to a boundary value problem for a system of higher order fractional differential equations. Precisely, we investigate the following problem:

$$\begin{cases} {}^cD^{\sigma_1}_{0^+}u(t) = \psi_1(t)f_1(u(t),v(t)), & t\in(0,1),\\ {}^cD^{\sigma_2}_{0^+}v(t) = \psi_2(t)f_2(u(t),v(t)), & t\in(0,1),\\ u(1)=u'(0)=\ldots=u^{n-2}(0)=u^{n-1}(0)=0, & \\ v(1)=v'(0)=\ldots=v^{n-2}(0)=v^{n-1}(0)=0, & \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative of order $\alpha\in(n-1,n], n\geq 2$ $(\alpha=\sigma_1,\sigma_2)$ and it is assumed that
(H1) $f_i\in C([0,1],[0,\infty)\times[0,\infty))$ and $\psi_i\in C([0,1],[0,+\infty)), i=1,2$.

The rest of the manuscript is arranged as follows. Section 2 contains the preliminary material. Main results are presented in Section 3, while an illustrative example is discussed in Section 4.

2. Preliminaries

In this section, we present some preliminary material that will be used in the proofs of the main results.

Definition 2.1. *Let X be a real Banach space. A non-empty closed set $P\subset X$ is called a cone of X if it satisfies the following conditions:*
(1) $x\in P, \mu\geq 0$ implies $\mu x\in P$,
(2) $x\in P, -x\in P$ implies $x=0$.

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Definition 2.2. *([12]) The fractional derivative of $f \in C^n[a,b]$ in the Caputo sense is defined as*

$$ {}^cD_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha \le n,\, n = [\alpha]+1. $$

In the following lemma, we present the solution of problem (1.1) in terms of Green's function.

Lemma 2.1. *Assume that the hypothesis $(H1)$ holds. Then, the boundary value problem (1.1) has a unique solution:*

$$ \begin{cases} u(t) = \int_0^1 G_1(t,s)\psi_1(s) f_1(u(s),v(s))ds, \\ v(t) = \int_0^1 G_2(t,s)\psi_2(s) f_2(u(s),v(s))ds, \end{cases} \tag{2.1} $$

where

$$ G_i(t,s) = \frac{1}{\Gamma(\sigma_i)} \begin{cases} (1-s)^{\sigma_i-1} - (t-s)^{\sigma_i-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\sigma_i-1}, & 0 \le t \le s \le 1, \end{cases} \quad (i=1,2). \tag{2.2} $$

Proof. The proof is similar to that of [13, Lemma 2.8] and is omitted. □

Proposition 2.1. *For $t,s \in [0,1]$, we obtain*

$$ 0 \le G_i(t,s) \le G_i(s,s) \le \frac{1}{\Gamma(\sigma)} \quad (i=1,2). $$

Proposition 2.2. *Let $\theta \in (0,\frac{1}{2})$, then for all $s \in [0,1]$, we have*

$$ \min_{\theta \le t \le 1-\theta} G_i(t,s) \ge \Big[1-(1-\theta)^{\sigma-1}\Big] G_i(s,s), \quad i=1,2. $$

Proof. For $\theta \in (0,\frac{1}{2})$ and $i=1,2$, we get

$$ \begin{aligned} \min_{\theta \le t \le 1-\theta} G_i(t,s) &= \frac{1}{\Gamma(\sigma_i)} \begin{cases} (1-s)^{\sigma_i-1} - (1-\theta-s)^{\sigma_i-1}, & s \in [0,\theta], \\ \min\Big\{(1-s)^{\sigma_i-1} - (1-\theta-s)^{\sigma_i-1}, (1-s)^{\sigma_i-1}\Big\} & \\ = (1-s)^{\sigma_i-1} - (1-\theta-s)^{\sigma_i-1}, & s \in [\theta, 1-\theta], \\ (1-s)^{\sigma_i-1}, & s \in [1-\theta,1]. \end{cases} \\ &= \frac{1}{\Gamma(\sigma_i)} \begin{cases} (1-s)^{\sigma_i-1} - (1-\theta-s)^{\sigma_i-1}, & s \in [0,1-\theta], \\ (1-s)^{\sigma_i-1}, & s \in [1-\theta,1]. \end{cases} \end{aligned} $$

Since $\theta \in (0,\frac{1}{2})$ and $\sigma_i > 1$ $(i=1,2)$, we get

$$ \left(1-\frac{s}{1-\theta}\right)^{\sigma_i-1} \le (1-s)^{\sigma_i-1}, $$

which consequently yields

$$ \begin{aligned} (1-s)^{\sigma_i-1} - (1-\theta-s)^{\sigma_i-1} &\ge (1-s)^{\sigma_i-1} - (1-\theta)^{\sigma_i-1}(1-s)^{\sigma_i-1} \\ &\ge [1-(1-\theta)^{\sigma_i-1}](1-s)^{\sigma_i-1}, \quad \text{for } s \in [0,1-\theta], \\ (1-s)^{\sigma_i-1} &\ge [1-(1-\theta)^{\sigma_i-1}](1-s)^{\sigma_i-1}, \quad \text{for } s \in [1-\theta,1]. \end{aligned} $$

Thus, for $s \in [0,1]$, we have

$$ \min_{\theta \le t \le 1-\theta} G_i(t,s) \ge [1-(1-\theta)^{\sigma_i-1}] \frac{(1-s)^{\sigma_i-1}}{\Gamma(\sigma_i)} = [1-(1-\theta)^{\sigma_i-1}] G_i(s,s),\, i=1,2. $$

□

Remark 2.1. *Let $\theta = \frac{1}{4}$ and $s \in [0,1]$. Then, by Proposition 2.2, we obtain*

$$ \min_{\frac{1}{4} \le t \le \frac{3}{4}} G_i(t,s) \ge \Big[1-(\frac{3}{4})^{\sigma_i-1}\Big] G_i(s,s),\, i=1,2. $$

Lemma 2.2. *If the assumption (H1) holds, then the unique solution u of the problem (1.1) satisfies the inequalities:*

(i): *$u(t) \ge 0$, for $t \in [0,1]$,*

(ii): $\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\|u\|$ *and* $\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} v(t) \geq \Big(1-(\frac{3}{4})^{\sigma_2-1}\Big)\|v\|$.

Proof. (i) By Proposition 2.1, it is obvious that $G_i(t,s)\geq 0,\ i=1,2$, so we get $u(t)\geq 0$.

(ii) From Remark 2.1, for $t\in\Big[\frac{1}{4},\frac{3}{4}\Big]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)\psi_1(s)f_1(u(s),v(s))ds \\ &\geq \Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\int_0^1 G_1(s,s)\psi_1(s)f_1(u(s),v(s))ds \geq \Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\|u\|. \end{aligned}$$

Therefore, we get $\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t)\geq \Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\|u\|$. In a similar manner, it can be shown that

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} v(t)\geq \Big(1-(\frac{3}{4})^{\sigma_2-1}\Big)\|v\|.$$

□

Define

$$K_1=\{u\in C([0,1])|u(t)\geq 0,\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t)\geq \Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\|u\|\}, \tag{2.3}$$

$$K_2=\{u\in C([0,1])|v(t)\geq 0,\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} v(t)\geq \Big(1-(\frac{3}{4})^{\sigma_2-1}\Big)\|v\|\}. \tag{2.4}$$

Observe that K_1 and K_2 are cones. Define an operator $T:K\to K$ as

$$T(u,v)(t)=(T_1(u,v)(t),T_2(u,v)(t)),$$

where

$$T_i(u,v)(t)=\int_0^1 G_i(t,s)\psi_i(s)f_i(u(s),v(s))ds,\quad i=1,2, \tag{2.5}$$

and $K=K_1\times K_2$ is a cone in $C([0,1])\times C([0,1])$.

Notice that that the existence of a positive solution for the system (1.1) is equivalent to that of a nontrivial fixed point of T in K.

Lemma 2.3. *Suppose that the condition (H1) holds. Then $T(K)\subseteq K$ and $T:K\to K$ is completely continuous.*

Proof. For any $(u,v)\in K$, by (2.3) and (2.4), we obtain $T_i(u,v)(t)\geq 0$ and, for $t\in[0,1]$ and $i=1,2$,

$$T_i(u,v)(t)=\int_0^1 G_i(t,s)\psi_i(s)f_i(u(s),v(s))ds\leq\int_0^1 G_i(s,s)\psi_i(s)f_i(u(s),v(s))ds.$$

Thus, $\|T_i(u,v)\|\leq\int_0^1 G_i(s,s)\psi_i(s)f_i(u(s),v(s))ds$.

On the other hand, for $t\in\Big[\frac{1}{4},\frac{3}{4}\Big]$ and $i=1,2$, we have

$$\begin{aligned} T_i(u,v)(t) &= \int_0^1 G_i(t,s)\psi_i(s)f_i(u(s),v(s))ds \\ &\geq \Big(1-(\frac{3}{4})^{\sigma_i-1}\Big)\int_0^1 G_i(s,s)\psi_i(s)f_i(u(s),v(s))ds \\ &\geq \Big(1-(\frac{3}{4})^{\sigma_i-1}\Big)\|T_i(u,v)\|. \end{aligned}$$

So $T(K)\subseteq K$. By conventional arguments and Ascoli-Arzela theorem, one can show that $T:K\to K$ is completely continuous. □

Our main results are based on the following new alternative versions of Krasnosel'skiĭ-Precup fixed point theorem, which were recently proved in the article [11] and are stated below for the convenience of the reader.

For $r=(r_1,r_2)\in\mathbb{R}^2_+$ and $R=(R_1,R_2)\in\mathbb{R}^2_+$ with $0<r_i<R_i$, $i=1,2$, set

$$\begin{aligned}(K_i)_{r_i} &= \{u\in K_i:\ \|u\|<r_i\},\ (\overline{K}_i)_{r_i}=\{u\in K_i:\ \|u\|\le r_i\},\\ K_{r,R} &= \{(u,v)\in K:\ r_1<\|u\|<R_1,\ r_2<\|v\|<R_2\}.\end{aligned}$$

Theorem 2.1. *([11, Theorem 2.8]) Assume that $T=(T_1,T_2):\overline{K}_{r,R}\to K$ is a compact map and there exists $h_i\in K_i\setminus\{0\}$ for each $i\in\{1,2\}$ such that one of the following conditions is satisfied in $\overline{K}_{r,R}$:*

- **(a):** *$T_i(u)+\mu h_i\neq u_i$, if $\|u_i\|=r_i$ and $\mu\ge 0$, and $T_i(u)\neq\lambda u_i$ if $\|u_i\|=R_i$ and $\lambda\ge 1$;*
- **(b):** *$T_i(u)\neq\lambda u_i$, if $\|u_i\|=r_i$ and $\lambda\ge 0$, and $T_i(u)+\mu h_i\neq u_i$ if $\|u_i\|=R_i$ and $\mu\ge 1$.*

Then, T has at least one fixed point $u=(u_1,u_2)\in K$ with $r_1<\|u\|<R_1,\ r_2<\|v\|<R_2$.

Theorem 2.2. *([11, Theorem 2.11]) Let X and Y be normed linear spaces, $K_1\subset X$ and $K_2\subset Y$ be two cones, $K=K_1\times K_2$ and $r^{(j)}=(r_1^{(j)},r_2^{(j)})\in\mathbb{R}^2_+$ and $R^{(j)}=(R_1^{(j)},R_2^{(j)})\in\mathbb{R}^2_+$ with $0<r_i^{(j)}<R_i^{(j)}$ $(i=1,2,\ j=1,2,3)$. Assume the sets $\overline{K}_{r^{(j)},R^{(j)}}$ are such that*

$$\overline{K}_{r^{(1)},R^{(1)}}\bigcup\overline{K}_{r^{(2)},R^{(2)}}\subset\overline{K}_{r^{(3)},R^{(3)}}\quad\text{and}\quad\overline{K}_{r^{(1)},R^{(1)}}\bigcap\overline{K}_{r^{(2)},R^{(2)}}=\emptyset.$$

Moreover, assume that $T=(T_1,T_2):\overline{K}_{r^{(3)},R^{(3)}}\to K$ is a compact map and there exists $h_i^j\in K_i\setminus\{0\}$ for each $i\in\{1,2\}$ and $j\in\{1,2,3\}$ such that one of the following conditions is satisfied in $\overline{K}_{r^{(j)},R^{(j)}}$:

- **(a):** *$T_i(u)+\mu h_i^j\neq u_i$, if $\|u_i\|=r_i^{(j)}$ and $\mu\ge 0$, and $T_i(u)\neq\lambda u_i$ if $\|u_i\|=R_i^{(j)}$ and $\lambda\ge 1$;*
- **(b):** *$T_i(u)\neq\lambda u_i$, if $\|u_i\|=r_i^{(j)}$ and $\lambda\ge 0$, and $T_i(u)+\mu h_i^j\neq u_i$ if $\|u_i\|=R_i^{(j)}$ and $\mu\ge 1$.*

Then, T has at least three fixed points $\widetilde{u}^j=(\widetilde{u}_1^j,\widetilde{u}_2^j)\in K$ $(j\in\{1,2,3\})$ such that

$$\widetilde{u}^1\in K_{r^{(1)},R^{(1)}},\ \ \widetilde{u}^2\in K_{r^{(2)},R^{(2)}},\ \ \widetilde{u}^3\in K_{r^{(3)},R^{(3)}}\setminus\left(\overline{K}_{r^{(1)},R^{(1)}}\bigcup\overline{K}_{r^{(2)},R^{(2)}}\right).$$

3. Main results

For $\varpi=(\varpi_1,\varpi_2)$ and $\varrho=(\varrho_1,\varrho_2)$ with $\varpi_i,\varrho_i>0$, $\varpi_i\neq\varrho_i$, $i=1,2$, we set the notation:

$$\begin{aligned}M_i &= \int_{\frac14}^{\frac34}G_i(s,s)\psi_i(s)ds,\ \ \widetilde{M}_i=\int_0^1G_i(s,s)\psi_i(s)ds, i=1,2,\\ \Theta_1^{\varpi,\varrho} &= \min\left\{f_1(u,v):\Big(1-(\frac34)^{\sigma_1-1}\Big)\varrho_1\le u\le\varrho_1,\ \Big(1-(\frac34)^{\sigma_2-1}\Big)r_2\le v\le R_2\right\},\\ \Theta_1^{\varpi,\varrho} &= \min\left\{f_2(u,v):\Big(1-(\frac34)^{\sigma_1-1}\Big)r_1\le u\le R_1,\ \Big(1-(\frac34)^{\sigma_2-1}\Big)\varrho_2\le v\le\varrho_2\right\},\\ \Lambda_1^{\varpi,\varrho} &= \max\left\{f_1(u,v):0\le u\le\varpi_1,\ 0\le v\le R_2\right\},\\ \Lambda_1^{\varpi,\varrho} &= \max\left\{f_2(u,v):0\le u\le R_1,\ 0\le v\le\varpi_2\right\},\end{aligned}$$

where $r_i=\min\{\varpi_i,\varrho_i\}$ and $R_i=\max\{\varpi_i,\varrho_i\}$.

Now, we present our main results.

Theorem 3.1. *Suppose that (H1) holds and there exist positive constant $\varpi_i,\varrho_i>0$, $\varpi_i\neq\varrho_i$, $i=1,2$, such that*

$$\Big(1-(\frac34)^{\sigma_i-1}\Big)M_i\Theta_i^{\varpi,\varrho}>\varrho_i,\quad\widetilde{M}_i\Lambda_i^{\varpi,\varrho}<\varpi_i\ \ (i=1,2).\tag{3.1}$$

Then, the system (1.1) has at least one positive solution $(u,v)\in K$ such that $r_1<\|u\|<R_1$ and $r_2<\|v\|<R_2$.

Proof. Let $h_i:=1\in K_i\setminus\{0\}, i=1,2$. The proof will be completed in two steps.
(1) We verify the condition (a) of Theorem 2.1 as $T_1(u,v)+\mu1\neq u$ and $T_2(u,v)+\mu1\neq v$, if $\|u\|=\varrho_1$, $\|v\|=\varrho_2$ and $\mu\ge 0$. To this end, assume that there exists $(u,v)\in\overline{K}_{r,R}$ with $\|u\|=\varrho_1$, $\|v\|=\varrho_2$ and $\mu\ge 0$ such that $T_1(u,v)+\mu1=u$ and $T_2(u,v)+\mu1=v$ for the sake of contradiction. Then we have

$$\begin{cases}u(t)=\int_0^1G_1(t,s)\psi_1(s)f_1(u(s),v(s))ds+\mu,\\ v(t)=\int_0^1G_2(t,s)\psi_2(s)f_2(u(s),v(s))ds+\mu.\end{cases}$$

Since $(u,v)\in \overline{K}_{r,R}\subseteq K$ with $\|u\|=\varrho_1$, $\|v\|=\varrho_2$, therefore,

$$\begin{cases}\Big(1-(\tfrac{3}{4})^{\sigma_1-1}\Big)\varrho_1\le u(t)\le \varrho_1,\\ \Big((\tfrac{3}{4})^{\sigma_2-1}\Big)r_2\le v(t)\le R_2,\end{cases}\quad \text{for all } t\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Hence, for $t\in\left[\frac14,\frac34\right]$, by (3.1) and Remark 2.1, we obtain

$$\begin{aligned} u(t) &\ge \int_{\frac14}^{\frac34} G_1(t,s)\psi_1(s)f_1(u(s),v(s))ds \ge \Theta_1^{\varpi,\varrho}\int_{\frac14}^{\frac34} G_1(t,s)\psi_1(s)ds\\ &\ge \Theta_1^{\varpi,\varrho}\Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)\int_{\frac14}^{\frac34} G_1(s,s)\psi_1(s)ds \ge \Theta_1^{\varpi,\varrho}\Big(1-(\frac{3}{4})^{\sigma_1-1}\Big)M_1>\varrho_1,\end{aligned}$$

which is a contradiction.

(2) We verify the condition (b) of Theorem 2.1 as $T_1(u,v)\ne\lambda u$ and $T_2(u,v)\ne\lambda v$, if $\|u\|=\varpi_1$, $\|v\|=\varpi_2$ and $\lambda\ge 0$. Equivalently, it will be shown that $\|T_i(u,v)\|<\varpi_i$ for all $(u,v)\in\overline{K}_{r,R}\subseteq K$ with $\|u\|=\varpi_1$, $\|v\|=\varpi_2$. So we have

$$\begin{cases}0\le u(t)\le\varpi_1,\\ 0\le v(t)\le R_2,\end{cases}\quad\text{for all } t\in[0,1].$$

Hence, in view of (3.1) and Proposition 2.1, one can get

$$T_i(u,v)(t)=\int_0^1 G_i(t,s)\psi_i(s)f_i(u(s),v(s))ds\le\Lambda_i^{\varpi,\varrho}\int_0^1 G_i(s,s)\psi_i(s)ds\le\Lambda_i^{\varpi,\varrho}\overline{M}_i<\varpi_i.$$

Consequently, $\|T_i(u,v)\|<\varpi_i$. Therefore, by Theorem 2.1 with $r_i=\min\{\varpi_i,\varrho_i\}$ and $R_i=\max\{\varpi_i,\varrho_i\}$, we have the conclusion. □

Theorem 3.2. *Suppose that (H1) and the following conditions hold:*

(H2) there exist positive constant $\varpi_i^j,\varrho_i^j>0$, $\varpi_i^j\ne\varrho_i^j$, $i=1,2$, $j=1,2,3$ *such that* $\varpi_i^1,\varpi_i^2,\varrho_i^1,\varrho_i^2\in[r_i^3,R_i^3]$ *for* $i\in\{1,2\}$*;*

(H3) there exists $i\in\{1,2\}$ *such that* $R_i^1<r_i^2$*, where* $r_i^j=\min\{\varpi_i^j,\varrho_i^j\}$, $R_i^j=\max\{\varpi_i^j,\varrho_i^j\}$*;*

$$(H4)\quad \Big(1-(\frac{3}{4})^{\sigma_i-1}\Big)M_i\Theta_i^{\varpi^j,\varrho^j}>\varrho_i^j,\quad \widetilde{M}_i\Lambda_i^{\varpi^j,\varrho^j}<\varpi_i^j,\quad i=1,2,\quad j=1,2,3.$$

Then, the system (1.1) has at least three positive solutions.

Proof. We shall apply Theorem 2.2 to the operator $T=(T_1,T_2):\overline{K}_{r^3,R^3}\to K$, where T_1,T_2 are defined in (2.5). By (H2), we have $\overline{K}_{r^3,R^1}\bigcup\overline{K}_{r^2,R^2}\subset\overline{K}_{r^3,R^3}$. On the other hand, it follows by (H3) that $\overline{K}_{r^1,R^1}\bigcap\overline{K}_{r^2,R^2}=\emptyset$.

Also, by the condition (H4) and the arguments used in the proof of Theorem 3.1, we can verify the assumptions (a) and (b) of Theorem 2.2. Therefore, by Theorem 2.2, we get the conclusion. □

4. Application

Consider the following system of fractional differential equations with the boundary conditions:

$$\begin{cases} {}^cD_{0^+}^{\frac32}u(t)=\psi_1(t)f_1(u(t),v(t)), & t\in(0,1),\\ {}^cD_{0^+}^{\frac32}v(t)=\psi_2(t)f_2(u(t),v(t)), & t\in(0,1),\\ u(1)=u'(0)=0, v(1)=v'(0)=0, \end{cases}\tag{4.1}$$

where $n=2$, $\sigma_i=\frac32$, $f_1(u,v)=u^3(1+\sin(u)\cos(v))$, $f_2(u,v)=v^2(1+\sin(u)\cos(v))$ and $\psi_1(t)=\psi_2(t)=t$. Using the given values, we find that $M_i=0.18889$, $\widetilde{M}_i=0.3009$, $\Big(1-(\frac34)^{\sigma_i-1}\Big)=0.1339$ for $i=1,2$. Moreover, by the condition (H4), we have $\varrho_1^1=\varrho_1^3=0.00024$, $\varpi_1^1=0.3$, $\varrho_1^2=130$, $\varpi_1^2=\varpi_1^3=1120$, $\varpi_2^j=3$ and $\varrho_2^j=1100$ for $j=1,2,3$. So, by Theorem 3.2, the problem (1.1) has at least three positive solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) satisfying $0.00024<\|u_1\|<0.3, 3<\|v_1\|<1100, 130<\|u_2\|<1120, 3<\|v_2\|<1100, 0.3<\|u_3\|<130, 3<\|v_3\|<1100$.

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Conflict of interest. The authors declare that they have no conflict of interest.

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