The multiplicity of periodic solutions for distributed delay differential systems

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Abstract

In this paper, we study the existence of periodic solutions of the following differential delay system

$$x'(t) = -f(\int_0^1 x(t-s)ds),\ x \in \mathbb{R}^{\mathbb{N}},$$

where $f \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $N \in \mathbb{N}$. We transform the problem of searching for 2-periodic solutions of the above system into that of finding 2-periodic solutions of an associated Hamiltonian system with the special symmetry. Using the critical point theory and the pseudoindex theory, we estimate the number of periodic solutions which is related to the discrepancies between the eigenvalues of asymptotic linear matrices at the origin and at infinity. At last, we give an example to illustrate our result.

Keywords: Distributed delay, critical point theory, periodic solutions, multiplicity

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1 Introduction

Time delay equations have a wide range of applications in the biological, mechanical and social sciences, and many other modeling applications. The presence of periodic solutions in mathematical models has aroused interest due to their theoretical and practical importances. In particular, the last three years have seen extensive research on dynamical systems of the spread of COVID-19, and time delay differential equations have played an important role in controlling the spread of this pandemic (cf. [5, 27]). For example, a newly confirmed case adds distributed delay to the dynamical system of a stochastic process (cf. [5])

$$X'(t) = \alpha(t) + \int_0^{\tau_0} \gamma(t,s)X(t-s)\,ds + \sum_{j=1}^{J} \beta_j(t)U_j(t-\tau_j) + Z(t),$$

with an initial condition $X(t) = g(t)$, $t \in [t_0 - \tau_0, t_0]$. Other emerging fields of research, such as real neural networks (cf. [3]), Dengue diseases (cf. [24]), and biological network motifs (cf. [9]) also widely use time-delay equations. For example, Glass and Jin (cf. [9]) constructed a discrete delay differential equation as follows

$$Y'(T) = \epsilon + \frac{\eta}{1 + X^n(T-\gamma)} - Y(T)$$

to model biological network motifs.

There are two widespread types of time delay: discrete and distributed. Indeed, in reality, some things have intermittent processes such as daily, monthly, and yearly occurrences that lend themselves to the application of delay differential equations with discrete delays (DDEs, for short). Other things have a series of continuous and uninterrupted change processes, such as chemical reactions, biological aggregation, economic systems, and cellular neural networks that lend themselves to the application of distributed delay differential equations (DDDEs, for short).

For periodic solutions to DDEs, the first known result was obtained by Jones in 1962. In [14], Jones used a type of fixed point theorem to prove the existence of a periodic solution for DDEs. Many researchers have since used all types of fixing point theorems to study the existence of periodic solutions for DDEs. There are other methods used by mathematicians to study the existence of periodic solutions to such kinds of equations. For example, Heiden (cf. [13]) considered the following equation

$$x''(t) + (a+b)x'(t) + abx(t) = -f(x(t-1)) \tag{1}$$

where a and b are positive constants, $f \in C(\mathbb{R}, \mathbb{R})$. Dividing the equation into two first-order differential equations, in which only one of them has a delay, he analyzed the associative characteristic equation and obtained the existence of a nonconstant m-periodic solution of (1), where $m > 2$.

In 1974, Kaplan and Yorke (cf. [17]) introduced a new technique to study the following equation

$$x'(t) = -f(x(t-1)) - f(x(t-2)), x \in \mathbb{R}. \tag{2}$$

They transferred the existence of periodic solutions of (2) to that of periodic solutions of an associated ordinary differential system. Later, Li and his cooperators extended the results to the following equation

$$x'(t) = -f(x(t-r_1)) - \cdots - f(x(t-r_{m-1})), x \in \mathbb{R}. \tag{3}$$

They (cf. [18]) showed that the associated system of (3) is a Hamiltonian system when m is even and a generalized Hamiltonian system when m is odd. Results of this study are available in [6, 7, 22, 28, 29, 31, 32].

In 2005, Guo and Yu (cf. [10]) first built the variational structure for the following system

$$x'(t) = -f(x(t-1)), x \in \mathbb{R}^N. \tag{4}$$

Using critical point theory directly, they proved the existence of multiple periodic solutions to (4). Later, C. Guo and Z. Guo (cf. [8]) studied the existence and multiplicity of 2η-periodic solutions to differential delay equations

$$x''(t) = -f(x(t-\eta)), x \in \mathbb{R}^N,$$

where the constant $\eta > 0$. Results of this study are available in [11, 12, 16, 19, 33, 34]

Since continuously distributed delay can more accurately describe the time effect of a real phenomenon than a discrete delay, DDDEs are receiving increasing attention. However, DDDEs are more difficult to solve than DDEs, and many DDDEs cannot be solved or analyzed theoretically. Hence, they must typically be reduced to discrete models for numerical simulation. Fortunately, there are some methods with excellent properties that can obtain solutions of DDDEs.

In 2007, Azevedoa, Gadottia and Ladeira (cf. [1]) considered two-dimensional differential equations with distributed delay

$$\begin{cases} \dot{x}_1 = \int_{-1}^{0} f_1(x_2(t+\theta))\, d\theta \\ \dot{x}_2 = \int_{-1}^{0} f_2(x_1(t+\theta))\, d\theta, \end{cases} \tag{5}$$

where f_1 and f_2 are restricted to be continuous derivable functions and their derivatives equal at the origin. Solutions of (5) are transformed into that of a second-order ordinary differential system. Verifying the properties of the corresponding characteristic equations, they proved that (5) has a special symmetric 2-periodic solution under the negative feedback assumption.

In 2014, Kennedy (cf. [15]) researched a nonlinear differential equation with distributed delay

$$x' = f\left[\int_{t-1}^{t-d} g(x(s))\,ds\right],\ d \in [0,1).$$

The functions f and g are bounded, smooth, and odd, which satisfy the positive and negative feedback conditions, respectively. Kennedy also described certain growth conditions as the functions f and g approach zero and proved the existence of nontrivial periodic solutions with period $2+2d$ using the elementary immovable points theory.

In 2020, Nakata (cf. [25]) studied the following differential system with distributed delay

$$\frac{d}{dt}x(t) = rx(t)(1 - \int_0^{-1} x(t-s)\,ds), \tag{6}$$

where $r > \frac{\pi^2}{2}$. Following the idea proposed by Kaplan and Yorke (cf. [17]), Nakata concluded that (6) has a 2-periodic solution. To identify periodic solutions, he expressed the periodic solutions in terms of Jacobi elliptic functions.

A year later, Nakata (cf. [26]) studied the following equation

$$x'(t) = -f(\int_0^1 x(t-s)\,ds), \quad x \in \mathbb{R}. \tag{7}$$

He proved that (7) possesses a periodic solution when $f(x) = r\sin(x)$. The author made use of numerical simulation and obtained two periodic solutions of (7). How many solutions does (7) have? Motivated by this, Xiao and Guo (cf. [30]) made use of the variational method and the index theory to restudy (7) where $f(x)$ is a general continuous function. They obtained the lower bound of 2-periodic solutions for (7) under some sufficient conditions.

In this paper, we study the following system

$$x'(t) = -f(\int_0^1 x(t-s)ds),\ x \in \mathbb{R}^N, N \in \mathbb{N}. \tag{8}$$

Throughout this paper, we assume that f satisfies the following assumptions.

(A1) $f \in C(\mathbb{R}^N, \mathbb{R}^N)$ is odd, and there exists a continuously differentiable function F, such that $F(0) = 0$ and the gradient of F is f, that is, for any

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$x \in \mathbb{R}^N$, $\nabla F(x) = f(x)$.
(**A2**)

$$f(x) = A_0 x + o(|x|) \text{ as } |x| \to 0,$$
$$f(x) = A_\infty x + o(|x|) \text{ as } |x| \to +\infty,$$

where A_0 and A_∞ are symmetric $N \times N$ matrices.

We transform the problem of searching for 2-periodic solutions of (8) into that of finding 2-periodic solutions of an associated Hamiltonian system with the special symmetry. Then the variational functional G on the Sobolev space $\mathcal{K}$ is built for the associated Hamiltonian system. By linearizing the functional G at the origin and at infinity, we obtain two linear operators L_0 and L_∞. The space $\mathcal{K}$ can be decomposed according to operators L_0 and L_∞. Denote by $W^+(W^0, W^-)$ and $V^+(V^0, V^-)$ the positive (null, negative) eigenspaces of L_0 and L_∞ respectively. The lower bound of 2-periodic solutions of (8) equals to $\max\{\dim(W^-) - \dim(V^-), \dim(V^-) - \dim(W^- \oplus W^0)\}$.

Since $\dim W^-$ and $\dim V^-$ are both infinite, it is complicated. Fortunately, we can convert the problem of calculating $\dim W^-$ and $\dim V^-$ to that of calculating the number of negative eigenvalues to the matrix $i(2j-1)\pi J + \overline{A}_0$ and $i(2j-1)\pi J + \overline{A}_\infty$(for $j \in \mathbb{N}$). As we can see in Lemma 4.2, for a given eigenvalue of A_0 and $j \in \mathbb{N}$, $i(2j-1)\Pi J + \overline{A}_0$ has two eigenvalues. One of them must be positive. The other may be positive, negative or null eigenvalue. However, the second eigenvalue is positive only for finite $j \in \mathbb{N}$. Therefore, we count all those j and use its negative value to define an index.

For any $a \in \mathbb{R}$, we set

$$j(a) = \begin{cases} -\lfloor \frac{\sqrt{2a}+\pi}{2\pi} \rfloor, & \text{if } a > 0 \\ 0, & \text{if } a \le 0 \end{cases}, \tag{9}$$

where $\lfloor b \rfloor = \max\{k \in \mathbb{Z} | k \le b\}$. Denote by $\lambda_1, \lambda_2, \cdots, \lambda_N$ all eigenvalues of A. Define the index as follows

$$\#(A) = \sum_{i=1}^{N} j(\lambda_i).$$

The primary results are stated as follows.

Theorem 1 *Assume that f satisfies (**A1**), (**A2**) and*
*(**A3**)*

$$\left\{ \frac{(2n-1)^2\pi^2}{2} | \; n \in \mathbb{N} \right\} \cap \sigma(A_\infty) = \emptyset$$

where $\sigma(B)$ is the set of all eigenvalues of matrix B.
Then, (8) has at least $|\#(A_0) - \#(A_\infty)|$ nonconstant 2-periodic solutions.

The primary contents of the different sections are as follows. In Section 2, we transform the existence of periodic solutions of (8) to that of some symmetric periodic solutions of a coupled Hamiltonian system. Then, we establish the variational functional of the associated ordinary differential system. In Section 3, we provide preliminary results. In Section 4, we prove the main result. In Section 5, we provide an example to show the proposed result.

2 Variational structure

First, we transform the problem of searching for 2-periodic solutions of (8) satisfying $x(t+1) = -x(t)$ to that of finding 2-periodic solutions of an associated ordinary differential system.

Set

$$y_1(t) = x(t), \quad y_2(t) = \int_0^1 x(t-s)\,ds, \quad y(t) = (y_1(t)^\tau, y_2(t)^\tau)^\tau, \tag{10}$$

and

$$D(y) = |y_1(t)|^2 + F(y_2), \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{11}$$

where I is a $N \times N$ identity matrix and τ is denoted by the transpose. We can intuitively obtain

$$y'(t) = J\nabla D(y). \tag{12}$$

The conversion still has the required properties.

Lemma 1 (1) If $x(t)$ is a 2-periodic solution of (8) with $x(t-1) = -x(t)$, then $y(t)$, where y is given by (10), is a 2-periodic solution of (12) with the symmetric structure $y(t-1) = -y(t)$.

(2) If $y(t) = (y_1(t)^\tau, y_2(t)^\tau)^\tau$ is a 2-periodic solution of (12) with the symmetric structure $y(t-1) = -y(t)$, then $x(t) = y_1(t)$ is a 2-periodic solution of (8) satisfying $x(t-1) = -x(t)$.

Remark 1 From Lemma 1, we have identified the existence of 2-periodic solutions of (8) satisfying $x(t-1) = -x(t)$ with the existence of 2-periodic solutions of (12) satisfying $y(t-1) = -y(t)$.

We now take $S^1 = \mathbb{R}/(2\mathbb{Z})$ and $C^\infty(S^1, \mathbb{R}^{2N})$, which is the space of 2-periodic infinitely differentiable vector valued functions with dimension $2N$. The element $y \in C^\infty(S^1, \mathbb{R}^{2N})$ is convergent in the space $L^2(S^1, \mathbb{R}^{2N})$ with the Fourier series

$$y(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ij\pi t}, \tag{13}$$

where $a_j \in \mathbb{C}^{2N}$ and $a_{-j} = \overline{a_j}$.

There are $z_1, z_2 \in L^2(S^1, \mathbb{R}^{2N})$. If every $z \in C^\infty(S^1, \mathbb{R}^{2N})$ satisfies

$$\int_0^2 (z_1, z'(t))dt = -\int_0^2 (z_2, z)dt,$$

we call z_2 a weak derivative of z_1 and denote by $\dot{z}_1$.

A Sobolev space $\mathcal{H}$ is given by

$$\mathcal{H} = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N}) = \left\{ y \in L^2(S^1, \mathbb{R}^{2N}) | 2\left|a_0\right|^2 + 4\sum_{j=1}^{+\infty}(1+j\pi)\left|a_j\right|^2 < +\infty \right\},$$

where $|\cdot|$ is the norm in $\mathbb{C}^{2N}$. For $z(t) = \sum_{j=-\infty}^{+\infty} b_j e^{ij\pi t}$, the space $\mathcal{H}$ is equipped with the inner product

$$< y, z >= 2(a_0, b_0) + 2\sum_{j=1}^{+\infty}(1+j\pi)\left[(a_j, b_j) + (a_{-j}, b_{-j})\right],$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{C}^{2N}$. The inner product has the associated norm

$$\|y\|^2 = 2|a_0|^2 + 4\sum_{j=1}^{+\infty}(1+j\pi)|a_j|^2.$$

For the next step, we define a functional G on $\mathcal{H}$ for consideration

$$G(y) = \int_0^2 [\frac{1}{2}(J\dot{y}(t), y(t)) + D(y(t))]dt, \forall y \in \mathcal{H}. \tag{14}$$

Following assumptions (A1) and (A2), G is well defined.

We now define an operator as follows

$$\phi(y) = \int_0^2 D(y(t))\, dt, \forall y \in \mathcal{H}.$$

By a standard argument as in (cf. [11]), we know

Lemma 2 Assume that f satisfies (**A1**) and (**A2**). G is a continuously differentiable functional on $\mathcal{H}$. $G'(y)$ is given by

$$\langle G'(y), z\rangle = \int_0^2 \left[(J\dot{y}(t), z(t)) + (\nabla D(y(t)), z(t))\right] dt, \forall z \in \mathcal{H}. \tag{15}$$

Correspondingly, $\phi' : \mathcal{H} \to \mathcal{H}^*$ is a compact mapping defined to be

$$\langle \phi'(y), z\rangle = \int_0^2 (\nabla D(y(t)), z(t))\, dt, \forall z \in \mathcal{H}.$$

We define an operator T such that $Ty(t) = y(t-1)$, where T is a bounded linear isometric operator. We let $\mathcal{K}$ be a subspace of $\mathcal{H}$ as follows

$$\mathcal{K} = \{y \in \mathcal{H} | Ty(t) = -y(t)\}.$$

For any $y \in \mathcal{K}$ satisfies $Ty(t) = y(t-1) = -y(t)$ and $\|Ty(t)\| = \|y(t)\|$, which makes any $a_j = (-1)^{j+1} a_j$. Thus, $a_j = 0$ as j is even. Now, $y \in \mathcal{K}$ has the following Fourier expansion

$$y(t) = \sum_{j=-\infty}^{+\infty} a_j e^{i(2j-1)\pi t}.$$

Lemma 3 The existence of 2-periodic solutions y of (12) satisfying $y(t-1) = -y(t)$ is equivalent to the existence of critical points of G restricted to $\mathcal{K}$.

Proof Because we have

$$\nabla D(Ty(t)) = T\nabla D(y(t))$$

and

$$\begin{aligned}\langle TG'(y), z(t)\rangle &= \int_0^2 (J\dot{y}(t-1)) + \nabla D(y(t-1)), z(t))\, dt \\ &= -\int_0^2 ((J\dot{y}(t) + \nabla D(y(t)), z(t)))\, dt \\ &= \langle -G'(y), z(t)\rangle,\end{aligned}$$

we find $\phi'(x) \in \mathcal{K}$ and $G'(x) \in \mathcal{K}$.

If $y \in \mathcal{K}$ is a critical point of G, then $< G'(y), z >= 0$ for all $z \in \mathcal{K}$. In addition, $< G'(y), z >= 0$ for all $z \in \mathcal{K}^{\perp}$. Thus, $< G'(y), z >= 0$ for all $z \in \mathcal{H}$. Now, we can say that critical points of G that are restricted to $\mathcal{K}$ are also critical points of G on the entire space $\mathcal{H}$, which corresponds to 2-periodic solutions of (12) satisfying symmetric structure $y(t-1) = -y(t)$. □

3 Preliminary results

Before we prove the primary results of this study, we must state certain required lemma. Let $\mathbb{K}$ be a Hilbert space. Following index theory, we let Θ be the family of closed subsets of $\mathbb{K}$ that are symmetric with respect to 0, i.e.

$$\Theta = \{Q \subset \mathbb{K} |\ Q \text{ is closed and } x \in Q \text{ if and only if } -x \in Q\}$$

and Ω be the set of odd and continuous mappings $\mathbb{K} \to \mathbb{K}$. For any $Q \in \Theta$, the Z_2-geometrical index $\omega(Q)$ of Q is defined by

$$\omega(Q) = \begin{cases} 0, & \text{if } Q = \emptyset, \\ \min\{t \in \mathbb{N} |\ \text{there is a odd map } s \in C(Q, \mathbb{R}^t \setminus \{0\})\}, \\ \infty, & \text{if there is no finite integer } t, \text{ such that there exists } s \in C(Q, \mathbb{R}^t \setminus \{0\}). \end{cases}$$

We let $\Psi : \mathbb{K} \to \mathbb{R}$ be a functional of the form

$$\Psi(y) = \frac{1}{2} < Ly, y > +\Phi(y),$$

where $L : \mathbb{K} \to \mathbb{K}$ is linear, continuous, symmetric and invariant, and $\Phi : \mathbb{K} \to \mathbb{R}$ is of class C^1 and $\nabla\Phi : \mathbb{K} \to \mathbb{K}$ are compact. We define

$$\mathit{\Lambda} = \{e^{kL} |\ k \in \mathbb{R}\}.$$

We let Ω^* denote a set of mappings such that for any $s \in \Omega^*$
($\mathfrak{b}1$) $s \in \Omega$, i.e., s is continuous and odd;
($\mathfrak{b}2$) s is a homeomorphism of the form $h + \tau$, where τ is compact and $h \in \Lambda$.
If we assume that $V^+ \in \Theta$ is a L-invariance subspace of $\mathcal{K}$, we set

$$\hat{\omega}^* = \{\Theta^*, \omega^*\}, \quad \text{where } \omega^*(Q) = \min_{s\in\Omega^*} \omega(s(Q) \cap V^+).$$

$\hat{\omega}^*$ is called a pseudo-index with respect to the Z_2-geometrical index ω.

We denote $\sigma_\epsilon(L)$ as the essential spectrum of L and let $0 \notin \sigma_\epsilon(L)$; thus, 0 is an isolated eigenvalue of finite multiplicity or 0 is one of the solutions. Therefore, $\mathbb{K}$ can be decomposed as follows

$$\mathbb{K} = V^+ \oplus V^0 \oplus V^-,$$

where $V^0 = \ker(L)$.

An operator P is a compact, promoted by the theorem in (cf. [10]), $\sigma_\epsilon = \sigma_\epsilon(L + P)$. Then, $\mathbb{K}$ has another decomposition of the direct sum

$$\mathbb{K} = W^- \oplus W^0 \oplus W^+,$$

where, $W^0 = \ker(L + P)$.

We require the following lemma from (cf. [2]) before proving the primary results of this study.

Lemma 4 We let $\mathbb{K}$ be a Hilbert space and let $\Psi \in C^1(\mathbb{K}, \mathbb{R})$ be a functional that satisfies the following assumptions
(**a1**) $\Psi(y) = \frac{1}{2} < Ly, y > +\Phi(y)$, where L is a bounded self-adjoint operator and Φ' is compact.
(**a2**) Ψ is even in the sense $\Psi(-y) = \Psi(y)$, for any $y \in \mathbb{K}$.
(**a3**) $0 \notin \sigma_\varepsilon(L)$.
(**a4**) $\Phi(0) = 0$;
(**a5**) There exists a compact linear operator P, such that $\Phi'(y) = Py + o(\|y\|)$ as $\|y\| \to 0$;
(**a6**) $\lim\limits_{\|y\|\to+\infty} \frac{\|\Phi'(y)\|}{\|y\|} = 0$;
(**a7**) Every sequence $\{y_n\}$ such that $\Psi(y_n) \to c$ and $\|\Psi'(y_n)\| \to 0$ is bounded as $n \to +\infty$.

If the integer

$$\overline{k} = \dim(V^+ \cap W^-) - \mathrm{codim}_{\mathcal{K}}(V^+ + W^-)$$

is well defined and positive, then the numbers

$$\varepsilon_k = \inf_{\hat{\omega}^*(Q)\geq k} \sup_{y\in Q} \Psi(y), \text{ for } k = 1, 2, \cdots, \overline{k}$$

are critical values of Ψ and $-\infty < \varepsilon_1 \leq \cdots \leq \varepsilon_{\overline{k}} < 0$. if $\varepsilon = \varepsilon_k = \cdots = \varepsilon_{k+l}$, then $\omega(\mathcal{K}_\varepsilon) \geq l + 1$, where $\mathcal{K}_\varepsilon = \{y \in \mathcal{K} |\ \Psi(y) = \varepsilon, \Psi'(y) = 0\}$.

4 Proofs of the primary results

Let $\mathcal{L}, \mathcal{A}_0, \mathcal{A}_\infty : \mathcal{K} \to \mathcal{K}$ by extending the bilinear forms

$$\begin{aligned}
< \mathcal{L}y, z > &= \int_0^2 (J\dot{y}(t), z(t))dt,\\
< \mathcal{A}_0 y, z > &= \int_0^2 (\overline{A}_0 y(t), z(t))dt,\\
< \mathcal{A}_\infty y, z > &= \int_0^2 (\overline{A}_\infty y(t), z(t))dt, \forall\ y, z \in \mathcal{K},
\end{aligned}$$

where $\overline{A}_0 = \mathrm{diag}(2I, A_0), \overline{A}_\infty = \mathrm{diag}(2I, A_\infty)$ are symmetric matrices. Then $\mathcal{L}, \mathcal{A}_0, \mathcal{A}_\infty$ are bounded self-adjoint operators. According to the Sobolev inequalities, $\mathcal{A}_0, \mathcal{A}_\infty$ are compact.

We set

$$D_0(y) = D(y) - \frac{1}{2}(\overline{A}_0 y, y),\ D_\infty(y) = D(y) - \frac{1}{2}(\overline{A}_\infty y, y).$$

Then the two operators, defined as follows

$$\phi_0(y) = \int_0^2 D_0(y(t))dt, \quad \phi_\infty(y) = \int_0^2 D_\infty(y(t))dt,$$

satisfies

$$\phi_0(y) = \phi(y) - \frac{1}{2} < \mathcal{A}_0 y, y >, \phi_\infty(y) = \phi(y) - \frac{1}{2} < \mathcal{A}_\infty y, y > .$$

We reasonably determine that

$$\phi_0'(y) = \phi'(y) - \mathcal{A}_0 y,\ \phi_\infty'(y) = \phi'(y) - \mathcal{A}_\infty y.$$

The compactness of ϕ' and $\mathcal{A}_0, \mathcal{A}_\infty$ implies that both ϕ_0' and ϕ_∞' are compact.

We also define two operators L_0 and L_∞ at the origin and at infinity, respectively, as follows

$$L_0 y = \mathcal{L}y + \mathcal{A}_0 y, \quad L_\infty y = \mathcal{L}y + \mathcal{A}_\infty y. \tag{16}$$

The former variational functional $G(y)$ can be rewritten as

$$G(y) = \frac{1}{2} < L_0 y, y > + \phi_0(y) = \frac{1}{2} < L_\infty y, y > + \phi_\infty(y).$$

Using Lemma 4, we can determine the existence of the critical points of functional G as well as its multiplicity. We should verify that (**a1**)-(**a7**) are valid with the equipment of assumptions (**A1**)-(**A3**).

Lemma 5 We assume that f satisfies (**A1**) and (**A2**). Then, G satisfies the assumptions (**a1**)-(**a6**).

Proof Based on (16), L_∞ is a bounded self-adjoint operator, while $\mathcal{L}$ and $\mathcal{A}_\infty$ are bounded, self-adjoint operators. Because $\phi'_\infty(y)$ is compact, (**a1**) holds.

According to assumption (**A1**)

$$D(-y) = |-y_1|^2 + F(-y_2) = D(y).$$

Thus, $D(-y) = D(y)$, which implies that (**a2**) is satisfied.

Directly, the essential spectrum of the operator $\mathcal{L}$ is ± 1. Since L_∞ is compact perturbations of $\mathcal{L}$, based on well-known theorems (cf. [2]), we have (**a3**) that $0 \notin \sigma_\varepsilon(L_\infty)$.

From the definition of ϕ_∞, we can determine that $\phi_\infty(0) = 0$, which satisfies (**a4**).

Because of (**a5**), for any $\epsilon_1 > 0$, we discuss similar as Xiao and Guo (cf. [30]) and obtain

$$|\nabla D(y) - \overline{A}_0 y| \le \varepsilon_1 |y| + c_1 |y|^2, \ \forall\ y \in \mathbb{R}^{2N}$$

where $c_1 = \max\{\varepsilon_1 \delta_1^{-2}, \|\overline{A}_0\|_M + \|\overline{A}_\infty\|_M + 1\}$ and $\|\cdot\|_M$ denotes the usual matrix norm. Using the Sobolev embedding theorem, for some positive $c_2 > 0$, $\|y\|_{L^2} \le \|y\|$ and $\|y\|_{L^4} \le c_2 \|y\|$, where $\|\cdot\|_{L^p}$ is regarded as the norm in $L^p(S^1, \mathbb{R}^{2N})$, and $p = 2, 4$. Then, we get

$$\begin{aligned} |\int_0^2 (\nabla D(y) - \overline{A}_0 y, z) dt| &\le \int_0^2 |\nabla D(y) - \overline{A}_0 y|\ |z| dt \\ &\le \varepsilon_1 \int_0^2 |y|\ |z| dt + c_1 \int_0^2 |y|^2\ |z| dt \\ &\le \varepsilon_1 \|y\| \cdot \|z\| + c_1 \|y\|_{L^4}^2 \cdot \|z\|, \\ &\le \varepsilon_1 \|y\| \cdot \|z\| + c_1 c_2^2 \|y\|^2 \cdot \|z\|. \end{aligned}$$

Let

$$\mathcal{A} y = \mathcal{A}_0 y - \mathcal{A}_\infty y.$$

Thus, we obtain

$$\begin{aligned} |< \phi'_\infty(y) - \mathcal{A}(y), z >| &= |\int_0^2 ((\nabla D(y) - \overline{A}_\infty y) - (\overline{A}_0 - \overline{A}_\infty) y, z) dt| \\ &= |\int_0^2 (\nabla D(y) - \overline{A}_0 y, z) dt| \end{aligned}$$

$$\leq \varepsilon_1\|y\| \cdot \|z\| + c_1 c_2^2 \|y\|^2 \cdot \|z\|.$$

Thus,

$$\limsup_{\|y\|\to 0} \frac{\|\phi'_\infty(y) - \mathcal{A}y\|}{\|y\|} = \limsup_{\|y\|\to 0} \sup_{\|z\|=1} \frac{|<\phi'_\infty(y) - \mathcal{A}y, z>|}{\|y\|} \leq \varepsilon_1.$$

where ε_1 has arbitrariness, and (**a5**) is satisfied.

For any $\varepsilon_2 > 0$, there exists $c_3 > 0$ such that

$$|\nabla D(y) - \overline{A}_\infty y| < \varepsilon_2 \mathrm{y} + c_3, \quad \forall\, y \in \mathbb{R}^{2N}.$$

Using the Hölder's inequality

$$|<\phi'_\infty(y), z>| \leq |\int_0^2 (\nabla D(y(t)) - \overline{A}_\infty y(t), z(t))dt| \leq \varepsilon_2\|y\|\|z\| + c_3\sqrt{2}\|z\|.$$

Assuming that $\|z\| = 1$, then

$$\|\phi'_\infty(y)\| \leq \varepsilon_2\|y\| + \sqrt{2}c_3.$$

Thus,

$$\limsup_{\|y\|\to +\infty} \frac{\|\phi'_\infty(y)\|}{\|y\|} \leq \varepsilon_2.$$

The arbitrariness of ε_2 implies that (**a6**) holds. □

Regarding (**a7**), we must decompose the space $\mathcal{K}$ by its positive, negative and null eigenvalues. We denote

$$\mathbb{L}^2_{odd}(S^1, \mathbb{R}^{2N}) = \{x \in L^2(S^1, \mathbb{R}^{2N}) |\ y(t-1) = -y(t) \text{ a.e. for } t \in [0,2]\},$$
$$\mathcal{H}^1_{odd}(S^1, \mathbb{R}^{2N}) = \{x \in \mathbb{L}^2_{odd}(S^1, \mathbb{R}^{2N}) | \int_0^2 [|y(t)|^2 + |\dot{y}(t)|^2]dt < \infty\}.$$

Referring to (cf. [10]), we know that $\mathcal{H}^1_{odd}(S^1, \mathbb{R}^2) \subset \mathcal{K} \subset \mathbb{L}^2_{odd}(S^1, \mathbb{R}^2)$. We thus define two operators $\mathcal{L}_0, \mathcal{L}_\infty : \mathcal{H}^1_{odd} \subset \mathbb{L}^2_{odd} \to \mathbb{L}^2_{odd}$ as follows

$$\mathcal{L}_0 y(t) = J\dot{y}(t) + \overline{A}_0 y(t), \quad \mathcal{L}_\infty y(t) = J\dot{y}(t) + \overline{A}_\infty y(t), \text{ for any } y \in \mathcal{H}^1_{odd}.$$

Each $y \in \mathcal{H}^1_{odd}$ has the following Fourier expansion

$$y(t) = \sum_{j=-\infty}^{\infty} a_j e^{i(2j-1)\pi t}, \ \overline{a_j} = a_{-j+1}.$$

$y(t)$ is substituted into $\mathcal{L}_\infty y(t)$ such that

$$\mathcal{L}_\infty y(t) = \sum_{j=-\infty}^{\infty} (i(2j-1)\pi J + \overline{A}_\infty) a_j e^{i(2j-1)\pi t}.$$

We assume that ξ is an eigenvalue of L_∞ and $0 \neq y \in \mathcal{H}^1_{odd}$ is an eigenvector corresponding to ξ. Then, for any $z \in \mathcal{K}$, $z(t) = \sum\limits_{j=-\infty}^{\infty} b_j e^{i(2j-1)\pi t}$, $\overline{b}_j = b_{-j+1}$, we obtain

$$< L_\infty y, z >= \xi < y, z >= 2\xi \sum_{j=1}^{\infty} [1 + (2j-1)\pi][(a_j, b_j) + (a_{-j+1}, b_{-j+1})].$$

Correspondingly, $< L_\infty y, z >$ has another form as follows

$$\begin{aligned} &< L_\infty y, z >= \int_0^2 (\mathcal{L}_\infty y, z) dt \\ &= 2\sum_{j=1}^{\infty} [((i(2j-1)\pi J + \overline{A}_\infty) a_j, b_j) + ((i(-2j+1)\pi J + \overline{A}_\infty) a_{-j+1}, b_{-j+1})]. \end{aligned}$$

For any $j \in \mathbb{N}$, we use $z_k(t) = \frac{1}{2}(e^{i(2j-1)\pi t} + e^{-i(2j-1)\pi t}) e_k$, $z_k(t) = \frac{i}{2}(e^{i(2j-1)\pi t} - e^{-i(2j-1)\pi t}) e_k$, $k = 1, 2, \cdots, 2N$, where $\{e_1, e_2, \cdots, e_{2N}\}$ forms a canonical basis of $\mathbb{R}^{2N}$. Then, we have

$$Re[(i(2j-1)\pi J + \overline{A}_\infty) a_j] = \xi[1 + (2j-1)\pi] Re(a_j),$$

where $Re(a_j)$ is the real part of complex vector a_j.

We denote by $Im(a_j)$ the imaginary part of complex vector a_j. Using a similar argument as above, we obtain

$$Im[(i(2j-1)\pi J + \overline{A}_\infty) a_j] = \xi[1 + (2j-1)\pi] Im(a_j).$$

Therefore, for some $j \in \mathbb{N}$, $\xi[1+(2j-1)\pi]$ is an eigenvalue of $i(2j-1)\pi J + \overline{A}_\infty$. Based on the definition of the space $\mathcal{K}$, ξ is an eigenvalue of L_∞ if and only if $\xi[1 + (2j-1)\pi]$ is an eigenvalue of $i(2j-1)\pi J + \overline{A}_\infty$ for some j.

Thus, for any $j \in \mathbb{N}$, we must study the following eigenvalue problem

$$\begin{cases} [i(2j-1)\pi J + \overline{A}_\infty] y = \xi[1 + (2j-1)\pi] y \\ y \in \mathbb{C}^{2N} \end{cases}.$$

Let $u_{j,1}, u_{j,2}, \cdots, u_{j,2N}$ be eigenvalues of $i(2j-1)\pi J + A_\infty$ and $v_{j,1}, v_{j,2}, \cdots, v_{j,2N}$ be the corresponding eigenvectors, which form an orthogonal basis of $\mathbb{C}^{2N}$. Then, we can say

$$\begin{aligned} &e_{j,k}^{(Re)}(t) = v_{j,k} e^{i(2j-1)\pi t} + \overline{v_{j,k}} e^{-i(2j-1)\pi t}, \\ &e_{j,k}^{(Im)}(t) = i[v_{j,k} e^{i(2j-1)\pi t} - \overline{v_{j,k}} e^{-i(2j-1)\pi t}],\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N. \end{aligned}$$

$\{e_{j,k}^{(Re)},\ e_{j,k}^{(Im)} |\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}$ thus forms a complete orthogonal basis of $\mathcal{K}$.

Similarly, the operator L_0 and the corresponding eigenvalue are as follows

$$\begin{cases} [i(2j-1)\pi J + \overline{A}_0]y = \xi[1 + (2j-1)\pi]y \\ y \in \mathbb{C}^{2N} \end{cases}.$$

Let $\tilde{u}_{j,1}, \tilde{u}_{j,2}, \cdots, \tilde{u}_{j,2N}$ be eigenvalues of $i(2j-1)\pi J + \overline{A}_0$ and $\tilde{v}_{j,1}, \tilde{v}_{j,2}, \cdots, \tilde{v}_{j,2N}$ be the corresponding eigenvectors, which form an orthogonal basis of $\mathbb{C}^{2N}$. In this case, another orthogonal basis of $\mathcal{K}$ of the form

$$\tilde{e}_{j,k}^{(Re)}(t) = \tilde{v}_{j,k}e^{i(2j-1)\pi t} + \overline{\tilde{v}_{j,k}}e^{-i(2j-1)\pi t},$$
$$\tilde{e}_{j,k}^{(Im)}(t) = i[\tilde{v}_{j,k}e^{i(2j-1)\pi t} - \overline{\tilde{v}_{j,k}}e^{-i(2j-1)\pi t}],\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N.$$

$\{\tilde{e}_{j,k}^{(Re)},\ \tilde{e}_{j,k}^{(Im)} |\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}$ thus forms a complete orthogonal basis of $\mathcal{K}$.

We can decompose the space $\mathcal{K}$ into a subspace as follows

$$V^+ = \overline{\text{span}\{e_{j,k}^{(Re)}, e_{j,k}^{(Im)} |\ u_{j,k} > 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}},$$
$$V^0 = \overline{\text{span}\{e_{j,k}^{(Re)}, e_{j,k}^{(Im)} |\ u_{j,k} = 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}},$$
$$V^- = \overline{\text{span}\{e_{j,k}^{(Re)}, e_{j,k}^{(Im)} |\ u_{j,k} < 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}}$$

and

$$W^+ = \overline{\text{span}\{\tilde{e}_{j,k}^{(Re)}, \tilde{e}_{j,k}^{(Im)} |\ \tilde{u}_{j,k} > 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}},$$
$$W^0 = \overline{\text{span}\{\tilde{e}_{j,k}^{(Re)}, \tilde{e}_{j,k}^{(Im)} |\ \tilde{u}_{j,k} = 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}},$$
$$W^- = \overline{\text{span}\{\tilde{e}_{j,k}^{(Re)}, \tilde{e}_{j,k}^{(Im)} |\ \tilde{u}_{j,k} < 0,\ j \in \mathbb{N}, k = 1, 2, \cdots, 2N\}}.$$

Now we are preparing to calculate the index in Lemma 4.

Lemma 6 Assume that (**A1**)-(**A3**) hold.

$$\dim(V^+ \cap W^-) - \text{codim}_{\mathcal{K}}(V^+ + W^-) = 2\#(A_0) - 2\#(A_\infty);$$
$$\dim(V^- \cap W^+) - \text{codim}_{\mathcal{K}}(V^- + W^-) = 2\#(A_\infty) - 2\#(A_0).$$

Proof For any $j \in \mathbb{N}$, we denote

$$\mathcal{K}_j = \{ae^{(i(2j-1)\pi t} + \overline{a}e^{-i(2j-1)\pi t} |\ a \in \mathbb{C}^{2N}\}.$$

Clearly, $\mathcal{K}_j$ is a subspace of $\mathcal{K}$ with dimensions $4N$ and $\mathcal{K} = \bigoplus_{j=1}^{+\infty} \mathcal{K}_j$.

We set

$$V_j^{\pm} = \mathcal{K}_j \cap V^{\pm}, \quad W_j^{\pm} = \mathcal{K}_j \cap W^{\pm}.$$

For $2N \times 2N$ real symmetric matrix P, define

$$\Sigma(P) = \text{numbers of negative eigenvalues of } \ P,$$
$$\Sigma^*(P) = \text{numbers of nonpositive eigenvalues of } \ P.$$

Then one obtains

$$\dim_{\mathcal{K}_j} V_j^+ = 4N - 2\Sigma^*(i(2j-1)\pi J + \overline{A}_\infty), \ \dim W_j^- = 2\Sigma(i(2n-1)\pi J + \overline{A}_0).$$

Therefore,

$$\dim(V_j^+ + W_j^-) = \dim V_j^+ + \dim W_j^- - \dim(V_j^+ \cap W_j^-),$$

and, because V_j^+ and $W_j^- \subset \mathcal{K}_j$

$$\dim(V_j^+ + W_j^-) = 4N - \operatorname{codim}_{\mathcal{K}_j}(V_j^+ + W_j^-).$$

Thus

$$\begin{aligned}
&\dim(V^+ \cap W^-) - \operatorname{codim}_{\mathcal{K}}(V^+ + W^-) \\
&\quad = \sum_{j=1}^{+\infty} [\dim(V_j^+ \cap W_j^-) - \operatorname{codim}_{\mathcal{K}_j}(V_j^+ + W_j^-)] \\
&\quad = \sum_{j=1}^{+\infty} [2\Sigma(i(2j-1)\pi J + \overline{A}_0) - 2\Sigma^*(i(2j-1)\pi J + \overline{A}_\infty)] \\
&\quad = \sum_{j=1}^{+\infty} [\Sigma(i(2j-1)\pi J + \overline{A}_0) - \Sigma(i(2j-1)\pi J + \overline{A}_\infty)]. \qquad (17)
\end{aligned}$$

The last equality holds because of (**A3**), which implies that $V^0 = \{0\}$.

Let's compute eigenvalues of $i(2j-1)\pi J + \overline{A}_0$. Denote $I_{2N} = \operatorname{diag}(I, I)$. Set

$$\det(\lambda I_{2N} - (i(2j-1)\pi J + \overline{A}_0)) = 0$$

Claim: $\lambda = 2$ is not an eigenvalue of $i(2j-1)\pi J + \overline{A}_0$.
Suppose to the opposite that 2 is an eigenvalue of $i(2j-1)\pi J + \overline{A}_0$. Then

$$\det(2I_{2N} - (i(2j-1)\pi J + \overline{A}_0)) = \det \begin{pmatrix} 0 & i(2j-1)\pi I \\ -i(2j-1)\pi I & \lambda I - A_0 \end{pmatrix} = 0,$$

that is, $-(2j-1)^2\pi^2 = 0$, which is impossible. Thus the claim holds.
It follows that

$$\begin{aligned}
\det(\lambda I_{2N} - (i(2j-1)\pi J + \overline{A}_0)) &= \det \begin{pmatrix} (\lambda-2)I & i(2j-1)\pi I \\ -i(2j-1)\pi I & \lambda I - A_0 \end{pmatrix} \\
&= \det \begin{pmatrix} (\lambda-2)I & i(2j-1)\pi I \\ 0 & \lambda I - A_0 - \frac{(2j-1)^2\pi^2 I}{\lambda - 2} \end{pmatrix} \\
&= (\lambda-2)^{N-1} \det((\lambda-2)(\lambda I - A_0) - (2j-1)^2\pi^2 I) = 0. \qquad (18)
\end{aligned}$$

Assume that $\lambda_{01}, \lambda_{02}, \cdots, \lambda_{0N}$ are also eigenvalues of A_0. To solve (18), we can equivalently solve the following equation

$$(\lambda-2)(\lambda-\lambda_{0i}) - (2j-1)^2\pi^2 = 0,$$

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and obtain

$$\lambda = \frac{(\lambda_{0i}+2) \pm \sqrt{(\lambda_{0i}-2)^2 + 4(2j-1)^2\pi^2}}{2}. \tag{19}$$

We conclude that for each eigenvalue λ_{0i} of A_0, $i(2j-1)\pi J + \overline{A}_0$ has two eigenvalues, given by (19).
Case I: $\lambda_{0i} \le 0$, then $i(2j-1)\pi J + \overline{A}_0$ has one positive and one negative eigenvalues.
Case II: $\lambda_{0i} > 0$, three subcases happen.
Subcase i: $0 < j < \lfloor \frac{\sqrt{2\lambda_{0i}}+\pi}{2\pi} \rfloor$, then $\lambda_{0i}+2 > (\lambda_{0i}-2)^2 + 4(2j-1)^2\pi^2/2$ and $i(2j-1)\pi J + \overline{A}_0$ has two positive eigenvalues.
Subcase ii: $j > \lfloor \frac{\sqrt{2\lambda_{0i}}+\pi}{2\pi} \rfloor$, then $\lambda_{0i}+2 < (\lambda_{0i}-2)^2 + 4(2j-1)^2\pi^2/2$ and $i(2j-1)\pi J + \overline{A}_0$ has one positive and one negative eigenvalues.
Subcase iii: $j = \lfloor \frac{\sqrt{2\lambda_{0i}}+\pi}{2\pi} \rfloor$, then $\lambda_{0i}+2 < (\lambda_{0i}-2)^2 + 4(2j-1)^2\pi^2/2$ and $i(2j-1)\pi J + \overline{A}_0$ has one positive and one null eigenvalues.

The same discussion can be done for $i(2j-1)\pi J + \overline{A}_\infty$. Following (17), we conclude that

$$\begin{aligned}
&\dim(V^+ \cap W^-) - \operatorname{codim}_{\mathcal{K}}(V^+ + W^-) \\
&\quad = \sum_{j=1}^{+\infty} [2\Sigma(i(2j-1)\pi J + \overline{A}_0) - 2\Sigma(i(2j-1)\pi J + \overline{A}_\infty)] \\
&\quad = 2\sum_{i=1}^{N} j(\lambda_{0i}) - 2\sum_{i=1}^{N} j(\lambda_{\infty i}) = 2\#(A_0) - 2\#(A_\infty).
\end{aligned}$$

Arguing similarly as the above procedure, we obtain

$$\begin{aligned}
&\dim(V^- \cap W^+) - \operatorname{codim}_{\mathcal{K}}(V^- + W^-) \\
&\quad = \sum_{j=1}^{+\infty} [\dim(V_j^- \cap W_j^+) - \operatorname{codim}_{\mathcal{K}_j}(V_j^- + W_j^+)] \\
&\quad = \sum_{j=1}^{+\infty} [2\Sigma(i(2j-1)\pi J + \overline{A}_\infty) - 2\Sigma^*(i(2j-1)\pi J + \overline{A}_0)] \\
&\quad = 2\#(A_\infty) - 2\#(A_0).
\end{aligned}$$

□

Recalling (**a7**), a sequence $\{y_n\}$ is called a $(PS)_c$ sequence of G if $G(y_n) \to c$ and $G'(y_n) \to 0$ for the definition.

Lemma 7 Assume that f satisfies (**A1**), (**A2**) and (**A3**). Then, each $(PS)_c$ sequence is bounded and (**a7**) holds.

Proof $\{y_n\} \subset \mathcal{K}$ is a $(PS)_c$ sequence. The assumption (**A3**) implies that $V^0 = \{0\}$. Then, we can set $y_n = y_n^+ + y_n^-$, where $y_n^+ \in V^+$ and $y_n^- \in V^-$. There exists $c_4 > 0$ such that

$$< L_\infty y, y > \le -c_4\|y\|, \forall y \in V^-,$$

$$< L_\infty y, y > \geq c_4\|y\|, \forall y \in V^+.$$

Subsequently,

$$< L_\infty y_n, y_n^+ - y_n^- > = < L_\infty y_n^+, y_n^+ > - < L_\infty y_n^-, y_n^- > \geq c_4\|y_n\|^2. \tag{20}$$

For an arbitrary $\varepsilon_3 > 0$, there exists a constant $c_5 > 0$ such that

$$|\nabla D(y) - \overline{A}_\infty y| \leq \varepsilon_3|y| + c_5, \forall\ y \in \mathbb{R}^{2N}.$$

Following from the Hölder inequality that

$$\begin{aligned} < L_\infty y_n, y_n^+ - y_n^- > &= < G'(y_n), y_n^+ - y_n^- > - \int_0^2 (\nabla D(y_n) - \overline{A}_\infty y_n, y_n^+ - y_n^-)dt \\ &\leq \varepsilon_4\|y_n\| + \int_0^2 (\varepsilon_3|y_n| + c_5)|y_n^+ - y_n^-|dt \\ &\leq \varepsilon_4\|y_n\| + \varepsilon_3\|y_n\|^2 + \sqrt{2}c_5\|y_n\|, \end{aligned} \tag{21}$$

where $\|G'(y_n)\| \leq \varepsilon_4$ as $n \to \infty$ and ε_4 can be sufficiently small. Because ε_3 is arbitrary, (20) and (21) imply that $\{y_n\}$ is bounded. □

Proof of Theorem 1. According to Lemmas 5 and 7, all assumptions of Lemma 4 are satisfied under the hypotheses (**A1**)-(**A3**). When $|\#(A_\infty)| > |\#(A_0)|$, Lemma 6 states that

$$\overline{k} = \dim(V^+ \cap W^-) - \mathrm{codim}_{\mathcal{K}}(V^+ + W^-) = 2\#(A_0) - 2\#(A_\infty).$$

We let two constants $\varepsilon_0 < \varepsilon_\infty < 0$ such that the numbers

$$\varepsilon_k = \inf_{\hat{\omega}^*(Q)\geq k} \sup_{y\in Q} G(y), \ \text{for } k = 1, 2, \cdots, \overline{k}$$

are critical values of G and $\varepsilon_0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_{\overline{k}} < 0$.

If $\varepsilon = \varepsilon_k = \cdots = \varepsilon_{k+r}$, $\omega(\mathcal{K}_\varepsilon) \geq r + 1$, where $\mathcal{K}_\varepsilon = \{y \in \mathcal{K}|\ G(y) = \varepsilon, G'(y) = 0\}$. Then, we can say that G possesses at least $2\#(A_0) - 2\#(A_\infty)$ critical values.

If $\overline{y}$ is a critical point, whose critical value is ε_k, it is a nonconstant 2-periodic solution because $G(\overline{y}) = \varepsilon_k < 0$. We have at least $2\#(A_0) - 2\#(A_\infty)$ nonzero periodic solutions of (8) satisfying $y(t-1) = -y(t)$. Since every nonzero 2-periodic solutions satisfying $y(t-1) = -y(t)$ must be nonconstant 2-periodic solutions, thus, we have at least $2\#(A_0) - 2\#(A_\infty)$ nonconstant periodic solutions of (8). However, if y is a solution of (8), then $-y$ is also a solution of (8) and satisfies $y(t-1) = -y(t)$ for all $t \in \mathbb{R}$. Identifying y with $-y$, we obtain $\#(A_0) - \#(A_\infty)$ nonconstant 2-periodic solutions of (8).

If $|\#(A_\infty)| < |\#(A_0)|$, then we replace G by $-G$ and repeat the above procedure, and obtain that (8) possesses at least $\#(A_\infty) - \#(A_0)$ nonconstant 2-periodic solutions of (8). This finishes the proof of this theorem. □

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5 Example

In this section, we consider the distributed delay system as follows

$$x'(t) = -f(\int_0^1 x(t-s)\,ds), \tag{22}$$

where $\rho_1 = \frac{1}{4}$ and $\rho_2 = 15$,

$$f(x) = \begin{cases} A_0 x + (x_1^5, x_2^5, x_3^5)^\tau & x \in A_{\rho_1} \\ A_\infty x + (x_1^{-\frac{1}{3}}, x_2^{-\frac{1}{3}}, x_3^{-\frac{1}{3}})^\tau & x \notin A_{\rho_2} \end{cases}.$$

We set

$$A_0 = \begin{bmatrix} 11 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad A_\infty = \begin{bmatrix} 250 & 2 & 1 \\ 2 & 50 & 1 \\ 1 & 1 & 40 \end{bmatrix}.$$

Computing directly, we obtain that

$$\sigma(A_0) = \{1.58, 3.731, 11.687\}, \sigma(A_\infty) = \{39.898, 50.077, 250.02\}.$$

We get

$$j(1.58) = -\lfloor \frac{\sqrt{2*1.58}+\pi}{2\pi} \rfloor = 0, \qquad j(3.731) = -\lfloor \frac{\sqrt{2*3.731}+\pi}{2\pi} \rfloor = 0,$$
$$j(11.687) = -\lfloor \frac{\sqrt{2*11.687}+\pi}{2\pi} \rfloor = -1, \qquad j(39.898) = -\lfloor \frac{\sqrt{2*39.898}+\pi}{2\pi} \rfloor = -1,$$
$$j(50.077) = -\lfloor \frac{\sqrt{2*50.077}+\pi}{2\pi} \rfloor = -2, \qquad j(250.02) = -\lfloor \frac{\sqrt{2*250.02}+\pi}{2\pi} \rfloor = -3.$$

Then, $\#(A_0) = -1$ and $\#(A_\infty) = -6$. Subsequently,

$$\dim(V^+ \cap W^-) - \mathrm{codim}_{\mathcal{K}}(V^+ + W^-) = 2\#(A_0) - 2\#(A_\infty) = 10.$$

Thus, (22) has at least 5 nonconstant 2-periodic solutions.

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