OSCILLATION ANALYSIS OF SOLUTIONS OF FIRST ORDER SUBLINEAR AND SUPERLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we study the oscillation and nonoscillation behaviors of the solutions of first order sublinear and superlinear difference equations with general retarded argument of the form

$$\Delta x(n) + p(n)x^{\alpha}(\tau(n)) = 0, \quad n \in \mathbb{N},$$

where α is a quotient of odd positive integers, $(p(n))$ is a sequence of nonnegative real numbers, $(\tau(n))$ is a sequence of integers such that

$$\tau(n) \leq n \text{ for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty$$

and Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$. Examples illustrating the results are also given.

Keywords: Nonoscillatory solution, oscillatory solution, retarded argument, sublinear difference equation, superlinear difference equation.

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1. INTRODUCTION

Consider the first order nonlinear difference equation of the form

$$\Delta x(n) + p(n)x^{\alpha}(\tau(n)) = 0, \quad n \in \mathbb{N}, \tag{E}$$

where $\alpha \in (0, \infty)$ is a ratio of odd positive integers, $(p(n))$ is a sequence of nonnegative real numbers, $(\tau(n))$ is a sequence of integers such that

$$\tau(n) \leq n \text{ and } \lim_{n\to\infty} \tau(n) = \infty \tag{1.1}$$

and, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.

If $0 < \alpha < 1$, then (E) is called *sublinear equation* (see [17] and the references cited therein), while, if $\alpha > 1$, then (E) is called *superlinear equation* (see [17] and the references cited therein).

In case where $\alpha = 1$, (E) reduces to the linear retarded difference equation (see [1−13, 15, 18-19] and the references cited therein)

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}. \tag{1.2}$$

The problem of establishing sufficient conditions for the oscillation of all solutions of (1.2) has been the subject of many investigations. See [1-20] and the references cited therein.

In 1998, Zhang and Tian [20] proved that, if $(\tau(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} p(n) > 0 \text{ and } \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.3}$$

then all solutions of (1.2) oscillate.

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In 2008, Chatzarakis, Koplatadze and Stavroulakis [3, 4], when $(\tau(n))$ is not necessarily monotone, studied the equation (1.2) and proved that, if one of the following conditions

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}p(j)>1,\ \text{ where }\ h(n)=\max_{0\le s\le n}\tau(s),\ n\ge 0, \tag{1.4}$$

or

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)<\infty\ \text{ and }\ \liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)>\frac{1}{e} \tag{1.5}$$

is satisfied, then all solutions of (1.2) oscillate.

In [15], Öcalan proved that if $(\tau(n))$ is not necessarily monotone and

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)>\frac{1}{e}, \tag{1.6}$$

then all solutions of (1.2) oscillate.

In 2019, Karpuz [10] obtained that if $(\tau(n))$ is not necessarily monotone and

$$\sum_{j=\tau(n)}^{n}p(j)\le\frac{1}{e}\ \text{ for all large } n,$$

then (1.2) has a nonoscillatory solution.

Define

$$k=-\min_{n\ge 0}\tau(n).$$

(Clearly, k is a positive integer.)

By a solution of the difference equation (E), we mean a sequence of real numbers $(x(n))_{n\ge -k}$ which satisfies (E) for all $n\ge 0$. It is clear that, for each choice of real numbers $c_{-k},c_{-k+1},...,c_{-1},c_0$, there exists a unique solution $(x(n))_{n\ge -k}$ of (E) which satisfies the initial conditions $x(-k)=c_{-k},\ x(-k+1)=c_{-k+1},...,\ x(-1)=c_{-1},\ x(0)=c_0$.

A solution $(x(n))_{n\ge -k}$ of the difference equation (E) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

Strong interest in Eq. (E) is motivated by the fact that it represents a discrete analogue of the differential equation

$$x'(t)+p(t)x^{\alpha}\left(\tau(t)\right)=0,\qquad t\ge t_0, \tag{1.7}$$

where $p\in C([t_0,\infty),[0,\infty))$, $\tau\in C([t_0,\infty),\mathbb{R})$, $\tau(t)<t$ and $\alpha\in(0,\infty)$. See [6 (page 168-175), 13 (page 90-91), 17] and the references cited therein.

If $\tau(n)=n-\ell$ where $\ell\in\mathbb{N}$, Eq. (E) takes the form

$$\Delta x(n)+p(n)x^{\alpha}\left(n-\ell\right)=0,\qquad n\in\mathbb{N}. \tag{E$'$}$$

In 2001, Tang and Liu [17] studied for the first time the difference equation (E$'$) and established the following theorems:

Theorem 1.1 (See [17, Theorem 1]). *Assume that $0<\alpha<1$. Then all solutions of (E$'$) oscillate if and only if*

$$\sum_{n=0}^{\infty}p(n)=\infty. \tag{1.8}$$

Note that condition (1.8) shows that the oscillation of all solutions of the sublinear equation (E′) is determined only by the coefficient $p(n)$, and is independent of the retarded argument ℓ.

Theorem 1.2 (See [17, Theorem 2])**.** Assume that $\alpha > 1$. Then the following conclusions hold.

(i) If there exists a $\lambda > \ell^{-1} \ln \alpha$ such that

$$\liminf_{n\to\infty} \left[p(n) \exp(-e^{\lambda n})\right] > 0, \tag{1.9}$$

then all solutions of (E′) oscillate.

(ii) If

$$(p_n, p_{n+1}, \dots, p_{n+\ell-1}) \not\equiv 0 \qquad \textit{for large } n \tag{1.10}$$

and there exists a $\mu < \ell^{-1} \ln \alpha$ such that

$$\limsup_{n\to\infty} \left[p(n) \exp(-e^{\mu n})\right] < \infty, \tag{1.11}$$

then (E′) has an eventually positive solution.

In this paper, our aim is to study further (E) and present some results on the oscillatory and nonoscillatory behavior of the solutions. These results are the improved and generalized discrete analogues of the results for the corresponding differential equation, which was studied in 2001 by Tang and Liu [17] and in 2002 by Tang [18]. Examples illustrating the results are also given.

2. SUBLINEAR EQUATION

In this section we investigated the oscillatory and nonoscillatory behavior of Eq. (E) in the case where $0 < \alpha < 1$.

Theorem 2.1. Assume that (1.1) holds and $(\tau(n))$ is not necessarily monotone. Further assume that $0 < \alpha < 1$. Then all solutions of (E) oscillate if and only if

$$\sum_{n=0}^{\infty} p(n) = \infty. \tag{2.1}$$

Proof. Sufficiency. Suppose to the contrary that $(x(n))$ is an eventually positive solution of (E). Then there exists a $n_1 \in \mathbb{N}$ such that $x(n)$, $x(\tau(n)) > 0$ and $\Delta x(n) \le 0$ for $n \ge n_1$. Therefore, (E) and (2.1) imply that $\lim_{n\to\infty} x(n) = 0$. On the other hand, by means of the mean value theorem, we have

$$x^{1-\alpha}(n) - x^{1-\alpha}(n+1) \ge (1-\alpha)x^{-\alpha}(n)\left[x(n) - x(n+1)\right]. \tag{2.2}$$

Since $(x(n))$ is nonincreasing, from equation (E), we obtain

$$x(n) - x(n+1) = p(n)x^{\alpha}\left(\tau(n)\right) \ge p(n)x^{\alpha}\left(n\right). \tag{2.3}$$

So, by (2.2) and (2.3), we get

$$x^{1-\alpha}(n) - x^{1-\alpha}(n+1) \ge (1-\alpha)x^{-\alpha}(n)\left[x(n) - x(n+1)\right] \ge (1-\alpha)p(n), \quad n \ge n_1. \tag{2.4}$$

Summing (2.4) from n_1 to ∞ and using (2.1), we obtain

$$x^{1-\alpha}(n_1) \ge (1-\alpha)\sum_{n=n_1}^{\infty} p(n) = \infty.$$

This is a contradiction.

Necessity. Suppose to the contrary that (2.1) is not true. Then there exists a $n_2 \in \mathbb{N}$ such that

$$\sum_{n=n_2}^{\infty} p(n) \leq \frac{1}{2}. \tag{2.5}$$

Define a sequence $(y(n))$ as follows:

$$y(n) = \frac{1}{2} + \sum_{i=n}^{\infty} p(i), \quad n \geq n_2. \tag{2.6}$$

From (2.5) and (2.6), we have $1/2 \leq y(n) \leq 1$ for $n \geq n_2$ and

$$y(n) \geq \frac{1}{2} + \sum_{i=n}^{\infty} p(i) y^{\alpha}(\tau(i)), \quad n \geq n_3 \geq n_2. \tag{2.7}$$

From the proof of Lemma 2.2 in [14] and (2.7), it is not difficult to show that the corresponding equation

$$x(n) = \frac{1}{2} + \sum_{i=n}^{\infty} p(i) x^{\alpha}(\tau(i)), \quad n \geq n_3 \tag{2.8}$$

has an eventually positive solution $(x(n))$. From (2.8), we can write that

$$x(n+1) = \frac{1}{2} + \sum_{i=n+1}^{\infty} p(i) x^{\alpha}(\tau(i)), \quad n \geq n_3. \tag{2.9}$$

Obviously, from (2.8) and (2.9), we have equation (E). Therefore, we get that $(x(n))$ is also an eventually positive solution of (E), leading to a contradiction, and so the proof is complete.

Example 2.1. Consider the difference equation

$$\Delta x(n) + p(n) x^{1/3}(\tau(n)) = 0, \qquad n \geq 1. \tag{2.10}$$

Here,

$$\tau(n) = \begin{cases} n, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases}.$$

Clearly, $\tau(n) \leq n$ and $\lim_{n\to\infty} \tau(n) = \infty$, i.e., (1.1) holds. If we take $p(n) = \frac{1}{n}$, then it is easy to see that $\sum\limits_{n=1}^{\infty} p(n) = \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$, which means that (2.1) holds. Thus all conditions of Theorem 2.1 are satisfied and therefore all solutions of (2.10) oscillate. On the other hand, if we take $p(n) = \frac{1}{n^2}$, then $\sum\limits_{n=1}^{\infty} p(n) = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$, which due to Theorem 2.1, we obtain that every solution of (2.10) is nonoscillatory.

3. SUPERLINEAR EQUATION

In this section we investigated the oscillatory and nonoscillatory behavior of Eq. (E) in the case where $\alpha > 1$. To prove the following theorems, we need the following lemmas.

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Lemma 3.1. Assume that (1.1) holds with $p(n) \geq 0,\ p(n) \not\equiv 0,\ \alpha > 0,\ \alpha$ is a quotient of odd positive integers. Then (E) has an eventually positive solution if and only if the corresponding inequality

$$\Delta x(n) + p(n)x^{\alpha}(\tau(n)) \leq 0, \quad n \in \mathbb{N}$$

has an eventually positive solution.

Also, Eq. (E) has an eventually negative solution if and only if the corresponding inequality

$$\Delta x(n) + p(n)x^{\alpha}(\tau(n)) \geq 0, \quad n \in \mathbb{N}$$

has an eventually negative solution.

Proof. Sufficiency. This part is the same as [14, Lemma 2.2].

.Necessity This part is trivial, since any eventually positive solution of (E) satisfies $\Delta x(n) + p(n)x^{\alpha}(\tau(n)) \leq 0, \quad n \in \mathbb{N}$ too.

Moreover, the proof of second chapter of the lemma is obtained in a similar way to the proof of first chapter, which we omit it. The proof is complete

Associated with (E), we consider the following equation

$$\Delta x(n) + q(n)x^{\alpha}(\tau(n)) = 0, \quad n \in \mathbb{N}, \tag{3.1}$$

where $(q(n))$ is a sequence of nonnegative real numbers.

Applying Lemma 3.1, we have the following.

Lemma 3.2. Assume that (1.1) holds with $p(n) \geq 0,\ p(n) \not\equiv 0$ and

$$p(n) \leq q(n).$$

If every solution of (E) oscillates, then every solution of (3.1) oscillates.

Theorem 3.1. Assume that (1.1) holds and that $\alpha > 1$, $\Delta\tau(n) \geq 0$. Further suppose that there exists a sequence $(\varphi(n))$ such that

$$\Delta\varphi(n) > 0 \ \text{ and } \ \lim_{n\to\infty} \varphi(n) = \infty, \tag{3.2}$$

$$\limsup_{n\to\infty} \frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)} < 1, \tag{3.3}$$

and

$$\liminf_{n\to\infty} \left[p(n) \frac{e^{-\varphi(n)}}{\Delta\varphi(n)} \right] > 0. \tag{3.4}$$

Then all solutions of (E) oscillate.

Proof. By (3.2), (3.3) and Discrete l'Hospital's rule [1, Theorem 1.8.7] , we have

$$\limsup_{n\to\infty} \frac{\alpha\varphi(\tau(n))}{\varphi(n)} \leq \limsup_{n\to\infty} \frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)} < 1. \tag{3.5}$$

It follows from (3.5) that there exists $0 < k < 1$, $n \geq n_1$ such that

$$\frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)} \leq k, \quad \text{and} \quad \frac{\alpha\varphi(\tau(n))}{\varphi(n)} \leq k. \tag{3.6}$$

Because of (3.4), there exists a $n_2 \geq n_1$ such that

$$p(n)\frac{e^{-\varphi(n)}}{\Delta\varphi(n)} \geq c > 0 \ \text{ for } n \geq n_2,$$

and so we have

$$p(n) \geq c\Delta\varphi(n)e^{\varphi(n)}, \quad n \geq n_2. \tag{3.7}$$

Set

$$q(n)=c\Delta\varphi(n)e^{\varphi(n)}. \tag{3.8}$$

By Lemma 3.2, it is sufficient to prove that every solution of equation (3.1) oscillates. Assume the contrary, and let $(x(n))$ be an eventually positive solution of (3.1). Then there exists a $n_3\geq n_2$ such that $\Delta x(n)\leq 0$, for $n\geq n_3$, which means that $(x(n))$ is nonincreasing and has a limit $l\geq 0$. Now, we claim that $\lim_{n\to\infty}x(n)=0$, otherwise, $\lim_{n\to\infty}x(n)=l>0$. Hence, by using this facts, we have

$$0<l-\varepsilon\leq x(n)\leq l+\varepsilon,\quad n\geq n_4, \tag{3.9}$$

where ε is an arbitrary real number. Since $\alpha>1$ and $(x(n))$ is nonincreasing, from (3.1), (3.8) and (3.9), we get

$$\Delta x(n)+c\Delta\varphi(n)e^{\varphi(n)}x\left(n\right)\leq 0,\quad \text{for } n\geq n_4,$$

or

$$\Delta x(n)+c(l-\varepsilon)\Delta\varphi(n)\leq 0,\quad \text{for } n\geq n_4. \tag{3.10}$$

Summing up (3.10) from a to ∞, and since $\lim_{n\to\infty}\varphi(n)=\infty$, then we obtain

$$l-a+c(l-\varepsilon)\left[\infty-\varphi(a)\right]\leq 0,$$

which is a contradiction, and so our claim is true.

Let $y(n)=-\ln x(n)$ for $n\geq n_5$. So, since $(x(n))$ is nonincreasing and $\lim_{n\to\infty}x(n)=0$, it follows that $(y(n))$ is nondecreasing and $\lim_{n\to\infty}y(n)=\infty$. Then, from (3.1) we have

$$1-e^{y(n)-y(n+1)}=q(n)e^{y(n)-\alpha y(\tau(n))}\geq 0,\quad n\geq n_5. \tag{3.11}$$

Consequently, we obtain

$$\Delta y(n)\geq q(n)e^{y(n)-\alpha y(\tau(n))},\quad n\geq n_5. \tag{3.12}$$

Therefore, since $(y(n))$ is nondecreasing, we have the following two possible cases.

Case 1. $y(n)-\alpha y\left(\tau(n)\right)\leq 0$ for $n\geq n_6$. Then, from (3.6), we obtain

$$\frac{y(n)}{\varphi(n)}\leq\frac{\alpha y\left(\tau(n)\right)}{\varphi(n)}=\frac{\alpha\varphi(\tau(n))}{\varphi(n)}\frac{y\left(\tau(n)\right)}{\varphi(\tau(n))}\leq k\frac{y\left(\tau(n)\right)}{\varphi(\tau(n))},\quad n\geq n_6. \tag{3.13}$$

Set $z(n)=\frac{y(n)}{\varphi(n)}$. Then, from (3.13), we have

$$z(n)\leq kz(\tau(n)),\quad n\geq n_6, \tag{3.14}$$

which implies that $z(\tau(n))\geq z(n)$ for $n\geq n_6$. Thus, it follows from this facts that all subsequences of $(z(n))$ are nonincreasing. Now, we claim that

$$\lim_{n\to\infty}z(n)=0. \tag{3.15}$$

Otherwise, there exists a sequence $\{n_p\}$ such that $n_p\to\infty$ as $p\to\infty$ and $\lim_{p\to\infty}z(n_p)=b>0$. Hence, from (3.14) we have

$$z(n_p)\leq kz(\tau(n_p)),\quad n\geq n_6. \tag{3.16}$$

By taking limit $p\to\infty$ 1 in (3.16), we get

$$b\leq kb.$$

Since $0<k<1$, this is a contradiction. Therefore, (3.15) is true. From (3.15), it follows that

$$y(n)<\frac{1}{1+\alpha}\varphi(n),\quad n\geq n_7. \tag{3.17}$$

Thus, since $(y(n))$ is nondecreasing, from (3.8), (3.12) and (3.17), we obtain

$$\Delta y(n) \geq q(n)e^{-(\alpha-1)y(\tau(n))} \geq q(n)e^{-(\alpha-1)\varphi(\tau(n))/(1+\alpha)} > \Delta\varphi(n), \quad n \geq n_7.$$

It follows that

$$y(n) > \varphi(n) + y(n_7) - \varphi(n_7), \quad \text{for } n \geq n_7. \tag{3.18}$$

Therefore, from (3.18), we get

$$\frac{y(n)}{\varphi(n)} > 1 + \frac{y(n_7) - \varphi(n_7)}{\varphi(n)}, \quad n \geq n_7. \tag{3.19}$$

Taking the limit as $n \to \infty$ in (3.19), we obtain

$$\lim_{n\to\infty} z(n) = 0 \geq 1,$$

which is a contradiction.

Case 2. $y(n) - \alpha y(\tau(n)) > 0$ for $n \geq n_6$. Thus, we have $y(n) > \alpha y(\tau(n))$ for $n \geq n_6$. Now, we consider the following possible case for $\tau(n)$; for some $n \geq n_7$ (or for all $n \geq n_7$) $n-1 \leq \tau(n) \leq n$ and for some $n \geq n_7$ (or for all $n \geq n_7$) $\tau(n) < n-1$.

First, we consider, for some $n \geq n_7$ (or for all $n \geq n_7$) $n-1 \leq \tau(n) \leq n$. It is clear that since $(\varphi(n))$ is increasing, we get $\varphi(n-1) \leq \varphi(\tau(n))$ and from (3.6) $\frac{\alpha\varphi(\tau(n))}{k} \leq \varphi(n)$, where $\frac{\alpha}{k} > 1$. Thus, for $n \geq n_7$ we obtain

$$\begin{aligned}\Delta\varphi(n-1) &= \varphi(n) - \varphi(n-1) \geq \frac{\alpha\varphi(\tau(n))}{k} - \varphi(n-1)\\ &\geq \frac{\alpha\varphi(\tau(n))}{k} - \varphi(\tau(n)) \geq \varphi(\tau(n))\left[\frac{\alpha}{k} - 1\right] > 0,\end{aligned}$$

and so we have

$$\Delta\varphi(n) \geq \varphi(\tau(n+1))\left[\frac{\alpha}{k} - 1\right] > 0, \quad n \geq n_8. \tag{3.20}$$

Thus, since $\lim_{n\to\infty}\varphi(n) = \infty$, from (3.20), we obtain

$$\lim_{n\to\infty} \Delta\varphi(n) = \infty.$$

Secondly, we consider, for some $n \geq n_7$ (or for all $n \geq n_7$) $\tau(n) < n-1$.Now, we can find a sequence $(\varphi(n))$ such that conditions (3.2) and (3.6) are satisfied. Indeed, if we take $\varphi(n) = e^{\frac{\alpha}{k}n}$, then it is clear that (3.2) is satisfied. Moreover, since

$$\frac{\varphi(n)}{\varphi(\tau(n))} = e^{\frac{\alpha}{k}[n-\tau(n)]} \geq \frac{\alpha}{k}[n-\tau(n)]e > \frac{\alpha}{k}, \quad \text{for } n \geq n_7,$$

we have the condition (3.6). On the other hand, we observe that

$$\Delta\varphi(n) = \varphi(n+1) - \varphi(n) = e^{\frac{\alpha}{k}(n+1)}\left(\frac{e-1}{e}\right), \quad \text{for } n \geq n_7,$$

and we get

$$\lim_{n\to\infty} \Delta\varphi(n) = \infty. \tag{3.21}$$

Then for every case of $(\tau(n))$ we have that there is a sequence $(\varphi(n))$ such that (3.21) holds. It follows from (3.8), (3.11) and (3.21) that

$$1 > q(n)e^{y(n)-\alpha y(\tau(n))} > q(n) > 1, \quad n \geq n_8.$$

This is a contradiction. If there exists an eventually negative solution $(x(n))$ of (E), then the proof can be done similarly as above. The proof is complete.

Theorem 3.2. Assume that (1.1) holds and that $\alpha > 1$, $\Delta\tau(n) \geq 0$. Further suppose that there exists a sequence $(\psi(n))$ such that

$$\Delta\psi(n) > 0 \quad \text{and} \quad \lim_{n\to\infty} \psi(n) = \infty, \tag{3.22}$$

$$\liminf_{n\to\infty} \frac{\alpha\Delta\psi(\tau(n))}{\Delta\psi(n)} > 1, \tag{3.23}$$

and

$$\limsup_{n\to\infty} \left[p(n)\frac{e^{-\psi(n)}}{\Delta\psi(n)}\right] < \infty. \tag{3.24}$$

Then (E) has a nonoscillatory solution.

Proof. By (3.23) and Discrete l'Hospital's rule [1, Theorem 1.8.7], we get

$$\liminf_{n\to\infty} \frac{\alpha\psi(\tau(n))}{\psi(n)} \geq \liminf_{n\to\infty} \frac{\alpha\Delta\psi(\tau(n))}{\Delta\psi(n)} > 1, \quad n \geq n_1,$$

and

$$\frac{\alpha\psi(\tau(n))}{\psi(n)} \geq L > 1, \quad n \geq n_1. \tag{3.25}$$

By (3.24), we have

$$p(n) \leq \frac{L}{L-1}\Delta\psi(n)e^{L\psi(n)}, \quad n \geq n_1, \tag{3.26}$$

Let $x(n) = e^{-\frac{L}{L-1}\psi(n)}$ for $n \geq n_1$. Thus,

$$\begin{aligned}
& x(n+1) - x(n) + p(n)x^{\alpha}(\tau(n)) \\
= \;& e^{-\frac{L}{L-1}\psi(n+1)} - e^{-\frac{L}{L-1}\psi(n)} + p(n)e^{-\alpha\frac{L}{L-1}\psi(\tau(n))} \\
= \;& = e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - e^{\alpha\frac{L}{L-1}\psi(\tau(n)) - \frac{L}{L-1}[\psi(n)+\psi(n+1)]}\left(e^{\frac{L}{L-1}\psi(n+1)} - e^{\frac{L}{L-1}\psi(n)}\right)\right] \\
\leq \;& e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - e^{\frac{L}{L-1}[\alpha\psi(\tau(n))-\psi(n)]}\left(e^{\frac{L}{L-1}\psi(n+1)} - e^{\frac{L}{L-1}\psi(n)}\right)\right].
\end{aligned} \tag{3.27}$$

Thus, since $x \geq \ln x$ and $\ln(e^x - e^y) \geq \ln e^x - \ln e^y$ for $x > y > 0$, $n \geq n_2$, from (3.27) we get

$$\begin{aligned}
& x(n+1) - x(n) + p(n)x^{\alpha}(\tau(n)) \\
\leq \;& e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - e^{\frac{L}{L-1}[\alpha\psi(\tau(n))-\psi(n)]}\ln\left(e^{\frac{L}{L-1}\psi(n+1)} - e^{\frac{L}{L-1}\psi(n)}\right)\right] \\
\leq \;& e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - e^{\frac{L}{L-1}[\alpha\psi(\tau(n))-\psi(n)]}\left(\ln e^{\frac{L}{L-1}\psi(n+1)} - \ln e^{\frac{L}{L-1}\psi(n)}\right)\right] \\
= \;& e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - \frac{L}{L-1}\Delta\psi(n)e^{\frac{L}{L-1}[\alpha\psi(\tau(n))-\psi(n)]}\right] \\
= \;& e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n) - \frac{L}{L-1}\Delta\psi(n)e^{\left(\frac{\alpha\psi(\tau(n))}{\psi(n)}-1\right)\frac{L}{L-1}\psi(n)}\right].
\end{aligned} \tag{3.28}$$

Also, from (3.25) we have

$$\frac{\alpha\psi(\tau(n))}{\psi(n)} - 1 \geq L - 1$$

and

$$\left(\frac{\alpha\psi(\tau(n))}{\psi(n)} - 1\right)\frac{L}{L-1} \geq L. \tag{3.29}$$

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So, from (3.28) and (3.29) we obtain

$$\begin{aligned} & x(n+1)-x(n)+p(n)x^{\alpha}(\tau(n)) \\ \leq\ & e^{-\alpha\frac{L}{L-1}\psi(\tau(n))}\left[p(n)-\frac{L}{L-1}\Delta\psi(n)e^{L\psi(n)}\right] \\ \leq\ & 0, \quad n\geq n_2. \end{aligned} \tag{3.30}$$

This shows that the inequality (3.30) has an eventually positive solution. In view of Lemma 3.1, the corresponding equation (E) also has an eventually positive solution.

Using the same process above, it is easy to see that under the assumption (3.22), (3.23) and (3.24), if we choose $x(n)=-e^{-\frac{L}{L-1}\psi(n)}$ for $n\geq n_1$, then equation (E) has an eventually negative solution. The proof is complete.

Now, we have the following result.

Corollary 3.1. *Assume that* $\alpha>1$ and $\tau(n)=n-\ell$ where $\ell\in\mathbb{N}$. Then,

(a) If there exists a $\lambda>\ell^{-1}\ln\alpha$ such that $\liminf_{n\to\infty}\left[p(n)\exp(-e^{\lambda n})\right]>0$, then Theorem 3.1 implies Theorem 1.2 (i).

(b) If there exists a $\mu<\ell^{-1}\ln\alpha$ such that $\limsup_{n\to\infty}[p(n)\exp(-e^{\mu n})]<\infty$, then Theorem 3.2 implies Theorem 1.2 (ii).

Proof. (a) *Let* $\lambda_1\in\left(l^{-1}\ln\alpha,\lambda\right)$ and let $\varphi(n)=e^{\lambda_1 n}$. Then,

$$\Delta\varphi(n)=e^{\lambda_1 n}\left(e^{\lambda_1}-1\right)>0, \quad \lim_{n\to\infty}\varphi(n)=\infty,$$

and

$$\limsup_{n\to\infty}\frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)}=\frac{\alpha}{e^{\lambda_1 l}}<1.$$

These show that conditions (3.2) and (3.3) in Theorem 3.1 hold. In addition, it is easy to see that for large n

$$\lambda_1 n+e^{\lambda_1 n}<e^{\lambda n}.$$

Thus, from (1.9), we obtain

$$\begin{aligned} \liminf_{n\to\infty}\left[p(n)\frac{e^{-\varphi(n)}}{\Delta\varphi(n)}\right] &= \frac{1}{(e^{\lambda_1}-1)}\liminf_{n\to\infty}\left[p(n)\exp\left(-e^{\lambda_1 n}-\lambda_1 n\right)\right] \\ &\geq \frac{1}{(e^{\lambda_1}-1)}\liminf_{n\to\infty}\left[p(n)\exp\left(-e^{\lambda n}\right)\right]>0, \end{aligned}$$

which shows that condition (3.4) in Theorem 3.1 also holds. Hence, in view of Theorem 3.1, every solution of (E′) oscillates.

(b) *Let* $\mu_1\in\left(\mu,\ell^{-1}\ln\alpha\right)$ and let $\psi(n)=e^{\mu_1 n}$. Then,

$$\Delta\psi(n)=e^{\mu_1 n}\left(e^{\mu_1}-1\right)>0, \quad \lim_{n\to\infty}\psi(n)=\infty,$$

and

$$\liminf_{n\to\infty}\frac{\alpha\Delta\psi(\tau(n))}{\Delta\psi(n)}=\frac{\alpha}{e^{\mu_1 l}}>1.$$

These show that conditions (3.22) and (3.23) in Theorem 3.2 hold. On the other hand, from (1.11) we have

$$\begin{aligned} \limsup_{n\to\infty}\left[p(n)\frac{e^{-\psi(n)}}{\Delta\psi(n)}\right] &= \frac{1}{(e^{\mu_1}-1)}\limsup_{n\to\infty}\left[p(n)\exp\left(-e^{\mu_1 n}-\mu_1 n\right)\right] \\ &\leq \frac{1}{(e^{\mu_1}-1)}\limsup_{n\to\infty}\left[p(n)\exp\left(-e^{\mu n}\right)\right]<\infty, \end{aligned}$$

which shows that condition (3.24) in Theorem 3.2 also holds. Hence, in view of Theorem 3.2, (E′) has an eventually positive solution.

Example 3.1. Consider the difference equation

$$\Delta x(n) + e^n x^{5/3}(\tau(n)) = 0, \qquad n \geq 1, \tag{3.31}$$

Here, $\alpha = 3$, $p(n) = e^n$ and

$$\tau(n) = \begin{cases} \frac{n-1}{2}, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases}.$$

Clearly, $\tau(n) \leq n$, $\lim_{n\to\infty} \tau(n) = \infty$ and $\Delta\tau(n) \geq 0$. If we take $\varphi(n) = n$, then it is easy to see that

$$\Delta\varphi(n) = 1 > 0 \ \text{ and } \lim_{n\to\infty} \varphi(n) = \infty,$$

and

$$\limsup_{n\to\infty} \frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)} = \frac{5}{6} < 1,$$

and

$$\liminf_{n\to\infty} \left[p(n)\frac{e^{-\varphi(n)}}{\Delta\varphi(n)}\right] = \liminf_{n\to\infty} \left[e^n e^{-n}\right] = 1 > 0,$$

which means that (3.2), (3.3) and (3.4) hold. Thus all conditions of Theorem 3.1 are satisfied and therefore all solutions of (3.31) oscillate. We should point out that no paper in the literature answers this example.

Example 3.2. Consider the difference equation

$$\Delta x(n) + e^n x^{5/3}(\tau(n)) = 0, \qquad n \geq 1, \tag{3.32}$$

Here, $\alpha = 5/3$, $p(n) = e^n$ and

$$\tau(n) = \begin{cases} \left[n - \frac{1}{n}\right], & n \text{ is odd} \\ n, & n \text{ is even} \end{cases},$$

where $\left[n - \frac{1}{n}\right]$ denotes the greatest integer $m \leq \left(n - \frac{1}{n}\right)$, $n = 1, 3, \ldots$. Clearly, $\tau(n) \leq n$, $\lim_{n\to\infty} \tau(n) = \infty$ and $\Delta\tau(n) \geq 0$. If we take $\varphi(n) = n$, then it is easy to see that

$$\Delta\varphi(n) = 1 > 0 \ \text{ and } \lim_{n\to\infty} \varphi(n) = \infty,$$

and

$$\liminf_{n\to\infty} \frac{\alpha\Delta\varphi(\tau(n))}{\Delta\varphi(n)} = \frac{5}{3} > 1,$$

and

$$\limsup_{n\to\infty} \left[p(n)\frac{e^{-\varphi(n)}}{\Delta\varphi(n)}\right] = \limsup_{n\to\infty} \left[e^n e^{-n}\right] = 1 < \infty,$$

which means that (3.22), (3.23) and (3.24) hold. Thus all conditions of Theorem 3.2 are satisfied and therefore (3.32) has a nonoscillatory solution. We should point out that no paper in the literature answers this example.

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