Extended version of Fixed Point Theorems and their applications on an integro-differential equation containing Riemann-Liouville fractional derivative and integral in $\mathbb{C}([0,L])$ space

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ABSTRACT. In this work, using freshly made contraction operator, different types of new extended version of fixed point theorems have been established. Also, the application of these fixed point theorems to an integro-differential equation containing Riemann-Liouville fractional derivative and integral in a Banach space has been illustrated with an appropriate example.

Key Words: Measure of noncompactness ($\mathcal{MNC}$); Fixed point Theorem ($F\mathcal{P}T$); Integro-Differential Equation (IDE); Fractional Integral Equation ($\mathcal{F}IE$).

MSC subject classification no : 26A33 ; 45G05 ; 45G10 ; 47H10

1. Introduction

The integro-differential equation is one of the most valuable tool that have been used in solving real world problems .In recent times, integro-differential equations have gained a lots of significance because of their several applications in different fields. The fixed point theorem ($F\mathcal{P}T$) and measure of noncompactness ($\mathcal{MNC}$) are very important in solving Integro-differential Equation. Kuratowski [12] first defined the idea of $\mathcal{MNC}$ in 1930. In 1955, the Schauder's fixed point theorem was modified by G. Darbo [13] with the help of Kuratowski's $\mathcal{MNC}$. There are many new research projects related to the applications of $F\mathcal{P}T$ on integral equatins, differential equations and integro-differential equations has been established by several mathematicians (see [11, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25, 27, 28, 29, 30]).

Fractional calculus deals with the investigation and applications of derivatives and integrals of arbitrary order. It is a very important topic having interconnections with different types of problems of function theory, integral and differential equations, and other branches of analysis. It has been continually developed , stimulated by ideas and results in several fields of mathematical analysis. Fractioanl integro-differential equations are widely used to describe many important phenomena in various fields such as physics, biophysics, chemistry, biology, control theory, economy and so on; see [2, 3, 4, 6, 7, 8, 9, 10, 32, 33, 34]. Das et al. [35] and Arab et al. [36] used a measure of noncompactness for the infinite systems of integral equations. Banas and Lecko [37], Rzepka and Sadarangani [38] discussed the solvability of infinite systems of integral equations with the help of measure of noncompactness. Aghajani and Haghighi [39] using the techniques of measures of noncompactness and Darbo fixed point theorem, proved the existence results for solutions of systems of nonlinear equations in Banach spaces, and discussed the existence of solutions for a general system of nonlinear functional integral equations. Surang Sitho, Sotiris K Ntouyas and Jessada Triboon [40] proved the existence results for initial value problems for hybrid fractional integro-differential equations. Ahmed Bragdi, Assia Friour and Assia Guezane Lakoud [41] discussed the existence of solutions for boundary value problem of nonlinear sequential fractional integro-differential equations with the help of Krasnoselskii fixed point theorem.

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The main goal of this work is to obtain the existence of a solution of the integro-differential equation (1.1) containing Riemann-Liouville (RL) fractional derivative and integral by using an extended version of Darbo's FPT

$$\begin{cases} \mathfrak{D}^{\xi}\left[\dfrac{\mathcal{U}(\eta)-\mathcal{I}^{\zeta}\ \mathcal{K}(\eta,\mathcal{U}(\eta))}{\mathcal{G}(\eta,\mathcal{U}(\eta))}\right]=\mathcal{Q}(\eta,\mathcal{U}(\eta))\ ,\ \eta\in T=[0,L] \\ \mathcal{U}(0)=0\ , \end{cases} \tag{1.1}$$

where $\mathfrak{D}^{\xi}$ is the RL fractional derivative of order ξ , $0\leq\xi\leq 1$; $\mathcal{I}^{\zeta}$ is the RL fractional integral of order ζ , $\zeta>0$; $\mathcal{G}$ is a function from $T\times\mathbb{R}$ to $\mathbb{R}\backslash\{0\}$ and $\mathcal{Q},\mathcal{K}$ are functions from $T\times\mathbb{R}$ to $\mathbb{R}$. Also from the Lemma 5.4 , the above integro-differential equation (1.1) is equivalent to the following fractional integral equation ($\mathcal{FIE}$)

$$\mathcal{U}(\eta)\ =\ \frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)}\int_0^{\eta}(\eta-\mathfrak{z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}+\frac{1}{\Gamma(\zeta)}\int_0^{\eta}(\eta-\mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z} \tag{1.2}$$

Finally at the end , we discuss about the solvability of the following IDE

$$\begin{cases} \mathfrak{D}^{\frac{1}{5}}\left[\dfrac{\mathcal{U}(\eta)-\mathcal{I}^{\frac{1}{7}}\ \frac{\mathcal{U}(\eta)}{17+\eta}}{\frac{\mathcal{U}(\eta)+1}{19+\eta}}\right]=\dfrac{\mathcal{U}(\eta)}{21+\eta}\ ,\ \eta\in T=[0,L] \\ \mathcal{U}(0)=0\ , \end{cases} \tag{1.3}$$

2. Preliminaries

Assume , $(G,\|\ .\ \|)$ be a real Banach space and $\mathcal{B}(\theta,e_0)=\{t\in G:\|\ t-\theta\ \|\leq e_0\}$.
Let ,

- X_G is the collection of all non-empty bounded subsets of G and Y_G is the collection of all non-empty relatively compact subsets of G ,
- $\bar{\mathfrak{P}}$ and Conv$\mathfrak{P}$ denote the closure and the convex closure of $\mathfrak{P}$ respectively, where $\mathfrak{P}\subset G$.
- $\mathbb{R}=(-\infty,\infty)$,
  and
- $\mathbb{R}^+=[0,\infty)$.

Now , We consider the following fundamental theorems and definitions which are useful for the generalization of Darbo's Fixed point theorem :

Definition 2.1. [5] *A map $W:X_G\to\mathbb{R}^+$ is known as a $\mathcal{MNC}$ in G. If it holds the axioms given below,*

(i) $\forall\ \mathfrak{P}\in X_G$, *we get* $W(\mathfrak{P})=0$ *gives* $\mathfrak{P}$ *is relatively compact.*
(ii) *ker* $W=\{\mathfrak{P}\in X_G:W(\mathfrak{P})=0\}\neq\emptyset$ *and ker* $W\subset Y_G$.
(iii) $\mathfrak{P}\subseteq\mathfrak{P}_1\implies W(\mathfrak{P})\leq W(\mathfrak{P}_1)$.
(iv) $W(\bar{\mathfrak{P}})=W(\mathfrak{P})$.
(v) $W(Conv\mathfrak{P})=W(\mathfrak{P})$.
(vi) $W(\mathbb{A}\mathfrak{P}+(1-\mathbb{A})\mathfrak{P}_1)\leq\mathbb{A}W(\mathfrak{P})+(1-\mathbb{A})W(\mathfrak{P}_1)$ *for* $\mathbb{A}\in[0,1]$.
(vii) *if* $\mathfrak{P}_l\in X_G$, $\mathfrak{P}_l=\bar{\mathfrak{P}}_l$, $\mathfrak{P}_{l+1}\subset\mathfrak{P}_l$ *for* $l=1,2,3,4,...$ *and* $\lim_{l\to\infty}W(\mathfrak{P}_l)=0$ *then* $\bigcap_{l=1}^{\infty}\mathfrak{P}_l\neq\emptyset$.

The family $kerW$ is known as the *kernel of measure* W. Since $W(\mathfrak{P}_\infty)\leq W(\mathfrak{P}_l)$ for any l, we can say that $W(\mathfrak{P}_\infty)=0$. Then $\mathfrak{P}_\infty=\bigcap_{l=1}^{\infty}\mathfrak{P}_l\in kerW$.

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Theorem 2.2. *([Shauder][1]) Let V be a nonempty , bounded , closed and convex subset (NBCCS) of a Banach space G. Then $H : V \to V$ has at least one fixed point provided that H is a compact, continuous mapping.*

Theorem 2.3. *([Darbo][13]) Let V be a NBCCS of a Banach Space G and let $H : V \to V$. Assume that we have a constant $\hat{\mathcal{M}} \in [0, 1)$ such that*

$$W(HZ) \leq \hat{\mathcal{M}}\ W(Z),\ Z \subseteq V.$$

Then H has a fixed point in V provided that H is a continuous mapping.

Definition 2.4. *Let $\Delta : \mathbb{R}^n \to \mathbb{R}^+$ be a function whih satisfy :*

$$\Delta(s_1^{'}, s_2^{'}, ..., s_n^{'}) \leq max\{s_1^{'}, s_2^{'}, ..., s_n^{'}\}.$$

This class of functions is denoted by $\bar{\Delta}$.
For example ,

(1) $\Delta(s_1^{'}, s_2^{'}, ..., s_n^{'}) = max\{s_1^{'}, s_2^{'}, ..., s_n^{'}\}$,

(2) $\Delta(s_1^{'}, s_2^{'}, ..., s_n^{'}) = \frac{1}{n}\{s_1^{'} + s_2^{'} + ... + s_n^{'}\}$; $s_1^{'}, s_2^{'}, ..., s_n^{'} \in \mathbb{R}$.

Definition 2.5. *Let $\mathfrak{F}, \alpha, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be functions which satisfy:*

(1) *The family of $\mathfrak{F}$ is denoted by $\bar{\mathfrak{F}}$ where $\mathfrak{F}$ is nondecreasing and continuous satisfying $\mathfrak{F}(0) = 0 < \mathfrak{F}(s^{'})$; $s^{'} \in \mathbb{R}^+$.*

(2) *The family of all α is denoted by $\bar{\alpha}$ where α is continuous , $\alpha(0) = 0$ and is bounded by $s^{'}$ $(\alpha(s^{'}) < s^{'})$; $s^{'} \in \mathbb{R}^+$.*

(3) *The family of all ψ is denoted by $\bar{\psi}$ where ψ is a nondecreasing continuous mapping .*

Definition 2.6. [20] *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semicontinuous operator . Then , the following conditions are equivalent*
(1) $\lim_{n\to\infty} g^n(s^{'}) = 0$ *for every* $s^{'} > 0$.
(2) $g(s^{'}) < s^{'}$ *for every* $s^{'} > 0$.

Definition 2.7. [42] *Let Λ be the family of all operators $\lambda : \mathbb{R}^+ \to (1, \infty)$ such that:*

(λ_1) *λ is increasing and continuous ;*
(λ_2) $\lim\limits_{s\to\infty} t_s = 0$ *iff* $\lim\limits_{s\to\infty} \lambda(t_s) = 1\ \forall\{t_s\} \subseteq (0, \infty)$.

Definition 2.8. *Φ denotes the family of all operators $\phi : [1, \infty) \to [1, \infty)$ so that:*

(ϕ_1) *ϕ is non-decreasing and continuous ;*
(ϕ_2) $\lim\limits_{n\to\infty} \phi^n(s^{'}) = 1$ *for all* $s^{'} \in [1, \infty)$,

Definition 2.9. *$\bar{\gamma}$ denotes the family of all nondecreasing operators $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that* $\lim_{t\to\infty} \gamma^t\left(s^{'}\right) = 0$ *for every* $s^{'} \geq 0$.

For example , $\gamma(s^{'}) = \aleph$, $\aleph \in (0, 1)$, $s^{'} \in \mathbb{R}^+$.

3. New Generalized Darbo's Fixed point theorems

In the article [31], Deb et al. discussed about the following types of new fixed point theorems with the help of measure of noncompactness :

Theorem 3.1. [31] *Let $H : V \to V$ be a continuous mapping where V is a nonempty, bounded, closed and convex subset of G such that*

$$\hat{\mathcal{E}}(\hat{\Psi}(W(H^m Z))) \leq \hat{\mathfrak{F}}\big[\hat{\alpha}(\mathcal{M}_{m-1}(Z)), \hat{\beta}(\mathcal{M}_{m-1}(Z)), \hat{\Phi}(\mathcal{M}_{m-1}(Z)), \hat{\gamma}((\mathcal{M}_{m-1}(Z))\big], \tag{3.1}$$

where

$$\mathcal{M}_{m-1}(Z) = \max\{W(Z), W(HZ), \cdots, W(H^{m-1}Z)\},$$

for each $\emptyset \neq Z \subseteq V$, where W is an arbitrary $\mathcal{MNC}$, $(\hat{\Psi}, \hat{\Phi}) \in \Upsilon$, $\hat{\mathfrak{F}} \in \bar{\mathcal{Z}}$, $\hat{\mathcal{E}} \in \mathcal{A}$, $\hat{\alpha} \in \mathcal{A}'$, $\hat{\beta} \in \bar{\mathcal{A}}$ and $\hat{\gamma} : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for H in V.

Motivated by the above mentioned work, we have established the following types of new fixed point theorems.

Theorem 3.2. *Suppose V be a NBCCS of a Banach space G and $H : V \to V$ be a continuous mapping with*

$$\mathfrak{F}[W(H^p Z) + \psi(W(H^p Z))] \leq \alpha[\mathfrak{F}\{(O_{p-1}Z) + \psi(O_{p-1}Z)\}] \tag{3.2}$$

where

$$O_{p-1}(Z) = \Delta\left(W(Z), W(HZ), ..., W(H^{p-1}Z)\right)$$

for each $Z \subset V$, where W is an arbitrary $\mathcal{MNC}$ and $\mathfrak{F} \in \bar{\mathfrak{F}}, \alpha \in \bar{\alpha}$, $\psi \in \bar{\psi}$ and $\Delta \in \bar{\Delta}$. Then H has at least one fixed point in V.

Proof. Take $Z_0 = V$, $Z_{q+p} = \overline{conv(H^p Z_q)}$, for $q = 0, 1, 2, \ldots$.

Evidently, $\{Z_q\}_{q\in\mathbb{N}}$ is a NBCCS such that

$$Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_q \supseteq \cdots \supseteq Z_{q+p}.$$

If $N \in \mathbb{N}$ be an integer such that $W(Z_N) = 0$, then Z_N is relatively compact and by Schauder Theorem we can say that H has a fixed point.

So, we can take $W(Z_q) > 0$ for all $q \in \mathbb{N} \cup \{0\}$.

Since W is monotone, hence the sequence $W(Z_q)$ is nonnegative and nonincreasing, and we deduce that $W(Z_q) + \psi(W(Z_q)) \to a$ when $n \to \infty$, where $a \geq 0$ is a real number.

Now we cliam that $a = 0$. For this purpose from the equation (3.2) , we have

$$\begin{aligned}
\mathfrak{F}[W(Z_{q+p}) + \psi(W(Z_{q+p}))] &= \mathfrak{F}[W(\overline{H^p Z_q}) + \psi(W(\overline{H^p Z_q}))] \\
&= \mathfrak{F}[W(H^p Z_q) + \psi(W(H^p Z_q))] \\
&\leq \alpha[\mathfrak{F}\{(O_{p-1}Z_q) + \psi(O_{p-1}Z_q)\}] \\
&< \mathfrak{F}\{(O_{p-1}Z_q) + \psi(O_{p-1}Z_q)\} \\
&< \mathfrak{F}\{W(Z_q) + \psi(W(Z_q)\}
\end{aligned} \tag{3.3}$$

for $q = 0, 1, 2, \ldots$, where

$$\begin{aligned}
O_{p-1}(Z_q) &= \Delta\{W(Z_q), W(Z_{q+1}), \cdots, W(Z_{q+p-1})\} \\
&\leq max\{W(Z_q), W(Z_{q+1}), \cdots, W(Z_{q+p-1})\} \leq W(Z_q).
\end{aligned} \tag{3.4}$$

Now, considering the equation (3.3), we get

$$\begin{aligned}
&\lim_{q\to\infty} \mathfrak{F}[W(Z_{q+p}) + \psi(W(Z_{q+p}))] < \lim_{q\to\infty} \mathfrak{F}\{W(Z_q)) + \psi(W(Z_q))\} \\
&\Rightarrow \mathfrak{F}(a) < \mathfrak{F}(a) .
\end{aligned}$$

Which is a contradiction .

Thus $a = 0$. Hence , $lim_{q\to\infty} W(Z_q) + \psi(W(Z_q)) = 0$ implies $lim_{q\to\infty} W(Z_q) = 0$. Therefore we infer $W(Z_q) \to 0$ as $q \to \infty$. Therefore, by Definition 2.1 (vii), $Z_\infty = \bigcap_{q=0}^{\infty} Z_q$ is nonempty, convex and closed . Also, the set Z_∞ under the operator H is invariant and $Z_\infty \in kerW$. Thus by Schauder's theorem (Theorem 2.2), H has at least one fixed point in V . ∎

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Corollary 3.3. *Suppose V be a NBCCS of G and $H : V \to V$ be a continuous mapping with*

$$\mathfrak{F}[W(H^pZ) + \psi(W(H^pZ))] \leq \alpha[\mathfrak{F}\{(O_{p-1}Z) + \psi(O_{p-1}Z)\}] \tag{3.5}$$

where

$$O_{p-1}(Z) = max\{W(Z), W(HZ), ..., W(H^{p-1}Z)\}$$

for each $Z \subset V$ and W be an arbitrary $\mathcal{M}N\mathsf{C}$ and $\mathfrak{F} \in \bar{\mathfrak{F}}, \alpha \in \bar{\alpha}$ and $\psi \in \bar{\psi}$. Then there exists at least one fixed point for H in V.

Proof. Putting $\Delta(t_1, t_2, ..., t_p) = max\{t_1, t_2, ..., t_p\}$ in the equation (3.2) of the Theorem (3.2) , we can get the above result . □

Corollary 3.4. *Suppose V be a NBCCS of G and $H : V \to V$ be a continuous mapping with*

$$\mathfrak{F}[W(H^pZ) + \psi(W(H^pZ))] \leq \alpha[\mathfrak{F}\{(O_{p-1}Z) + \psi(O_{p-1}Z)\}] \tag{3.6}$$

where

$$O_{p-1}(Z) = \frac{1}{p}\{W(Z) + W(HZ) + ... + W(H^{p-1}Z)\}$$

for each $Z \subset V$ and W be an arbitrary $\mathcal{M}N\mathsf{C}$ and $\mathfrak{F} \in \bar{\mathfrak{F}}, \alpha \in \bar{\alpha}$ and $\psi \in \bar{\psi}$. Then there exists at least one fixed point for H in V.

Proof. Putting $\Delta(t_1, t_2, ..., t_p) = \frac{1}{p}\{t_1 + t_2 + ... + t_p\}$ in the equation (3.2) of the Theorem (3.2) , we can get the above result . □

Corollary 3.5. *Suppose V be a NBCCS of G and $H : V \to V$ be a continuous mapping with*

$$W(H^pZ) \leq \alpha(O_{p-1}Z) \tag{3.7}$$

where

$$O_{p-1}(Z) = \Delta(W(Z), W(HZ), ..., W(H^{p-1}Z))$$

for each $Z \subset V$ and W be an arbitrary $\mathcal{M}N\mathsf{C}$,$\alpha \in \bar{\alpha}$ and $\Delta \in \bar{\Delta}$. Then there exists at least one fixed point for H in V.

Proof. Putting $\mathfrak{F}(t) = t : \psi(t) = 0$ in the equation (3.2) of the Theorem (3.2) , we can get the above result . □

Corollary 3.6. *Suppose V be a NBCCS of G and $H : V \to V$ be a continuous mapping with*

$$W(HZ) \leq \hat{\mathcal{M}}\ W(Z) \quad , \qquad \hat{\mathcal{M}} \in [0, 1) \tag{3.8}$$

for each $Z \subset V$ and W be an arbitrary $\mathcal{M}N\mathsf{C}$. Then there exists at least one fixed point for H in V.

Proof. Putting $p = 1$; $\alpha(t) = \hat{\mathcal{M}}\ t$, $\hat{\mathcal{M}} \in [0, 1)$ in the equation (3.7) of the Corollary (3.5) , we can get the above result which is known as Darbo's $F\mathcal{P}T$. □

Theorem 3.7. *Suppose V be a NBCCS of G and $H : V \to V$ a continuous mapping such that*

$$\delta(W(H^pZ)) \leq \delta(O_{p-1}(Z)) - \mu(O_{p-1}(Z)), \tag{3.9}$$

where

$$O_{p-1}(Z) = \Delta(W(Z), W(HZ), \cdots, W(H^{p-1}Z)),$$

for each $\emptyset \neq Z \subseteq V$, where W be an arbitrary $\mathcal{M}N\mathsf{C}$, $\Delta \in \bar{\Delta}$ and functions $\delta, \mu : \mathbb{R}^+ \to \mathbb{R}^+$, such that δ is increasing and continuous and μ is decreasing and lower semicontinuous on $\mathbb{R}^+$. Also, $\mu(0) = 0$ and $\mu(t') > 0$ for $t' > 0$. Then there exists at least one fixed point for H in V.

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Proof. Take $Z_0 = V$, $Z_{q+p} = \overline{conv(H^p Z_q)}$, for $q = 0, 1, 2, \ldots$.

Evidently, $\{Z_q\}_{q\in\mathbb{N}}$ is a NBCCS such that

$$Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_q \supseteq \cdots \supseteq Z_{q+p}.$$

If for an integer $q_0 \in \mathbb{N}$ one has $W(Z_{q_0}) = 0$, then Z_{q_0} is relatively compact and by Schauder Theorem we can say that H has a fixed point .

So, we can take $W(Z_q) > 0$ for all $q \in \mathbb{N} \cup \{0\}$.

Since W is monotone, hence the sequence $W(Z_q)$ is nonnegative and nonincreasing, and we deduce that $W(Z_q) \to a_1$ when $q \to \infty$, where $a_1 \geq 0$ is a real number.

Now we cliam that $a_1 = 0$. For this purpose from 3.9 we have

$$\begin{aligned}\delta(W(Z_{q+p})) &= \delta(W(\overline{conv(H^p Z_q)})) \\ &= \delta(W(H^p Z_q)) \\ &\leq \delta(O_{p-1}(Z_q)) - \mu(O_{p-1}(Z_n)) \\ &\leq \delta(W(Z_q)) - \mu(W(Z_q)))\end{aligned} \tag{3.10}$$

for $q = 0, 1, 2, \ldots$, where

$$\begin{aligned}O_{p-1}(Z_q) &= \Delta(W(Z_q), W(Z_{q+1}), \cdots, W(Z_{q+p-1})\} \\ &\leq max\{W(Z_q), W(Z_{q+1}), \cdots, W(Z_{q+p-1}\} \\ &\leq W(Z_q).\end{aligned}$$

Then from (3.10), we get

$$\lim_{q\to\infty} \delta(W(Z_{q+p})) \leq \lim_{q\to\infty} \delta[W(Z_q)] - \lim_{q\to\infty} \mu[(W(Z_q)].$$

This yields $\delta(a_1) \leq \delta(a_1) - \mu(a_1)$. Consequently $\mu(a_1) = 0$ so $a_1 = 0$. Therefore we infer $W(Z_q) \to 0$ as $q \to \infty$. Therefore, by Definition 2.1 (vii), $Z_\infty = \bigcap_{q=0}^{\infty} Z_q$ is nonempty, convex and closed. Also, the set Z_∞ under the operator H is invariant and $Z_\infty \in kerW$. Thus , the proof is complete by using Theorem 2.2. □

Theorem 3.8. *Suppose V be a NBCCS of G and the mapping $H : V \to V$ be continuous so that satisfies in the following condition*

$$W(H^p Z) \leq \gamma(O_{p-1}(Z))$$

where

$$O_{p-1}(Z) = \Delta(W(Z), W(HZ), \cdots, W(H^{p-1}Z)),$$

for each $\emptyset \neq Z \subseteq V$, where W is an arbitrary $\mathcal{M}N\mathsf{C}$, $\Delta \in \bar{\Delta}$ and $\gamma \in \bar{\gamma}$. Then there exists at least one fixed point for H in V.

Proof. Just like the proof of the previous theorem, we consider the sequences $\{Z_q\}$ by induction, where $Z_0 = V$, $Z_{q+p} = \overline{conv(H^p Z_q)}$, for $q = 0, 1, \ldots$. Also, we can take $W(Z_q) > 0$ for all $q = 0, 1, \ldots$. In addition, according to our assumptions, for $m = 0, 1, \ldots, p-1$ and each $r \in \mathbb{N}$ one has

$$\begin{aligned}W(Z_{m+rp}) = W\left(Z_{m+(r-1)p+p}\right) &= W\left(\overline{conv\left(H^p\left(Z_{m+(r-1)p}\right)\right)}\right) \\ &= W(H^p\left(Z_{m+(r-1)p}\right)) \\ &\leq \gamma\left(O_{p-1}\left(Z_{m+(r-1)p}\right)\right),\end{aligned}$$

where

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$$
\begin{aligned}
O_{p-1}\left(Z_{m+(r-1)p}\right) &= \Delta\left(W\left(Z_{m+(r-1)p}\right), W\left(Z_{m+(r-1)p+1}\right), \cdots, W\left(Z_{m+rp-1}\right)\right) \\
&\leq max\{W\left(Z_{m+(r-1)p}\right), W\left(Z_{m+(r-1)p+1}\right), \cdots, W\left(Z_{m+rp-1}\right)\} \\
&\leq W\left(Z_{m+(r-1)p}\right).
\end{aligned}
$$

Hence, by using the mathematical induction method, we obtain

$$
\begin{aligned}
W(Z_{m+rp}) &\leq \gamma(W(Z_{m+(r-1)p})) \\
&\leq \gamma^2(W(Z_{m+(r-2)p})) \\
&\vdots \\
&\leq \gamma^r(W(Z_m)).
\end{aligned}
$$

Now, from the fact that $\gamma^r(W(Z_m)) \to 0$, as $r \to \infty$, we conclude that $W(Z_{m+rp}) \to 0$ as $r \to \infty$. On the other hand, for each $q \in \mathbb{N}$, by the division algorithm, we can write $n = km+p$, where $p = 0, 1, \ldots, m-1$. This shows that $W(Z_q) \to 0$ as $q \to \infty$. By Definition 2.1 (vii), $Z_\infty = \bigcap_{q=0}^{\infty} Z_q$ is a nonempty, convex and closed subset of Z. Also, the set Z_∞ under the operator H is invariant and $Z_\infty \in kerW$. Thus , the proof is complete by using Theorem 2.2. □

Theorem 3.9. *Suppose V be a NBBCS of G and $H : V \to V$ a continuous mapping such that*

$$
\lambda(\varphi(W(H^p Z))) \leq \frac{\lambda(\varphi(O_{p-1}(Z)))}{\lambda(\varphi(\sigma(O_{p-1}(Z))))} \tag{3.11}
$$

where

$$
O_{p-1}(Z) = \Delta(W(Z), W(HZ), \cdots, W(H^{p-1}Z)\},
$$

for each $\emptyset \neq Z \subseteq V$, where W be an arbitrary $\mathcal{MNC}$,$\lambda \in \Lambda$, $\Delta \in \bar{\Delta}$ and functions $\varphi, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$, such that φ is increasing and continuous and σ is decreasing and lower semicontinuous on $\mathbb{R}^+$. Also, $\sigma(0) = 0$ and $\sigma(t) > 0$ for $t > 0$. Then there exists at least one fixed point for H in V.

Proof. Just like the proof of Theorem 3.7, we define sequence $\{Z_q\}$ by induction. Moreover, from (3.11) we have

$$
\begin{aligned}
\lambda(\varphi(W(Z_{q+p}))) &= \lambda(\varphi(W(\overline{conv(H^q Z_q)}))) \\
&= \lambda(\varphi(W(H^q Z_q))) \\
&\leq \frac{\lambda(\varphi(O_{p-1}(Z_q)))}{\lambda(\varphi(\sigma(O_{p-1}(Z_q))))}
\end{aligned} \tag{3.12}
$$

where

$$
O_{p-1}(Z_q) = \Delta(W(Z_q), W(HZ_q), \cdots, W(H^{p-1}Z_q)),
$$

for $q = 0, 1, 2, \ldots$. Since the sequence $\{W(Z_q)\}$ is nonnegative and nonincreasing, we deduce that $W(Z_q) \to a_2$ when $q \to \infty$, where $a_2 \geq 0$ is a real number. On the other hand, considering

Equation (3.12), we get

$$\begin{aligned}\lambda(\varphi(a_2)) &= \lim_{q\to\infty}\lambda(\varphi(W(Z_{q+p})))\\ &\le \lim_{q\to\infty}\frac{\lambda(\varphi(O_{p-1}(Z_q)))}{\lambda(\varphi(\sigma(O_{p-1}(Z_q))))}\\ &\le \frac{\lambda(\varphi(a))}{\lambda(\varphi(\lim_{q\to\infty}\sigma(O_{p-1}(Z_q))))}\\ &\le \frac{\lambda(\varphi(a))}{\lambda(\varphi(\lim_{q\to\infty}\sigma(O_{p-1}(Z_q))))}\\ &\le \frac{\lambda(\varphi(a))}{\lambda(\varphi(\sigma(\lim_{q\to\infty}O_{p-1}(Z_q))))},\end{aligned}$$

where

$$\begin{aligned}O_{p-1}(Z_q) &= \Delta(W(Z_q),W(Z_{q+1}),\cdots,W(Z_{q+p-1}))\\ &\le max\{W(Z_q),W(Z_{q+1}),\cdots,W(Z_{q+p-1})\}\\ &\le W(Z_q)\to a_2 \quad (as\ q\to\infty).\end{aligned}$$

This yields $\lambda(\varphi(a_2)) \le \dfrac{\lambda(\varphi(a_2)}{\lambda(\varphi(\sigma(a_2)))}$. Consequently $\lambda(\varphi(\sigma(a_2))) = 1$ then $\varphi(\sigma(a_2)) = 0$ and $\sigma(a_2) = 0$ so $a_2 = 0$. Therefore we infer $W(Z_q) \to 0$ as $q \to \infty$. Now, considering that $Z_{q+1} \subset Z_q$, therefore, by Definition 2.1 (vii), $Z_\infty = \bigcap_{q=0}^{\infty} Z_q$ is nonempty , convex and closed . Also , the set Z_∞ under the operator H is invariant and $Z_\infty \in kerW$. Thus , the proof is complete by using Theorem 2.2 . □

4. Measure of noncompactness on $\mathbb{C}([0, L])$:

Assume that the space $G = \mathbb{C}(T)$ is the collection of all real valued continuous operators on $T = [0, L]$. Then , G is a Banach space with the norm

$$\| \mathcal{E} \| = \sup\{|\mathcal{E}(s)| : s \in T\},\ \mathcal{E} \in G.$$

Let $\Omega(\neq \emptyset) \subseteq G$ be bounded. For $\mathcal{E} \in \mathbb{C}(T)$ with $\varpi > 0$, the modulus of the continuity of $\mathcal{E}$ is denote by $\beta(\mathcal{E}, \varpi)$ i.e.,

$$\beta(\mathcal{E},\varpi) = \sup\{|\mathcal{E}(s_1) - \mathcal{E}(s_2)| : s_1, s_2 \in T, |s_2 - s_1| \le \varpi\}.$$

In addition, we define

$$\beta(\Omega,\varpi) = \sup\{\beta(\mathcal{E},\varpi) : \mathcal{E} \in \Omega\};\ \beta_0(\Omega) = \lim_{\varpi\to 0}\beta(\Omega,\varpi).$$

where , the function β_0 is known as a $\mathcal{M}N\mathsf{C}$ in G and the Hausdorff $\mathcal{M}N\mathsf{C}$ $\pounds$ is define as $\pounds(\Omega) = \frac{1}{2}\beta_0(\Omega)$ (see [5]).

5. Solvability of integro-differential equation

In this portion , first we consider the following notations and definitions (see [22, 26]) :

Definition 5.1. *The Riemann-Liouville (RL) fractional derivative of order $\xi > 0$ of continuous fuction $\mathfrak{H} : \mathbb{R}^+ \to \mathbb{R}$ is defined as*

$$\mathfrak{D}^\xi\ \mathfrak{H}(s) = \frac{1}{\Gamma(n-\xi)}\left(\frac{d}{ds}\right)^n \int_0^s (s-t)^{n-\xi-1}\mathfrak{H}(t)\ dt\ , \qquad n-1 < \xi < n, \tag{5.1}$$

where , $n = [\xi] + 1$,$[\xi]$ is the integral part of a real number ξ , whenever the right-hand side is point-wise defined on $\mathbb{R}^+$, where the gamma function Γ is defined as $\Gamma(\xi) = \int_0^\infty e^{-s}s^{\xi-1}ds$.

Definition 5.2. *The RL fractional integral of order $\zeta > 0$ of continuous fuction $\mathfrak{H} : \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$$\mathcal{I}^{\zeta}\ \mathfrak{H}(s) = \frac{1}{\Gamma(\zeta)} \int_0^s (s-t)^{\zeta-1} \mathfrak{H}(t)\ dt\ , \tag{5.2}$$

whenever the right-hand side is point-wise defined on $\mathbb{R}^+$.

Lemma 5.3. [22] *Let $\xi > 0$. Then for $\mathcal{U} \in \mathbb{C}[0, J] \cap \mathbb{L}[0, J]$ we have*

$$\mathcal{I}^{\xi}\mathfrak{D}^{\xi}\ \mathcal{U}(\eta) = \mathcal{U}(\eta) - \sum_{i=1}^{n} \frac{(\mathcal{I}^{n-\xi}\ \mathcal{U})^{(n-i)}(0)}{\Gamma(\xi - i + 1)}\ \eta^{\xi - i}\ , \tag{5.3}$$

where $n - 1 < \xi < n$.

Lemma 5.4. [40] *Suppose that $0 < \xi \le 1$ and functions $\mathcal{Q}, \mathcal{G}, \mathcal{K}$ satisfy the equation (1.1) . Then the unique solution of the fractional integro-differential equation (1.1) is given by*

$$\begin{aligned} \mathcal{U}(\eta)\ &= \frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)} \int_0^{\eta} (\eta - \mathfrak{z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z} \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}\ ,\ \eta \in T. \end{aligned} \tag{5.4}$$

Proof. Applying the Riemann-Liouville fractional integral of order ξ to both sides of (1.1) and using Lemma 5.3 , we have

$$\begin{aligned} &\left[\frac{\mathcal{U}(\eta) - \mathcal{I}^{\zeta}\mathcal{K}(\eta,\mathcal{U}(\eta))}{\mathcal{G}(\eta,\mathcal{U}(\eta))}\right] - \frac{\eta^{\xi-1}}{\Gamma(\xi)}\ \mathcal{I}^{1-\xi}\ \left[\frac{\mathcal{U}(\eta) - \mathcal{I}^{\zeta}\mathcal{K}(\eta,\mathcal{U}(\eta))}{\mathcal{G}(\eta,\mathcal{U}(\eta))}\right]_{\eta=0} \\ &= \mathcal{I}^{\xi}\ \mathcal{Q}(\eta,\mathcal{U}(\eta)). \end{aligned}$$

Since , $\mathcal{U}(0) = 0, \mathcal{K}(0, 0) = 0$ and $\mathcal{G}(0, 0) \neq 0$, it follows that

$$\begin{aligned} \mathcal{U}(\eta)\ &= \frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)} \int_0^{\eta} (\eta - \mathfrak{z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z} \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}\ ,\ \eta \in T. \end{aligned} \tag{5.5}$$

Thus (5.4) holds. The proof is completed. □

Now, we discuss how our results can be applied to find the solution of an integro-differential equation (IDE) in the $\mathbb{C}([0, L])$ space.

Take the following IDE :

$$\begin{cases} \mathfrak{D}^{\xi} \left[\dfrac{\mathcal{U}(\eta) - \mathcal{I}^{\zeta}\ \mathfrak{Y}(\eta,\mathcal{U}(\eta))}{\mathcal{G}(\eta,\mathcal{U}(\eta))}\right] = \mathcal{Q}(\eta,\mathcal{U}(\eta))\ ,\ \eta \in T = [0, L] \\ \mathcal{U}(0) = 0\ , \end{cases} \tag{5.6}$$

where $\mathfrak{D}^{\xi}$ is the Riemann-Liouville (RL) fractional derivative of order ξ, $0 \le \xi \le 1$, $\mathcal{I}^{\zeta}$ is the Riemann-Liouville (RL) fractional integral of order ζ, $\zeta > 0$. Also from the Lemma 5.4, we can say that the above integro-differential equation (5.6) is equivalent to the following ($\mathcal{F}IE$)

$$\begin{aligned} \mathcal{U}(\eta)\ &= \frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)} \int_0^{\eta} (\eta - \mathfrak{z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z} \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \mathfrak{z})^{\zeta-1}\ \mathfrak{Y}(\ \mathfrak{z}, \mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}. \end{aligned} \tag{5.7}$$

Let

$$\mathfrak{X}_{b_0} = \{\mathcal{U} \in G \ : \| \mathcal{U} \| \leq b_0\}.$$

Assume that

(I) $\mathcal{G} : T \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ be continuous and $\exists$ constant $\mathcal{G}_1 \geq 0$ satisfying

$$|\mathcal{G}(\eta,\mathcal{U}(\eta)) - \mathcal{G}(\eta,\mathcal{U}'(\eta))| \leq \mathcal{G}_1 |\mathcal{U}(\eta) - \mathcal{U}'(\eta)|,$$

for $\eta \in T$ and $\mathcal{U},\mathcal{U}' \in \mathbb{R}$.
Also,

$$\hat{\mathcal{G}} = \sup_{\eta \in T} \mathcal{G}(\eta, 0).$$

(II) $\mathcal{Q} : T \times \mathbb{R} \to \mathbb{R}$ be continuous and $\exists$ constant $\mathcal{Q}_1 \geq 0$ satisfying

$$|\mathcal{Q}(\eta,\mathcal{U}(\eta)) - \mathcal{Q}(\eta,\mathcal{U}'(\eta))| \leq \mathcal{Q}_1 |\mathcal{U}(\eta) - \mathcal{U}'(\eta)|,$$

for $\eta \in T$ and $\mathcal{U},\mathcal{U}' \in \mathbb{R}$.
Also,

$$\mathcal{Q}(\eta, 0) = 0.$$

(III) $\mathcal{K} : T \times \mathbb{R} \to \mathbb{R}$ be continuous and $\exists$ constant $\mathcal{K}_1 \geq 0$ satisfying

$$|\mathcal{K}(\eta,\mathcal{U}(\eta)) - \mathcal{K}(\eta,\mathcal{U}'(\eta))| \leq \mathcal{K}_1 |\mathcal{U}(\eta) - \mathcal{U}'(\eta)|,$$

for $\eta \in T$ and $\mathcal{U},\mathcal{U}' \in \mathbb{R}$.
Also,

$$\mathcal{K}(\eta, 0) = 0.$$

(IV) $\exists$ a positive number b_0 such that

$$\frac{(\mathcal{G}_1 \ b_0 + \hat{\mathcal{G}}) \ \mathcal{Q}_1}{\Gamma(\xi + 1)} \ L^{\xi} + \frac{\mathcal{K}_1}{\Gamma(\zeta + 1)} \ L^{\zeta} \leq 1. \tag{5.8}$$

Theorem 5.5. *There exists a solution of equation (5.7) in G whenever conditions (I)-(IV) are satisfied .*

Proof. We take the operator $\mathcal{F} : G \to G$ which is defined as given below :

$$(\mathcal{F}\mathcal{U})(\eta) = \frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)} \int_0^{\eta} (\eta - \mathfrak{z})^{\xi-1} \ \mathcal{Q}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z}) \ ) \ d\mathfrak{z} + \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \mathfrak{z})^{\zeta-1} \ \mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z}) \ ) \ d\mathfrak{z}.$$

Step (1): Here, we will prove that the operator $\mathcal{F}$ maps from $\mathfrak{X}_{b_0}$ into $\mathfrak{X}_{b_0}$. Let $\mathcal{U} \in \mathfrak{X}_{b_0}$.

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Now ,we have ,

$$
\begin{aligned}
&|(\mathcal{F}\mathcal{U})(\eta))| \\
&\leq \left|\frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}\right| + \left|\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}\right| \\
&\leq \frac{(|\mathcal{G}(\eta,\mathcal{U}(\eta))-\mathcal{G}(\eta,0)|+|\mathcal{G}(\eta,0)|)}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{z})^{\xi-1}\ (|\mathcal{Q}(\mathfrak{z},\mathcal{U}(\mathfrak{z}))-\mathcal{Q}(\mathfrak{z},0)|+|\mathcal{Q}(\mathfrak{z},0)|)\ d\mathfrak{z} \\
&+\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{z})^{\zeta-1}\ (|\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z}))-\mathcal{K}(\ \mathfrak{z},0)|+|\mathcal{K}(\ \mathfrak{z},0)|)\ d\mathfrak{z} \\
&\leq \frac{\mathcal{G}_1(\mathcal{U})}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{z})^{\xi-1}\ \mathcal{Q}_1(\mathcal{U})\ d\mathfrak{z}+\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{z})^{\zeta-1}\ \mathcal{K}_1(\mathcal{U})\ d\mathfrak{z} \\
&\leq \frac{(\mathcal{G}_1(\mathcal{U})+\hat{\mathcal{G}})\mathcal{Q}_1(\mathcal{U})}{\Gamma(\xi)\ \xi}\left[-(\eta-\mathfrak{z})^{\xi}\right]_0^\eta+\frac{\mathcal{K}_1(\mathcal{U})}{\Gamma(\zeta)\ \zeta}\left[-(\eta-\mathfrak{z})^{\zeta}\right]_0^\eta \\
&\leq \frac{(\mathcal{G}_1(\mathcal{U})+\hat{\mathcal{G}})\mathcal{Q}_1(\mathcal{U})}{\Gamma(\xi+1)}\ \eta^{\xi}\ +\ \frac{\mathcal{K}_1(\mathcal{U})}{\Gamma(\zeta+1)}\ \eta^{\zeta} \\
&\leq \frac{(\mathcal{G}_1(\mathcal{U})+\hat{\mathcal{G}})\mathcal{Q}_1(\mathcal{U})}{\Gamma(\xi+1)}\ L^{\xi}\ +\ \frac{\mathcal{K}_1(\mathcal{U})}{\Gamma(\zeta+1)}\ L^{\zeta}.
\end{aligned}
$$

Hence $\parallel \mathcal{U} \parallel \leq b_0$ gives

$$
\begin{aligned}
\parallel \mathcal{F}\mathcal{U} \parallel &\leq \frac{(\mathcal{G}_1\ b_0+\hat{\mathcal{G}})\ \mathcal{Q}_1\ b_0}{\Gamma(\xi+1)}\ L^{\xi}\ +\ \frac{\mathcal{K}_1\ b_0}{\Gamma(\zeta+1)}\ L^{\zeta}\leq b_0 \\
&\leq \frac{(\mathcal{G}_1\ b_0+\hat{\mathcal{G}})\ \mathcal{Q}_1}{\Gamma(\xi+1)}\ L^{\xi}+\frac{\mathcal{K}_1}{\Gamma(\zeta+1)}\ L^{\zeta}\leq 1.
\end{aligned}
$$

Due to the assumption (IV) , $\mathcal{F}$ maps $\mathfrak{X}_{b_0}$ to $\mathfrak{X}_{b_0}$.

Step (2): Now , we will show that $\mathcal{F}$ is continuous on $\mathfrak{X}_{b_0}$. Let $\varpi > 0$ and $\mathcal{U},\mathcal{U}_1 \in \mathfrak{X}_{b_0}$ such that $\parallel \mathcal{U}-\mathcal{U}_1 \parallel < \varpi$, for all $\eta \in [0, L]$.

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Now ,we have ,

$$|(\mathcal{F}\mathcal{U})(\eta) - (\mathcal{F}\mathcal{U}_1)(\eta)|$$

$$\leq \Big|\frac{\mathcal{G}(\eta,\mathcal{U}(\eta))}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )\ d\mathfrak{Z}$$

$$-\frac{\mathcal{G}(\eta,\mathcal{U}_1(\eta))}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{Z},\mathcal{U}_1(\mathfrak{Z})\ )\ d\mathfrak{Z}\Big|$$

$$+\ \Big|\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{Z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )\ d\mathfrak{Z}$$

$$-\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{Z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{Z},\mathcal{U}_1(\mathfrak{Z})\ )\ d\mathfrak{Z}\Big|$$

$$\leq \frac{|\mathcal{G}(\eta,\mathcal{U}(\eta))-\mathcal{G}(\eta,\mathcal{U}_1(\eta))|}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ \mathcal{Q}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )\ d\mathfrak{Z}$$

$$+\frac{\mathcal{G}(\eta,\mathcal{U}_1(\eta))}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ |\mathcal{Q}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )-\mathcal{Q}(\ \mathfrak{Z},\mathcal{U}_1(\mathfrak{Z})\ )|\ d\mathfrak{Z}$$

$$+\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{Z})^{\zeta-1}\ |\mathcal{K}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )-\mathcal{K}(\ \mathfrak{Z},\mathcal{U}_1(\mathfrak{Z})\ )|\ d\mathfrak{Z}$$

$$\leq \frac{\mathcal{G}_1|\mathcal{U}(\eta)-\mathcal{U}_1(\eta)|}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ (|\mathcal{Q}(\ \mathfrak{Z},\mathcal{U}(\mathfrak{Z})\ )-\mathcal{Q}(\ \mathfrak{Z},0\ )|+|\mathcal{Q}(\ \mathfrak{Z},0)|)\ d\mathfrak{Z}$$

$$+\frac{(|\mathcal{G}(\eta,\mathcal{U}_1(\eta))-\mathcal{G}(\eta,0)|+|\mathcal{G}(\eta,0)|)}{\Gamma(\xi)}\int_0^\eta (\eta-\mathfrak{Z})^{\xi-1}\ \mathcal{Q}_1|\mathcal{U}(\mathfrak{Z})\ )-\mathcal{U}_1(\mathfrak{Z})|\ d\mathfrak{Z}$$

$$+\frac{1}{\Gamma(\zeta)}\int_0^\eta (\eta-\mathfrak{Z})^{\zeta-1}\ \mathcal{K}_1|\mathcal{U}(\mathfrak{Z})\ )-\mathcal{U}_1(\mathfrak{Z})|\ d\mathfrak{Z}$$

$$\leq \frac{\mathcal{G}_1|\mathcal{U}(\eta)-\mathcal{U}_1(\eta)|\ \mathcal{Q}_1(\mathcal{U})}{\Gamma(\xi+1)}\ L^\xi+\frac{(\mathcal{G}_1(\mathcal{U}_1)+\hat{\mathcal{G}})\ \mathcal{Q}_1|\mathcal{U}(\mathfrak{Z})\ )-\mathcal{U}_1(\mathfrak{Z})|}{\Gamma(\xi+1)}\ L^\xi+\frac{\mathcal{K}_1|\mathcal{U}(\mathfrak{Z})\ )-\mathcal{U}_1(\mathfrak{Z})|}{\Gamma(\zeta+1)}\ L^\zeta.$$

Hence , $\|\mathcal{U}-\mathcal{U}_1\|<\varpi$ gives ,

$$|(\mathcal{F}\mathcal{U})(\eta)-(\mathcal{F}\mathcal{U}_1)(\eta)|<\frac{\mathcal{G}_1(\varpi)\ \mathcal{Q}_1(\mathcal{U})}{\Gamma(\xi+1)}\ L^\xi+\frac{(\mathcal{G}_1(\mathcal{U}_1)+\hat{\mathcal{G}})\ \mathcal{Q}_1(\varpi)}{\Gamma(\xi+1)}\ L^\xi+\frac{\mathcal{K}_1(\varpi)}{\Gamma(\zeta+1)}\ L^\zeta$$

i.e. As $\varpi\to 0$ we get $|(\mathcal{F}\mathcal{U})(\eta)-(\mathcal{F}\mathcal{U}_1)(\eta)|\to 0$.
Then, $\mathcal{F}$ is continuous on $\mathfrak{X}_{b_0}$.

Step (3): An estimate of $\mathcal{F}$ with respect to β_0. Taking $\mho(\neq\emptyset)\subseteq\mathfrak{X}_{b_0}$. Let $\varpi>0$ be arbitrary and choosing $\mathcal{U}\in\mho$ and $\eta_1,\eta_2\in[0,L]$ such as $|\eta_2-\eta_1|\leq\varpi$ with $\eta_2\geq\eta_1$.

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We have,

$$
\begin{aligned}
&|(\mathcal{F}\mathcal{U})(\eta_2) - (\mathcal{F}\mathcal{U})(\eta_1)| \\
&= \Big| \frac{\mathcal{G}(\eta_2, \mathcal{U}(\eta_2))}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} + \frac{1}{\Gamma(\zeta)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\zeta-1} \, \mathcal{K}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \\
&- \frac{\mathcal{G}(\eta_1, \mathcal{U}(\eta_1))}{\Gamma(\xi)} \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} - \frac{1}{\Gamma(\zeta)} \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\zeta-1} \, \mathcal{K}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \\
&\leq \Big| \frac{\mathcal{G}(\eta_2, \mathcal{U}(\eta_2))}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} - \frac{\mathcal{G}(\eta_1, \mathcal{U}(\eta_1))}{\Gamma(\xi)} \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \\
&+ \Big| \frac{1}{\Gamma(\zeta)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\zeta-1} \, \mathcal{K}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} - \frac{1}{\Gamma(\zeta)} \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\zeta-1} \, \mathcal{K}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \\
&\leq \mathcal{J}_1(\eta_2, \eta_1) + \mathcal{J}_2(\eta_2, \eta_1)
\end{aligned}
\tag{5.9}
$$

where ,

$$
\begin{aligned}
&\mathcal{J}_1(\eta_2, \eta_1) \\
&= \Big| \frac{\mathcal{G}(\eta_2, \mathcal{U}(\eta_2))}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \\
&- \frac{\mathcal{G}(\eta_1, \mathcal{U}(\eta_1))}{\Gamma(\xi)} \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \\
&\leq \frac{|\mathcal{G}(\eta_2, \mathcal{U}(\eta_2)) - \mathcal{G}(\eta_1, \mathcal{U}(\eta_1))|}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, |\mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,)| \, d\mathfrak{z} \\
&+ \frac{|\mathcal{G}(\eta_1, \mathcal{U}(\eta_1))|}{\Gamma(\xi)} \left(\Big| \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} - \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \right) \\
&\leq \frac{|\mathcal{G}(\eta_2, \mathcal{U}(\eta_2)) - \mathcal{G}(\eta_2, \mathcal{U}(\eta_1))| + |\mathcal{G}(\eta_2, \mathcal{U}(\eta_1)) - \mathcal{G}(\eta_1, \mathcal{U}(\eta_1))|}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, |\mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,)| \, d\mathfrak{z} \\
&+ \frac{|\mathcal{G}(\eta_1, \mathcal{U}(\eta_1)) - \mathcal{G}(\eta_1, 0)| + |\mathcal{G}(\eta_1, 0)|}{\Gamma(\xi)} \Big(\Big| \int_0^{\eta_1} (\eta_2 - \mathfrak{z})^{\xi-1} \, |\mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,)| \, d\mathfrak{z} \\
&+ \int_{\eta_1}^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} - \int_0^{\eta_1} (\eta_1 - \mathfrak{z})^{\xi-1} \, \mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) \, d\mathfrak{z} \Big| \Big) \\
&\leq \frac{\mathcal{G}_1 |\mathcal{U}(\eta_2) - \mathcal{U}(\eta_1)| + \beta_{\mathcal{G}}(b_0, \varpi)}{\Gamma(\xi)} \int_0^{\eta_2} (\eta_2 - \mathfrak{z})^{\xi-1} \, (|\mathcal{Q}(\, \mathfrak{z}, \mathcal{U}(\mathfrak{z}) \,) - \mathcal{Q}(\, \mathfrak{z}, 0)| + |\mathcal{Q}(\, \mathfrak{z}, 0)|) \, d\mathfrak{z}
\end{aligned}
$$

$$+\frac{\mathcal{G}_1\ ||\mathcal{U}(\eta_1)||+\hat{\mathcal{G}}}{\Gamma(\xi)}\ \Big(\int_{\eta_1}^{\eta_2}\ (\eta_2-\mathfrak{z})^{\xi-1}\ (|\mathcal{Q}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )-\mathcal{Q}(\ \mathfrak{z},0)|+|\mathcal{Q}(\ \mathfrak{z},0)|)\ d\mathfrak{z}$$

$$+\int_0^{\eta_1}\ \left[(\eta_2-\mathfrak{z})^{\xi-1}-(\eta_1-\mathfrak{z})^{\xi-1}\ \right](|\mathcal{Q}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )-\mathcal{Q}(\ \mathfrak{z},0)|+|\mathcal{Q}(\ \mathfrak{z},0)|)\ d\mathfrak{z}\Big)$$

$$\le\frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi)+\beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ \eta_2^{\xi}$$

$$+\frac{(\mathcal{G}_1\ ||\mathcal{U}||+\hat{\mathcal{G}})\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\Big((\eta_2-\eta_1)^{\xi}+(\eta_2^{\xi}-\eta_1^{\xi}-(\eta_2-\eta_1)^{\xi})\Big)$$

$$\le\frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi)+\beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ \eta_2^{\xi}+\frac{(\mathcal{G}_1\ ||\mathcal{U}||+\hat{\mathcal{G}})\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ \left(\eta_2^{\xi}-\eta_1^{\xi}\right)$$

where

$$\beta_{\mathcal{G}}(b_0,\varpi)=\sup\left\{|\mathcal{G}(\eta_2,\mathcal{U})-\mathcal{G}(\eta_1,\mathcal{U})|:|\eta_2-\eta_1|\le\varpi,\ \eta_1,\eta_2\in T,\ \|\ \mathcal{U}\ \|\le b_0\right\},$$

and ,

$$\beta(\mathcal{U},\varpi)=\sup\left\{|\mathcal{U}(\eta_2)-\mathcal{U}(\eta_1)|\le\varpi;|\eta_2-\eta_1|\le\varpi;\eta_1,\eta_2\in T\right\}$$

Again ,

$$\mathcal{J}_2(\eta_2,\eta_1)$$

$$=\Big|\ \frac{1}{\Gamma(\zeta)}\int_0^{\eta_2}\ (\eta_2-\mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}-\ \frac{1}{\Gamma(\zeta)}\int_0^{\eta_1}\ (\eta_1-\mathfrak{z})^{\zeta-1}\ \mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )\ d\mathfrak{z}\Big|$$

$$\le\ \frac{1}{\Gamma(\zeta)}\Big(\int_0^{\eta_1}\ (\eta_2-\mathfrak{z})^{\zeta-1}\ |\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )|\ d\mathfrak{z}+\int_{\eta_1}^{\eta_2}\ (\eta_2-\mathfrak{z})^{\zeta-1}\ |\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )|\ d\mathfrak{z}$$

$$-\int_0^{\eta_1}\ (\eta_1-\mathfrak{z})^{\zeta-1}\ |\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )|\ d\mathfrak{z}\Big)$$

$$\le\ \frac{1}{\Gamma(\zeta)}\Big(\int_{\eta_1}^{\eta_2}\ (\eta_2-\mathfrak{z})^{\zeta-1}\ (|\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )-\mathcal{K}(\ \mathfrak{z},0)|+|\mathcal{K}(\ \mathfrak{z},0)|)\ d\mathfrak{z}$$

$$+\int_0^{\eta_1}\ ((\eta_2-\mathfrak{z})^{\zeta-1}-(\eta_1-\mathfrak{z})^{\zeta-1})\ (|\mathcal{K}(\ \mathfrak{z},\mathcal{U}(\mathfrak{z})\ )-\mathcal{K}(\ \mathfrak{z},0)|+|\mathcal{K}(\ \mathfrak{z},0)|)\ d\mathfrak{z}\Big)$$

$$\le\ \frac{\mathcal{K}_1||\mathcal{U}||}{\Gamma(\zeta+1)}\Big((\eta_2-\eta_1)^{\zeta}+(\eta_2^{\zeta}-\eta_1^{\zeta}-(\eta_2-\eta_1)^{\zeta})\Big)$$

$$\le\ \frac{\mathcal{K}_1||\mathcal{U}||}{\Gamma(\zeta+1)}\left(\eta_2^{\zeta}-\eta_1^{\zeta}\right)$$

As $\varpi \to 0$, then $\eta_2 \to \eta_1$, so we get

$$\lim_{\varpi\to 0} \mathcal{J}_1(\eta_2, \eta_1) \to \frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi) + \beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ \eta_2^{\xi}$$

and ,

$$\lim_{\varpi\to 0} \mathcal{J}_2(\eta_2, \eta_1) \to 0$$

Hence, from the equation (5.9) , we get

$$|(\mathcal{F}\mathcal{U})(\eta_2) - (\mathcal{F}\mathcal{U})(\eta_1)| \leq \frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi) + \beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ \eta_2^{\xi}$$

i.e

$$\beta(\mathcal{F}\mathcal{U},\varpi) \leq \frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi) + \beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||\mathcal{U}||}{\Gamma(\xi+1)}\ L^{\xi} \qquad [since, \eta_2 \in T = [0,L]]$$

Since $||\mathcal{U}|| \leq b_0$, we get

$$\beta(\mathcal{F}\mathcal{U},\varpi) \leq \frac{(\mathcal{G}_1\ \beta(\mathcal{U},\varpi) + \beta_{\mathcal{G}}(b_0,\varpi))\ \mathcal{Q}_1||b_0||}{\Gamma(\xi+1)}\ L^{\xi}$$

By uniform continuty of $\mathcal{G}$ on $T \times [-b_0, b_0]$, we have $\lim_{\varpi\to 0} \beta_{\mathcal{G}}(b_0,\varpi) \to 0$ as $\varpi \to 0$. Taking $\sup_{\mathcal{U}\in\mho}$ and $\varpi \to 0$, we get

$$\beta_0(\mathcal{F}\ \mho) \leq \frac{\mathcal{G}_1\ \mathcal{Q}_1\ b_0\ L^{\xi}}{\Gamma(\xi+1)}\ \beta_0(\mho)$$

Thus

$$\beta_0(\mathcal{F}\ \mho) \leq \bar{\mathcal{M}}\ \beta_0(\mho)$$

Where , $\bar{\mathcal{M}} = \dfrac{\mathcal{G}_1\ \mathcal{Q}_1\ b_0\ L^{\xi}}{\Gamma(\xi+1)} < 1 \left(Since\ ,\ \dfrac{(\mathcal{G}_1\ b_0 + \hat{\mathcal{G}})\ \mathcal{Q}_1}{\Gamma(\xi+1)}\ L^{\xi} + \dfrac{\mathcal{K}_1}{\Gamma(\zeta+1)}\ L^{\zeta} \leq 1 \right)$

From Corollary 3.6 , $\mathcal{F}$ has a fixed point for in $\mho \subseteq \mathfrak{X}_{b_0}$
i,e the equation (5.7) has a solution in G.
Thus , we can say that the equation (5.6) has a solution in G.

□

Now , we take an example to illustrate the theorem 5.5.

Example 5.6. Taking the following IDE:

$$\begin{cases} \mathfrak{D}^{\frac{1}{5}} \left[\dfrac{\mathcal{U}(\eta) - \mathcal{I}^{\frac{1}{7}}\ \frac{\mathcal{U}(\eta)}{17+\eta}}{\dfrac{\mathcal{U}(\eta)+1}{19+\eta}} \right] = \dfrac{\mathcal{U}(\eta)}{21+\eta}\ , \\ \\ \mathcal{U}(0) = 0\ , \end{cases} \tag{5.10}$$

for $\eta \in [0,1] = T$,

which is a particular case of equation (5.6).
Here,

$$\xi = \frac{1}{5}\ ,\ \zeta = \frac{1}{7};$$

$$\mathcal{K}(\eta, \mathcal{U}(\eta)) = \frac{\mathcal{U}(\eta)}{17+\eta}\ ;$$

$$\mathcal{G}(\eta,\mathcal{U}(\eta)) = \frac{\mathcal{U}(\eta)+1}{19+\eta} \ ;$$

$$\mathcal{Q}(\eta,\mathcal{U}(\eta)) = \frac{\mathcal{U}(\eta)}{21+\eta} \ ;$$

and

$$L = 1.$$

Also, it is obvious that $\mathcal{K}, \mathcal{G}$ *and* $\mathcal{Q}$ are contineous satisfying

$$|\mathcal{K}(\eta,\mathcal{U}(\eta)) - \mathcal{K}(\eta,\mathcal{U}_1(\eta)| \leq \frac{|\mathcal{U}-\mathcal{U}_1|}{17} \ ;$$

$$|\mathcal{G}(\eta,\mathcal{U}(\eta)) - \mathcal{G}(\eta,\mathcal{U}_1(\eta)| \leq \frac{|\mathcal{U}-\mathcal{U}_1|}{19} \ ;$$

and

$$|\mathcal{Q}(\eta,\mathcal{U}(\eta)) - \mathcal{Q}(\eta,\mathcal{U}_1(\eta)| \leq \frac{|\mathcal{U}-\mathcal{U}_1|}{21} \ ;$$

Therefore , $\mathcal{K}_1 = \dfrac{1}{17}$, $\mathcal{G}_1 = \dfrac{1}{19}$, $\mathcal{Q}_1 = \dfrac{1}{21}$.
And

$$\hat{\mathcal{G}} = \sup_{\eta\in T} \ \mathcal{G}(\eta,0) = \frac{1}{19}$$

Now , putting these values , the inequality of assumption (IV) becomes ,

$$\frac{(\frac{b_0}{19}+\frac{1}{19})\times\frac{1}{21}}{\Gamma(\frac{1}{5}+1)} \ 1^{\frac{1}{5}} + \frac{\frac{1}{17}}{\Gamma(\frac{1}{7}+1)} \ 1^{\frac{1}{7}} \leq 1$$
$$\implies \ \frac{b_0+1}{19\times 21\times\Gamma(\frac{6}{5})} \leq \frac{17\ \Gamma(\frac{8}{7})-1}{17\ \Gamma(\frac{8}{7})}$$
$$\implies \ b_0 \leq \frac{(17\ \Gamma(\frac{8}{7})-1)\times 19\times 21\times\Gamma(\frac{6}{5})}{17\ \Gamma(\frac{8}{7})} - 1.$$

However, assumption (IV) is also satisfied for $b_0 = \dfrac{\left[(17\ \Gamma(\frac{8}{7})-1)\times 19\times 21\times\Gamma(\frac{6}{5})\right]-17\ \Gamma(\frac{8}{7})}{17\ \Gamma(\frac{8}{7})}$.

Thus , we have achieved all of the assumptions from (I) to (IV) in Theorem 5.5 .
From Theorem 5.5 , we can say that The equation (5.10) have solutions in $G = \mathbb{C}(T)$.

References

[1] R.P. Agarwal, D. O'Regan, Fixed point theory and applications, Cambridge University Press (2004).
[2] R. Arab, H.K. Nashine, N.H. Can, T.T. Binh , Solvability of functional-integral equations (fractional order) using measure of noncompactness, Advances in Difference Equations volume 2020, Article number: 12 (2020).
[3] J. Banas, B.C. Dhage, Global asymptotic stability of solutions of a functional integral equation, Nonlin. Anal. 69 (2008) 1945-1952.
[4] J. Banas, B. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, Appl. Math. Letters 16 (2003) 1-6.
[5] J. Banaś and K. Goebel, Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, 1980.

17 Sep 2023 21:45:42 PDT
230508-DasAnupam Version 3 - Submitted to Rocky Mountain J. Math.

[6] A. Das, B. Hazarika, P. Kumam, Some New Generalization of Darbo's Fixed Point Theorem and Its Application on Integral Equations, Mathematics 2019, 7(3), 214; https://doi.org/10.3390/math7030214.

[7] A. Das, B. Hazarika,R. Arab, M. Mursaleen, Applications of a fixed point theorem to the existence of solutions to the nonlinear functional integral equations in two variables, Rendiconti del Circolo Matematico di Palermo, 2019, 68(1), pp. 139-152.

[8] A. Das, B. Hazarika and M. Mursaleen, Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in $\ell_p(1 < p < \infty)$, RACSAM (2019) 113:31–40, https://doi.org/10.1007/s13398-017-0452-1

[9] A. Das, B. Hazarika, V. Parvaneh, M. Mursaleen, Solvability of generalized fractional order integral equations via measures of noncompactness, Mathematical Sciences, https://doi.org/10.1007/s40096-020-00359-0.

[10] B.C. Deuri , A. Das , Solvability of fractional integral equations via Darbo's fixed point theorem, j.Pseudo-Differ.Oper.Appl. (2022)13:26 , https://doi.org/10.1007/s11868-022-00458-7 .

[11] F. Jarad, T. Abdeljawad, K. Shah, On the weighted fractional operators of a function with respect to another function, Fractals, Vol. 28. No. 8(2020) 2040011, DOI:10.1142/S0218348X20400113 .

[12] K. Kuratowski, Sur les espaces complets, Fund. Math. 15(1930) 301-309.

[13] G. Darbo, Punti uniti in trasformazioni a codominio non compatto (Italian), Rend. Sem. Mat. Univ. Padova 24(1955)84-92.

[14] B. Hazarika, R. Arab, M. Mursaleen, Applications of Measure of Noncompactness and Operator Type Contraction for Existence of Solution of Functional Integral Equations, Complex Analysis and Operator Theory, vol. 13(2019), 3837–3851.

[15] H.K. Nashine, A. Das, Extension of Darbo's fixed point theorem via shifting distance functions and its application, Nonlinear Analysis: Modelling and Control, vol. 27(2022), 1–14.

[16] A. Das, I. Suwan, B.C. Deuri, T. Abdeljawad, On solution of generalized proportional fractional integral via a new fixed point theorem, Advances in Difference Equations (2021) 2021: 427.

[17] H.K. Nashine, R. Arab, R.P. Agarwal, A.S. Haghigh, Darbo type fixed and coupled fixed point results and its application to integral equation,Periodica Mathematica Hungarica volume 77, pages 94-107(2018).

[18] M. Rabbani,A. Das, B. Hazarika, R. Arab, Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations,Chaos, Solitons and Fractals 140 (2020) 110221, https://doi.org/10.1016/j.chaos.2020.110221.

[19] H.M. Srivastava, A. Das, B. Hazarika, S. A. Mohiuddine, Existence of Solution for Non-Linear Functional Integral Equations of Two Variables in Banach Algebra, Symmetry 2019, 11, 674; doi:10.3390/sym11050674.

[20] A. Aghajani, J. Banas', N. Sabzali, *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math. Soc. Simon Stevin, **20** (2013), no. 2, 15 pages, DOI:https://www.researchgate.net/publication/236151631.

[21] M. Jleli, E. Karapinar and B. Samet, *Further generalizations of the Banach contraction principle*, J. Inequal. Appl.,**2014** (2014), 439, 9 pages, DOI: http://www.journalofinequalitiesandapplications.com/content/2014/1/439.

[22] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo : Theory and applications of Fractional Differential Equations. North-Holland Mathematics Studies , vol. 204.Elsevier,Amsterdam(2006).

[23] K.S. Miller, B. Ross : An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993).

[24] V. Lakshmikantham, S. Leela, J. Vasundhara Devi : Theory and Dynamic Systems Cambridge Academic Publishers , Cambridge (2009).

[25] V. Lakshmikantham , A.S. Vatsala : Basic theory of fractional differential equations . Nonlinear Anal. 69(8), 2677-2682 (2008).

[26] I. Podlubny : Fractional Differential Equations . Academic Press , San Diego (1999).

[27] V. Parvaneh, S. Banaei, J.R. Roshan and M. Mursaleen, On tripled fixed point theorems via measure of noncompactness with applications to a system of fractional integral equations,Filomat,35(14),2021,4897-4915.

[28] S. Banaei, M. Mursaleen, V. Parvaneh, Some fixed point theorems via measure of noncompactness with applications to differential equations, Computational and Applied Mathematics(2020) 39:139. https://doi.org/10.1007/s40314-020-01164-0.

[29] B. Mohammadi, M. Paunović, V. Parvaneh, Existence of solution for some φ-Caputo fractional differential inclusions via Wardowski-Mizoguchi-Takahashi multi-valued contractions, Filomat,37:12(2023), 3777-3789, https://doi.org/10.2298/FIL2312777M.

[30] M. Paunovć, B. Mahammadi, V. Parvaneh, On weak wardowski contractions and solvability of p-caputo implicit fractional pantograph differential equation with generalized anti-periodic boundary conditions, Journal of nonlinear and convex analysis, 23(6),2022,1261-1274.

[31] S. Deb, H. Jafari, A. Das and V. Parvaneh, New fixed point theorems via measure of noncompactness and its application on fractional integral equation involving an operator with iterative relations, Journal of Inequalities and Applications (2023), 2023:106 , https://doi.org/10.1186/s13600-023-03003-2.

[32] L. Debnath, Recent applications of fractional calculus to science and engineering, Int.J.Math. Sci. 54, 3413-3442(2003).

[33] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore (2000).

[34] A. Shahram, D. Baleanu, R. Shhram : Analyzing transient response of the parallel RCL circuit by using the Caputo-Fabrizio fractional derivative, Adv.Differ.Equ.2020, 55(2020).

[35] A. Das, B. Hazarika, R. Arab, M. Mursaleen, Solvability of infinite system of integral equations in two variables i the sequence c_0 and ℓ_1, J. Comput. Appl. Math. 326(207) 183-192.

[36] R. Arab, R. Allahyari, A.S. Haghighi, Existence of solutions of infinite systems of integral equations in two variables via measure of noncompactness, Appl. Math. Comput. 246 (2014) 283-291.

[37] J. Banaś, M. Lecko, An existence theorem for a class of infinite systems of integral equations, Math. Comput. Model. 34(2001) 533-539.

[38] R. Rzepka, K. Sadarangani, On solutions of an infinite system of singular integral equations, Math. Comput. Model. 45(2007) 1265-1271.

[39] A. Aghajani, A.S. Haghighi, Existence of solutions for a system of integral equations via measure of noncompactness, Novi Sad J. Math. 44(1) (2004) 59-73.

[40] S. Sitho, S.K. Nstouyas , J. Tariboon , Existence results for hybrid fractional integro-differential equations, Boundary Value Problrms (2015) 2015:113 , DOI 10.1186/s13661-015-0376-7.

[41] A. Bragdi, A. Frioui , A.G. Lakoud , Existence of solutions for nonlinear fractional integro-differential equations, Advances in Difference Equations (2020) 2020:418, https://doi.org/10.1186/s13662-020-02874-9.

[42] V. Parvaneh, N. Hussain, A. Mukheimer, H. Aydi , 2019, On fixed point results for modified JS-contractions with applications, Axioms 8,no.3:84, https://doi.org/10.3390/axioms803008.

17 Sep 2023 21:45:42 PDT
230508-DasAnupam Version 3 - Submitted to Rocky Mountain J. Math.