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THE COMBINATORICS OF WEIGHTED COHOMOLOGY

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ABSTRACT. In this paper we introduce weighted simplicial cohomology with coefficients in $\mathbb{Q}[[\pi]]$. This cohomology is derived from a weighted coboundary operator incorporating simplex weights from $\mathbb{Q}[[\pi]]$. We provide a particular bi-partition for the set of n-simplices into μ^n and κ^n-simplices and then we establish a four-term exact sequence relating the torsion module of weighted cohomology with regular cohomology having coefficients in $\mathbb{Q}[[\pi]]$. Furthermore, we prove a structure theorem for the torsion, expressing its invariant factors as ratios of weights of distinguished (μ^{n-1}, κ^n)-simplex-pairs. We then employ this result to interpret the long homology sequence arising from a natural map connecting weighted and regular cohomology over $\mathbb{Q}[[\pi]]$. Secondly we leverage weighted homology by a bi-partition into μ_n- and κ_n-simplices with its torsion expressed via a pairing (κ_n, μ_{n-1}). We show that cohomological torsion is described by a pairing of the form (μ^{n-1}, μ_n), which gives rise to an isomorphism between weighted cohomological torsion and $\mathrm{Hom}(\mathrm{Im}\partial_n^v, R)$.

1. Introduction

Constructions incorporating weight parameters have been studied in the context of algebraic invariants of various spaces. For instance, in [1], parametric constructions of Čech and Vietoris-Rips complexes were described, while in [7] certain orbifold cohomologies of weighted projective spaces were computed. In [14] a notion of weighted cohomology groups was analyzed for arithmetic groups, and in [8] simplicial complexes associated to Coxeter systems were studied via weighted L^2-cohomology groups.

In [10], Dawson introduced weighted simplicial complexes and developed a weight dependent homology theory for them. His construction was a simplicial complex equipped with a weight function $v\colon X \to R$, where R was an integral domain, such that for simplices $\sigma, \tau \in X$ with $\sigma \subset \tau$, $v(\sigma) | v(\tau)$ held. He focused on establishing the Eilenberg-Steenrod [12, 9] axioms based on a weighted version of the Mayer-Vietoris [22] sequence and provided a category-theory centered treatment of the subject. The key difference between standard and weighted homology was in the definition of a novel boundary operator that incorporated the weight-function v

$$d_n^v(\sigma) = \sum_{i=0}^{n} \frac{v(\sigma)}{v(\hat{\sigma}_i)} (-1)^i \hat{\sigma}_i.$$

Subsequent contributions [20, 23] were more application focused: an extension of Dawson's framework to persistent homology was presented, and weighted Laplacians were introduced and studied as an approach to weighted cohomology. Using a non-essential difference in definition, [2] studied the weighted homology with coefficients in certain discrete valuation rings, for loop nerves arising from

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pairs of arc diagrams with shared vertices [4, 5]. This was done by connecting weighted simplicial homology to its classical counterpart via explicit chain maps, the latter being an instance of the former with constant weights. This was further extended to interaction structures in [15], and in [16, 3] general formulas for the torsion of weighted homology, as well as tractable computational and algorithmical principles were given. This was achieved in terms of the weights of certain pairings of simplices at subsequent dimensions. These parings were in general non-local, in the sense that the pair needn't have exhibited any face relations.

From an application perspective, weighted simplicial complexes naturally arise in the context of data analysis, with many real world data sets exhibiting simplicial structure [17, 18, 19] and indeed being organized as such [13, 21]. While current methods for topological data analysis abound [6, 21, 24], a prevalent feature of these data-sets is the presence of additional simplex-specific data [11], which now can be encoded using simplicial weights. Hence, the theory of such complexes merits further investigation.

This paper extends the previously mentioned theoretical frameworks, and is organised as follows: in Section 2 we introduce the construction of weighted functional simplicial cohomology with values in $R=\mathbb{Q}[[\pi]]$ and establish some basic properties. In Section 3 we prove an explicit structure theorem for the torsion part of the weighted cohomolgy modules over R. In Section 4 we prove an explicit structure theorem for a cohomological quotient in the context of co-chain maps arising between regular and weighted cohomology. In Section 5 we give a combinatorial interpretation of the isomorphism between weighted torsion and weighted co-torsion at subsequent simplicial dimensions. Finally, in Section 6 study weighted cohomology further, providing an interpretation via $\mathrm{Hom}(\mathrm{Im}\partial_n^v, R)$.

2. Definitions and basic properties

Let X be a simplicial complex and $\omega\colon X \to \mathbb{Z}$ satisfying

$$\sigma \subset \tau \implies \omega(\sigma) \geq \omega(\tau), \quad \forall \sigma,\tau \in X.$$

Given $R=\mathbb{Q}[[\pi]]$, a formal power series ring with transcendental variable π, ω induces a weight function $v\colon X \to R$ by $v(\sigma)=\pi^{\omega(\sigma)}$. The pair (X,v) is called a weighted simplicial complex. Adopting the definition in [4], the weighted boundary operator is given by

$$\partial_n^v\colon C_n(X) \to C_{n-1}(X), \quad \partial_n^v(\sigma) \doteq \sum_{i=0}^{n} \pi^{\omega(\hat{\sigma}_i)-\omega(\sigma)}(-1)^i \hat{\sigma}_i,$$

where $C_n(X)$ is the free R-module over all n-simplices of X. Then, the weighted homology of (X,v) are the R-modules $H_n^v(X) = \mathrm{Ker}\partial_n^v/\mathrm{Im}\partial_{n+1}^v$. Let $v',v\colon X \to R$ be such that $v'(\sigma)|v(\sigma), \forall \sigma \in X$. We have chain maps

$$\theta_n^{v',v}\colon C_n(X) \to C_n(X), \quad \theta_n^{v',v}(\sigma) = \frac{v(\sigma)}{v'(\sigma)}\sigma.$$

Denoting $\theta_n = \theta_n^{v',v}$, the following diagram commutes

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial_n^{v'}} & C_{n-1}(X) \\ \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n-1}} \\ C_n(X) & \xrightarrow{\partial_n^{v}} & C_{n-1}(X). \end{array}$$

For any R-module L, setting $\delta_v^{n-1}(f) = f \circ \partial_n^v$ and $\theta^n(f) = f \circ \theta_n$, the following diagram also commutes

$$\begin{array}{ccc} \mathrm{Hom}(C_n(X),L) & \xleftarrow[\delta_{v'}^{n-1}]{} & \mathrm{Hom}(C_{n-1}(X),L) \\ {\scriptstyle \theta^n}\uparrow & & {\scriptstyle \theta^{n-1}}\uparrow \\ \mathrm{Hom}(C_n(X),L) & \xleftarrow[\delta_{v}^{n-1}]{} & \mathrm{Hom}(C_{n-1}(X),L). \end{array}$$

The weighted cohomology of (X,v), is denoted by $H_v^n(X,L)$ and is the homology of $(\mathrm{Hom}(C_n(X),L),\delta_v^n)$. Let $\theta^n\colon H_v^n(X,L) \longrightarrow H_{v'}^n(X,L)$ denote the induced map. We set $\mathrm{Hom}(C_n(X/\theta),L) \doteq \frac{\mathrm{Hom}(C_n(X),L)}{\theta^n(\mathrm{Hom}(C_n(X),L))}$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(C_{n-1}(X),L) & \xrightarrow{\theta^{n-1}} & \mathrm{Hom}(C_{n-1}(X),L) & \xrightarrow{j} & \mathrm{Hom}(C_{n-1}(X/\theta),L) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_v^{n-1}} & & \downarrow{\scriptstyle \delta_{v'}^{n-1}} & & \downarrow{\scriptstyle \bar{\delta}_{v'}^{n-1}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(C_n(X),L) & \xrightarrow{\theta^n} & \mathrm{Hom}(C_n(X),L) & \xrightarrow{j} & \mathrm{Hom}(C_n(X/\theta),L) & \longrightarrow & 0 \end{array}$$

is commutative. Denoting by $H_{v'}^n(X/\theta,L)$, the homology of $(\mathrm{Hom}(C_n(X/\theta),L),\bar{\delta}_{v'}^n)$, we obtain the long exact sequence

$$H_{v'}^{n-1}(X,L) \xrightarrow{j} H_{v'}^{n-1}(X/\theta,L) \xrightarrow{\delta_{v'}^{n-1}} H_v^n(X,L) \xrightarrow{\theta^n} H_{v'}^n(X,L) \xrightarrow{j} H_{v'}^n(X/\theta,L).$$

In what follows we will fix a finitely generated simplicial complex X and restrict ourselves to the case $L = R$ and $v' \equiv 1_R$, i.e. weighted simplicial cohomology with coefficients in the base ring R, where the v' weighted simplicial cohomology reduces to standard simplicial cohomology with coefficients and values in R. We denote $H_v^n \doteq H_v^n(X,R)$, $C_n \doteq C_n(X)$, $C^n \doteq \mathrm{Hom}(C_n(X),R)$, $H_{v'}^n(X/\theta) \doteq H_{v'}^n(X/\theta,R)$, $C^n(X/\theta) \doteq \mathrm{Hom}(C_n(X/\theta),R)$ and shall omit the v' index label going forward. Finally, we write F_v^n, T_v^n and F_n^v, T_n^v for the free and torsion sub-modules of H_v^n and H_n^v respectively.

3. An explicit structure theorem for T_v^n

In this section we show that the invariant factors of T_v^n are in fact ratios of weights of distinguished pairs of simplices from the weighted complex.

Let $X^n \doteq \{\sigma^n\}$ be the n-simplices of X ordered in increasing ω-value. Clearly,

$$C^n = \langle \lambda_{\sigma^n}\colon C_n \to R \mid \lambda_{\sigma^n}(\sigma) = \delta_{\sigma\sigma^n} \rangle.$$

We bi-partition X^n as follows: let $\mathscr{M}^n = \mathscr{K}^n = \varnothing$. For each σ^n in increasing order we consider $\delta_v^n(\sum_{\sigma\in\mathscr{M}^n} r_\sigma\lambda_\sigma + \lambda_{\sigma^n}) = 0$. If a solution exists over R, we add σ^n to $\mathscr{K}^n$. Otherwise, we add it to $\mathscr{M}^n$. When this process terminates, by construction $X^n = \mathscr{M}^n \dot\cup \mathscr{K}^n \doteq \{\mu^n\}\dot\cup\{\kappa^n\}$.

Lemma 1. *The following assertions hold,*
(a) $\{\Lambda^v_{\kappa^n} = \sum_{\mu^n} r_{\mu^n}\lambda_{\mu^n} + \lambda_{\kappa^n}\}$ *is a* $\mathrm{Ker}\delta^n_v$*-basis where* $r_{\mu^n}v(\mu^n) = u^{\kappa^n}_{\mu^n}v(\kappa^n)$ *holds for* $u^{\kappa^n}_{\mu^n}\in R/(\pi)$.
(b) the following sequence is exact

$$0 \longrightarrow \mathrm{Ker}\delta^n_v \xrightarrow{\theta^n} \mathrm{Ker}\delta^n \longrightarrow \bigoplus_{\kappa^n\in\mathscr{K}^n} R/(v(\kappa^n)) \longrightarrow 0.$$

(c) there exists a $\mathrm{Ker}\delta^n$*-basis* $\{\Lambda_{\kappa^n}\}$*, such that* λ_{κ^n} *appears exclusively in* Λ_{κ^n}*, with coefficient one, and furthermore* $v(\kappa^n)\Lambda_{\kappa^n} = \theta^n(\Lambda^v_{\kappa^n})$ *holds.*

Proof. Clearly $\{\Lambda^v_{\kappa^n}\}$ is a $\mathrm{Ker}\delta^n_v$-basis. To prove (a), it suffices to show
Claim 1: for each $\Lambda^v_{\kappa^n} = \sum_{\mu^n} r_{\mu^n}\lambda_{\mu^n} + \lambda_{\kappa^n}$, we have $r_{\mu^n}v(\mu^n) = u^{\kappa^n}_{\mu^n}v(\kappa^n)$.
By construction $\delta^n_v(\Lambda^v_{\kappa^n}) = 0$, hence $\forall\sigma^{n+1}\in X$,

$$\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}r_{\mu^n}v(\mu^n) + c_{\kappa^n}v(\kappa^n) = 0, \text{for} \quad \kappa^n\subset\sigma^{n+1}$$
$$\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}r_{\mu^n}v(\mu^n) = 0, \text{for} \quad \kappa^n\not\subset\sigma^{n+1},$$

with $c_{\mu^n}, c_{\kappa^n}\in\{1,-1\}$. Writing $v(\mu^n) = \pi^{\omega(\mu^n)}$, $v(\kappa^n) = \pi^{\omega(\kappa^n)}$ and expanding $r_{\mu^n} = \sum_q x_{\mu^n,q}\pi^q$ where $x_{\mu^n,q}\in R/(\pi)$, we obtain

$$\sum_q\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}x_{\mu^n,q}\pi^{q+\omega(\mu^n)} + c_{\kappa^n}\pi^{\omega(\kappa^n)} = 0 \qquad \text{for} \quad \kappa^n\subset\sigma^{n+1}$$
$$\sum_q\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}x_{\mu^n,q}\pi^{q+\omega(\mu^n)} = 0 \qquad \text{for} \quad \kappa^n\not\subset\sigma^{n+1},$$

and taking $[\pi^{\omega(\kappa^n)}]$-coefficients,

$$\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}x_{\mu^n,\omega(\kappa^n)-\omega(\mu^n)} + c_{\kappa^n} = 0 \qquad \text{for} \quad \kappa^n\subset\sigma^{n+1}$$
$$\sum_{\mu^n\subset\sigma^{n+1}} c_{\mu^n}x_{\mu^n,\omega(\kappa^n)-\omega(\mu^n)} = 0 \qquad \text{for} \quad \kappa^n\not\subset\sigma^{n+1}.$$

We make the Ansatz:

$$\bar\Lambda^v_{\kappa^n} = \sum_{\mu^n}\bar r_{\mu^n}\lambda_{\mu^n} + \lambda_{\kappa^n}, \quad \bar r_{\mu^n} = x_{\mu^n,\omega(\kappa^n)-\omega(\mu^n)}\pi^{\omega(\kappa^n)-\omega(\mu^n)}.$$

$\bar\Lambda^v_{\kappa^n}$ is a cocycle with λ_{κ^n}-coefficient one. However, since $\{\delta^n_v(\lambda_{\mu^n})\}$ is a $\mathrm{Im}\delta^n_v$-basis, $\Lambda^v_{\kappa^n}$ is the unique $\mathrm{Ker}\delta^n_v$ element with λ_{κ^n}-coefficient one. As such $\bar\Lambda^v_{\kappa^n} = \Lambda^v_{\kappa^n}$, implying $u^{\kappa^n}_{\mu^n} = x_{\mu^n,\omega(\kappa^n)-\omega(\mu^n)}$ whence

(a). To prove (b) it suffices to show
Claim 2: the following diagram of exact sequences commutes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\delta^n_\nu & \longrightarrow & C^n & \xrightarrow{\delta^n_\nu} & \mathrm{Im}\delta^n_\nu & \longrightarrow & 0 \\
 & & \downarrow \theta^n & & \downarrow \theta^n & & \downarrow \theta^{n+1} & & \\
0 & \longrightarrow & \mathrm{Ker}\delta^n & \longrightarrow & C^n & \xrightarrow{\delta^n} & \mathrm{Im}\delta^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{\kappa^n\in\mathscr{K}^n} R/(\nu(\kappa^n)) & \longrightarrow & \bigoplus_{\sigma^n\in X^n} R/(\nu(\sigma^n)) & \longrightarrow & \bigoplus_{\mu^n\in\mathscr{M}^n} R/(\nu(\mu^n)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Exactness of the middle column is immediate. Note that $\theta^{n+1}(\lambda_{\mu^n}\circ\partial^\nu_{n+1}) = \lambda_{\mu^n}\circ\partial^\nu_{n+1}\circ\theta_{n+1} = \lambda_{\mu^n}\circ\theta_n\circ\partial_{n+1}$, hence $\theta^{n+1}(\delta^n_\nu(\lambda_{\mu^n})) = \nu(\mu^n)\delta^n(\lambda_{\mu^n})$. As such, to show exactness of the right column it suffices to verify that $\mathrm{rnk}(\mathrm{Im}\delta^n_\nu) = \mathrm{rnk}(\mathrm{Im}\delta^n)$. By Claim 1, $\Lambda^\nu_{\kappa^n}\circ\theta_n = \nu(\kappa^n)(\sum_{\mu^n} u^{\kappa^n}_{\mu^n}\lambda_{\mu^n}+\lambda_{\kappa^n})$. Since $\delta^n_\nu(\Lambda^\nu_{\kappa^n}) = \Lambda^\nu_{\kappa^n}\circ\partial^\nu_{n+1} = 0$, we have

$$\delta^n(\Lambda^\nu_{\kappa^n}\circ\theta_n) = (\Lambda^\nu_{\kappa^n}\circ\theta_n)\circ\partial_{n+1} = (\Lambda^\nu_{\kappa^n}\circ\partial^\nu_{n+1})\circ\theta_{n+1} = 0.$$

Therefore,

$$\delta^n(\nu(\kappa^n)(\sum_{\mu^n} u^{\kappa^n}_{\mu^n}\lambda_{\mu^n}+\lambda_{\kappa^n})) = \nu(\kappa^n)\delta^n(\sum_{\mu^n} u^{\kappa^n}_{\mu^n}\lambda_{\mu^n}+\lambda_{\kappa^n}) = 0.$$

Consequently $|\{\delta^n_\nu(\lambda_{\mu^n})\}| = |\{\delta^n(\lambda_{\mu^n})\}|$ as $\mathrm{Im}\delta^n_\nu$- and $\mathrm{Im}\delta^n$-bases respectively, and so exactness of the right column follows, whence (b). To prove (c) it suffices to show
Claim 3: $\{\Lambda_{\kappa^n} = \sum_{\mu^n} u^{\kappa^n}_{\mu^n}\lambda_{\mu^n}+\lambda_{\kappa^n}\}$ is a $\mathrm{Ker}\delta^n$-basis.
Clearly, $\nu(\kappa^n)\Lambda_{\kappa^n} = \Lambda^\nu_{\kappa^n}\circ\theta_n$. This means

$$\nu(\kappa^n)\delta^n(\Lambda_{\kappa^n}) = \delta^n(\nu(\kappa^n)\Lambda_{\kappa^n}) = \delta^n(\Lambda^\nu_{\kappa^n}\circ\theta_n) =$$

$$= (\Lambda^\nu_{\kappa^n}\circ\theta_n)\circ\partial_{n+1} = (\Lambda^\nu_{\kappa^n}\circ\partial^\nu_{n+1})\circ\theta_{n+1} = \delta^n_\nu(\Lambda^\nu_{\kappa^n})\circ\theta_{n+1} = 0,$$

and since $\delta^n_\nu(\Lambda^\nu_{\kappa^n}) = 0$, $\{\Lambda_{\kappa^n}\}\subset\mathrm{Ker}\delta^n$. The fact that they form a basis is immediate, whence (c) and the Lemma. □

Theorem 3.1. *There exists a bi-partition $\mathscr{K}^n = \mathscr{K}^n_-\dot\cup\mathscr{K}^n_+$, and a pairing $(\mu^{n-1},\kappa^n)\in\mathscr{M}^{n-1}\times\mathscr{K}^n_-$, such that the following sequence is exact*

$$0 \longrightarrow T^n_\nu \xrightarrow{\text{incl}} H^n_\nu \xrightarrow{\theta^n} H^n \longrightarrow \bigoplus_{\kappa^n\in\mathscr{K}^n_+} R/(\nu(\kappa^n)) \longrightarrow 0,$$

with $T^n_\nu \cong \bigoplus_{(\mu^{n-1},\kappa^n)\in\mathscr{M}^{n-1}\times\mathscr{K}^n_-} R/\left(\frac{\nu(\mu^{n-1})}{\nu(\kappa^n)}\right)$.

Proof. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{T}^n_v & \longrightarrow & \mathrm{Ker}\delta^n_v & \longrightarrow & \mathfrak{F}^n_v & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow p & \searrow^{q\circ p} & \downarrow p & & \\ 0 & \longrightarrow & T^n_v & \longrightarrow & H^n_v & \xrightarrow{q} & F^n_v & \longrightarrow & 0, \end{array}$$

where $p(\phi)\doteq\phi+\mathrm{Im}\delta^{n-1}_v$, q is the projection onto F^n_v and $\mathfrak{T}^n_v\doteq\mathrm{Ker}(q\circ p)$. As F^n_v is projective, $\mathrm{Ker}(\delta^n_v)=\mathfrak{F}^n_v\oplus\mathfrak{T}^n_v$ where by construction $p(\mathfrak{F}^n_v)=F^n_v$ and $p(\mathfrak{T}^n_v)=T^n_v$. Similarly, $\mathrm{Ker}\delta^n\doteq\mathfrak{F}^n\oplus\mathfrak{T}^n$. Let $\phi\in\mathfrak{T}^n$ and suppose $\exists r\in R,\exists\psi\in C^{n-1}$ such that $r\phi=\delta^{n-1}(\psi)=\psi\circ\partial_n$. Since $\partial_n(\sigma^n)$ produces only ±1 coefficients for σ^n-faces, we have $\phi=\delta^{n-1}(\bar\psi)\in\mathrm{Im}\delta^{n-1}$ where $\psi=r\bar\psi$, and $\mathfrak{T}^n=\mathrm{Im}\delta^{n-1}$ follows.

Claim 1: there exists $\mathscr{K}^n_-\subset\mathscr{K}^n$ such that the following sequence is exact

$$0\longrightarrow\mathfrak{T}^n_v\xrightarrow{\theta^n}\mathfrak{T}^n\longrightarrow\bigoplus_{\kappa^n\in\mathscr{K}^n_-}R/(v(\kappa^n))\longrightarrow 0\ .$$

We show $\theta^n(\mathfrak{T}^n_v)\subset\mathfrak{T}^n$. Note, $\forall\phi\in\mathfrak{T}^n_v,\exists r\in R,\exists\psi\in C^{n-1}$ such that $r\phi=\delta^{n-1}_v(\psi)$. As such,

$$r\theta^n(\phi)=\theta^n(r\phi)=(r\phi)\circ\theta_n=\delta^{n-1}_v(\psi)\circ\theta_n=\psi\circ\theta_{n-1}\circ\partial_n=\delta^{n-1}(\psi\circ\theta_{n-1}).$$

The Claim then follows from $\mathrm{rnk}(\mathfrak{T}^n)=\mathrm{rnk}(\mathrm{Im}\delta^n)=\mathrm{rnk}(\mathrm{Im}\delta^n_v)=\mathrm{rnk}(\mathfrak{T}^n_v)$ and $(\mathfrak{F}^n\oplus\mathfrak{T}^n)/\theta^n(\mathrm{Ker}\delta^n_v)\cong\mathfrak{F}^n/\theta^n(\mathfrak{F}^n_v)\oplus\mathfrak{T}^n/\theta^n(\mathfrak{T}^n_v)\cong\bigoplus_{\kappa^n\in\mathscr{K}^n}R/(v(\kappa^n))$.

By construction $\mathrm{Im}\delta^{n-1}_{(v)}\subset\mathfrak{T}^n_{(v)}$, and we have the following exact sequence

$$0\longrightarrow\mathfrak{T}^n_v/\mathrm{Im}\delta^{n-1}_v\xrightarrow{\theta^n}\mathfrak{T}^n/\theta^n(\mathrm{Im}\delta^{n-1}_v)\xrightarrow{q}\mathfrak{T}^n/\theta^n(\mathfrak{T}^n_v)\longrightarrow 0,$$

where $q(\phi+\theta^n(\mathrm{Im}\delta^{n-1}_v))\doteq\phi+\theta^n(\mathfrak{T}^n_v),\forall\phi\in\mathfrak{T}^n$. We use this to show

Claim 2: the following sequence is exact

$$0\longrightarrow\mathrm{Im}\delta^{n-1}_v\xrightarrow{\mathrm{incl}}\mathfrak{T}^n_v\longrightarrow\bigoplus_{\substack{(\mu^{n-1},\kappa^n)\in\\ \mathscr{M}^{n-1}\times\mathscr{K}^n_-}}R/\left(\frac{v(\mu^{n-1})}{v(\kappa^n)}\right)\longrightarrow 0.$$

Their quotient being full torsion, we can select bases for $\mathfrak{T}^n_v$ and $\mathrm{Im}\delta^{n-1}_v$, $\{\mathfrak{t}^v_j\}$ and $\{\mathfrak{i}_j\}$ respectively, such that $r_j\mathfrak{t}^v_j=\mathfrak{i}_j$ where the r_j's are the invariant factors of T^n_v. This induces the homomorphism $p\colon\mathfrak{T}^n_v\to\mathrm{Im}\delta^{n-1}_v,p(\mathfrak{t}^v_j)=\mathfrak{i}_j$. Furthermore, $\mathfrak{T}^n=\mathrm{Im}\delta^{n-1}$ implies $\mathfrak{T}^n/\theta^n(\mathrm{Im}\delta^{n-1}_v)\cong\bigoplus_{\mu^{n-1}\in\mathscr{M}^{n-1}}R/(v(\mu^{n-1}))$, and $\mathfrak{T}^n/\theta^n(\mathfrak{T}^n_v)\cong\bigoplus_{\kappa^n\in\mathscr{K}^n_-}R/(v(\kappa^n))$ follows from Claim 1. Together with q, these isomorphisms induce the homomorphism q' in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{T}^n_v & \xrightarrow{\theta^n} & \mathfrak{T}^n & \longrightarrow & \bigoplus_{\kappa^n\in\mathscr{K}^n_-}R/(v(\kappa^n)) & \longrightarrow & 0 \\ & & \downarrow p & & & & \uparrow q' & & \\ 0 & \longrightarrow & \mathrm{Im}\delta^{n-1}_v & \xrightarrow{\theta^n} & \mathfrak{T}^n & \longrightarrow & \bigoplus_{\mu^{n-1}\in\mathscr{M}^{n-1}}R/(v(\mu^{n-1})) & \longrightarrow & 0. \end{array}$$

Note, there exists a basis $\{\mathfrak{t}_j\}$ for $\mathfrak{T}^n$ such that $\theta^n(\mathfrak{t}_j^\nu)=\nu(\kappa^n)\mathfrak{t}_j$ and $\theta^n(\mathfrak{i}_j)=\nu(\mu^{n-1})\mathfrak{t}_i$. But then

$$r_j\nu(\kappa^n)\mathfrak{t}_j=r_j\theta^n(\mathfrak{t}_j^\nu)=\theta^n(\mathfrak{i}_j)=\nu(\mu^{n-1})\mathfrak{t}_j.$$

Thus, a pairing $(\mu^{n-1},\kappa^n)\in\mathscr{M}^{n-1}\times\mathscr{K}_-^n$, arises from q via q', which makes the diagram commutative, and for which $r_j\nu(\kappa^n)=\nu(\mu^{n-1})$ holds. Claim 2 then follows. Together with Claim 1 and Lemma 1, the Theorem is proved. □

4. The explicit structure of $H^n(X/\theta)$

Theorem 3.1 provides insight into the long exact sequence

$$H^n\xrightarrow{j}H^n(X/\theta)\xrightarrow{\delta^n}H_\nu^{n+1}\xrightarrow{\theta^{n+1}}H^{n+1}\xrightarrow{j}H^{n+1}(X/\theta).$$

Corollary 2.

$$H^n(X/\theta)\cong T_\nu^{n+1}\bigoplus_{\kappa^n\in\mathscr{K}_+^n}R/(\nu(\kappa^n)).$$

Proof. By construction $H^n(X/\theta)$ is full torsion, and is a direct sum of cyclic modules, being a quotient of $\mathrm{Ker}\bar{\delta}^n\le C^n(X/\theta)=\bigoplus_{\sigma^n\in X^n}R/(\nu(\sigma^n))$. Thus, any $R/(\nu(\sigma^n))$ summand of $H^n(X/\theta)$ is direct. Now, Theorem 3.1 gives rise to the exact sequence

$$0\longrightarrow F_\nu^n\xrightarrow{\theta^n}H^n\xrightarrow{j}\textstyle\bigoplus_{\kappa^n\in\mathscr{K}_+^n}R/(\nu(\kappa^n))\longrightarrow 0.$$

Then, $\mathrm{Ker}\delta^n=\mathrm{Im}\,j=\oplus_{\kappa^n\in\mathscr{K}_+^n}R/(\nu(\kappa^n))$ is a direct summand of $H^n(X/\theta)$, and $\mathrm{Im}\delta^n\cong H^n(X/\theta)/\mathrm{Ker}\delta^n$ is isomorphic to its direct complement. By the long exact sequence and Theorem 3.1, $\mathrm{Im}\delta^n=\mathrm{Ker}\theta^{n+1}=T_\nu^{n+1}$, and the Corollary follows. □

5. A combinatorial interpretation of $T_\nu^n\cong T_{n-1}^\nu$

Theorem 3.1 provides a combinatorial interpretation for the invariant factors of $T_\nu^n=\mathfrak{T}_\nu^n/\mathrm{Im}\delta_\nu^{n-1}$, namely $r_j=\nu(\mu^{n-1})/\nu(\kappa^n)$.

In [16] the homological counterpart to Theorem 3.1, was obtained via a pairing $(\kappa_{n-1},\mu_n)\in\mathscr{K}_{n-1}^-\times\mathscr{M}_n$, where the bi-partition $X^n=\mathscr{M}_n\dot\cup\mathscr{K}_n$ arose instead by checking ,in descending ω-order, the existence of solutions to $\partial_n^\nu(\sum_{\sigma\in\mathscr{M}_n}r_\sigma\sigma+\sigma_n)=0$. Namely T_{n-1}^ν is isomorphic to the last term in the short exact sequence,

$$0\longrightarrow\mathrm{Im}\partial_n^\nu\xrightarrow{\text{incl}}\mathfrak{T}_{n-1}^\nu\longrightarrow\bigoplus_{\substack{(\kappa_{n-1},\mu_n)\in\\ \mathscr{K}_{n-1}^-\times\mathscr{M}_n}}R/\left(\frac{\nu(\kappa_{n-1})}{\nu(\mu_n)}\right)\longrightarrow 0.$$

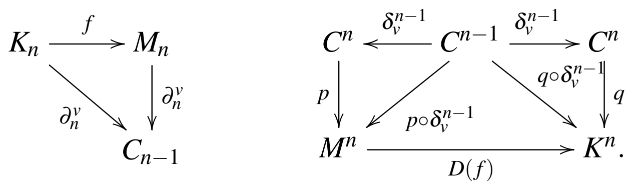
We will provide a combinatorial interpretation of $T_\nu^n\cong T_{n-1}^\nu$ via homological and cohomological μ-simplices.

Denote $M_n=\langle\mathscr{M}_n\rangle$ and $K_n=\langle\mathscr{K}_n\rangle$ as sub-modules of C_n, and $M^n=\mathrm{Hom}(M_n,R)$ and $K^n=\mathrm{Hom}(K_n,R)$.

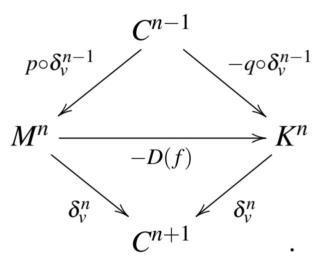
Lemma 3. *Given $\mathscr{M}_n$, there exists a bi-partition $X^n = \mathscr{M}^n \dot\cup \mathscr{K}^n$ such that*
(a) $\langle \delta_v^n(\lambda_{\mu^n}) \mid \mu^n \in \mathscr{M}^n \rangle = \mathrm{Im}\delta_v^n$ *as a basis.*
(b) $\mathscr{M}^n \subset \mathscr{K}_n$ *and* $\mathscr{M}_n \subset \mathscr{K}^n$.

Proof. To prove (a) it suffices to show
Claim: $\langle \{\delta_v^n(\lambda_{\kappa_n})\} \rangle = \mathrm{Im}\delta_v^n$.

Since, $\{\partial_n^v(\mu_n)\}$ is an $\mathrm{Im}\partial_n^v$-basis, there exists $f\colon K_n \to M_n$ such that the two diagrams below commute



In addition, $\delta_v^n \circ (p \circ \delta_v^{n-1}) + \delta_v^n \circ (q \circ \delta_v^{n-1}) = 0$. Consider



Since $(\delta_v^n \circ -D(f)) \circ (p \circ \delta_v^{n-1}) = \delta_v^n \circ (-q \circ \delta_v^{n-1}) = \delta_v^n \circ (p \circ \delta_v^{n-1})$, for $\mathrm{Im}(p \circ \delta_v^{n-1}) \subset M^n$-elements the lower triangle commutes. On the other hand

$$\delta_v^{n-1} = p \circ \delta_v^{n-1} + q \circ \delta_v^{n-1} = p \circ \delta_v^{n-1} + D(f) \circ p \circ \delta_v^{n-1} = (\mathrm{I} + D(f)) \circ (p \circ \delta_v^{n-1}),$$

with $I\colon M^n \to M^n$ being the identity. Thus

$$\mathrm{rnk}(\mathrm{Im}(p \circ \delta_v^{n-1})) = \mathrm{rnk}(\mathrm{Im}\delta_v^{n-1}) = |\mathscr{M}_n| = \mathrm{rnk}(M^n)$$

makes $M^n/\mathrm{Im}(p \circ \delta_v^{n-1})$ full torsion. Namely, for any $\phi \in M^n$ there exists $r \in R$ such that $r\phi \in \mathrm{Im}(p \circ \delta_v^{n-1})$. But then $(\delta_v^n \circ -D(f))(r\phi) = \delta_v^n(r\phi)$ holds. Since C^{n+1} is free we can simplify r and this shows the lower triangle is commutative for all M^n-elements, whence $\delta_v^n(M^n) \subset \delta_v^n(K^n)$ and the Claim. Then, via restricting from X^n to $\mathscr{K}_n$ and bi-partitioning as in Section 5, (a) follows. Since $\mathscr{M}^n \dot\cup \mathscr{K}^n = X^n = \mathscr{M}_n \dot\cup \mathscr{K}_n$, (b) follows from (a), and the Lemma is proved. □

The following statement is then immediate.

Corollary 4. *Denoting $\mathscr{N} \doteq \mathscr{K}^n \cap \mathscr{K}_n$, we have $X^n = \mathscr{M}^n \dot\cup \mathscr{M}_n \dot\cup \mathscr{N}$, where $\mathscr{M}^n, \mathscr{K}^n$ and $\mathscr{M}_n, \mathscr{K}_n$ are defined as in Lemma 3.*

Lemma 5. *The following assertions hold*
(a) $\langle \Lambda_v^v + \mathrm{Im}\delta_v^{n-1} | v \in \mathscr{N} \rangle = F_v^n$ *as a basis.*
(b) $\mathscr{N} = \mathscr{K}_+^n$.

Proof. *Claim* 1: we have the split exact sequence

$$0 \longrightarrow K^n \xrightarrow{s'} C^n/\mathrm{Im}\delta_\nu^{n-1} \xrightarrow{t} C^n/(s(K^n)+\mathrm{Im}\delta_\nu^{n-1}) \longrightarrow 0,$$

where $t(\psi+\mathrm{Im}\delta_\nu^{n-1}) \doteq \psi+\mathrm{Im}\delta_\nu^{n-1}+s(K^n)$ and $s'(\phi) \doteq s(\phi)+\mathrm{Im}\delta_\nu^{n-1}$ with $s(\phi) \doteq \psi\colon C_n \to R$ being the unique map such that $\psi|_{K_n} = \phi$ and $\psi|_{M_n} = 0$. Clearly, $\mathrm{Im}s' = \mathrm{Ker}t$ and t is surjective. Note, $\mathrm{rnk}(\mathrm{Im}\delta_\nu^{n-1}) = \mathrm{rnk}(\mathrm{Im}\partial_n^\nu) = \mathrm{rnk}(M_n)$. Furthermore we have $s(K^n)+\mathrm{Im}\delta_\nu^{n-1} \cong s(K^n) \oplus p(\mathrm{Im}\delta_\nu^{n-1})$, for p defined as in the proof of Lemma 3, and $\mathrm{rnk}(p(\mathrm{Im}\delta_\nu^{n-1})) = \mathrm{rnk}(\mathrm{Im}\delta_\nu^{n-1})$. This implies $C^n/(s(K^n)+\mathrm{Im}\delta_\nu^{n-1})$ is full torsion whence $\mathrm{rnk}(\mathrm{Ker}t) = \mathrm{rnk}(C^n/\mathrm{Im}\delta_\nu^{n-1})$. Therefore $\mathrm{rnk}(s'(K^n)) = \mathrm{rnk}(\mathrm{Ker}t) = \mathrm{rnk}(C^n/\mathrm{Im}\delta_\nu^{n-1}) = n - \mathrm{rnk}(\mathrm{Im}\partial_n^\nu) = \mathrm{rnk}(K^n)$, and since K^n is free s' is injective. Consider

$$t'\colon C^n/(s(K^n)+\mathrm{Im}\delta_\nu^{n-1}) \to C^n/\mathrm{Im}\delta_\nu^{n-1}$$

$$t'(\psi+s(K^n)+\mathrm{Im}\delta_\nu^{n-1}) \doteq p(\psi)+(D(f)\circ p)(\psi)+\mathrm{Im}\delta_\nu^{n-1},$$

which is a well defined morphism for p, f and D as introduced in the proof of Lemma 3. The above short exact sequence splits since

$$t\circ t'\colon C^n/(s(K^n)+\mathrm{Im}\delta_\nu^{n-1}) \to C^n/(s(K^n)+\mathrm{Im}\delta_\nu^{n-1})$$

is the identity map, and the Claim follows. By Corollary 4 we have $\mathscr{K}_n = \mathscr{N} \dot\cup \mathscr{M}^n$, and $\langle\{\Lambda_\nu^\nu = \sum_{\mu^n} r_{\mu^n}\lambda_{\mu^n}+\lambda_\nu\}\dot\cup\{\lambda_{\mu^n}\}\rangle = K^n$ as a basis. But s' is injective, and since $\mathscr{N} \subset \mathscr{K}^n$, by construction only the $\{\Lambda_\nu^\nu+\mathrm{Im}\delta_\nu^{n-1}\}$ portion of the $\mathrm{Im}s'$-basis persists in the quotient H_ν^n. Thus (a) follows, whence (b) and the Lemma is proved. □

The following statement is immediate, providing together with Lemmas 3 and 5 a combinatorial interpretation for the interplay between the weighted torsion and weighted co-torsion.

Corollary 6.

$$T_\nu^n \cong \bigoplus_{\substack{(\mu^{n-1},\mu_n)\in \\ \mathscr{M}^{n-1}\times\mathscr{M}_n}} R/\left(\frac{\nu(\mu^{n-1})}{\nu(\mu_n)}\right) \cong \bigoplus_{\substack{(\kappa_{n-1},\mu_n)\in \\ \mathscr{K}_{n-1}^{-}\times\mathscr{M}_n}} R/\left(\frac{\nu(\kappa_{n-1})}{\nu(\mu_n)}\right) \cong T_{n-1}^\nu.$$

6. A Hom-space interpretation of $\mathfrak{T}_\nu^n$

Consider the following restriction isomorphism and its induced embedding

$$\mathrm{res}\colon \mathrm{Im}\delta_\nu^{n-1} \to \bigoplus_{\mu^{n-1}} \langle\lambda'_{\mu^{n-1}}\rangle, \quad \mathrm{res}(\delta_\nu^{n-1}(\lambda_{\mu^{n-1}})) = \lambda'_{\mu^{n-1}} \doteq \lambda_{\mu^{n-1}}|_{\mathrm{Im}\partial_n^\nu},$$

$$\varepsilon\colon \mathrm{Im}\delta_\nu^{n-1} \to \mathrm{Hom}(\mathrm{Im}\partial_n^\nu, R), \quad \varepsilon(\delta_\nu^{n-1}(\phi)) = \phi' \doteq \phi|_{\mathrm{Im}\partial_n^\nu}.$$

Then, the following diagram commutes

$$\begin{array}{ccccccc}
\mathrm{Im}\partial_n^v & \xrightarrow{D} & \mathrm{Hom}(\mathrm{Im}\partial_n^v, R) & & & & \bigoplus_{\mu^{n-1}}\langle \mu^{n-1}\rangle \\
\uparrow \partial_n^v & & \uparrow \mathrm{incl} & \nwarrow \varepsilon & & & \downarrow D \\
\bigoplus_{\mu_n}\langle \mu_n\rangle & & \bigoplus_{\mu^{n-1}}\langle \lambda'_{\mu^{n-1}}\rangle & \xleftarrow{\mathrm{res}} & \mathrm{Im}\delta_v^{n-1} & \xleftarrow{\delta_v^{n-1}} & \bigoplus_{\mu^{n-1}}\langle \lambda_{\mu^{n-1}}\rangle.
\end{array}$$

Since $\delta_v^{n-1}(\phi)(z) = \phi'(\partial_n^v(z))$, we have

$$\varepsilon(\delta_v^{n-1}(\lambda_{\mu^{n-1}}))(\mu_n) = \lambda'_{\mu^{n-1}}(\partial_n^v(\mu_n)) = (\pm 1)\frac{v(\mu^{n-1})}{v(\mu_n)}.$$

This is tantamount to

$$\lambda'_{\mu^{n-1}} = \sum_{\mu^{n-1}\subset\mu^n} (\pm 1)\frac{v(\mu^{n-1})}{v(\mu_n)}\lambda_{\partial_n^v(\mu_n)}.$$

Smith Normalization of the above representation matrix implies the existence of a pairing $(\mu^{n-1}, \mu_n) \in \mathscr{M}^{n-1}\times\mathscr{M}_n$ such that the following sequence is exact

$$0 \longrightarrow \mathrm{Im}\delta_v^{n-1} \xrightarrow{\varepsilon} \mathrm{Hom}(\mathrm{Im}\partial_n^v, R) \longrightarrow \bigoplus_{\substack{(\mu^{n-1},\mu_n)\in \\ \mathscr{M}^{n-1}\times\mathscr{M}_n}} R/\left(\tfrac{v(\mu^{n-1})}{v(\mu_n)}\right) \longrightarrow 0.$$

The torsion module $\mathrm{Hom}(\mathrm{Im}\partial_n^v, R)/\mathrm{Im}\varepsilon$ is reminiscent of $T_v^n = \mathfrak{T}_v^n/\mathrm{Im}\delta_v^{n-1}$ which manifests a similar but potentially different pairing. Further investigation of this connection in fact yields the following statement

Theorem 6.1.

$$\iota_v\colon \mathrm{Hom}(\mathrm{Im}\partial_v^n, R) \cong \mathfrak{T}_v^n, \quad \iota_v(\phi) = \phi\circ\partial_n^v.$$

Proof. ι_v maps (injectively) into $\mathfrak{T}_v^n \subset \mathrm{Ker}\delta_v^n$, since $\mathrm{Hom}(\mathrm{Im}\partial_n^v, R)/\oplus_{\lambda'_{\mu^{n-1}}}\langle\lambda'_{\mu^{n-1}}\rangle$ is full torsion. Namely,

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{Im}\partial_n^v, R) & \xrightarrow{\iota_v} & \mathfrak{T}_v^n \\
\uparrow \mathrm{incl} & & \uparrow \mathrm{incl} \\
\oplus_{\lambda'_{\mu^{n-1}}}\langle\lambda'_{\mu^{n-1}}\rangle & \xrightarrow{\iota_v} & \mathrm{Im}\delta_v^{n-1}
\end{array}$$

where $\iota_v|_{\oplus_{\lambda'_{\mu^{n-1}}}\langle\lambda'_{\mu^{n-1}}\rangle}$ is an isomorphism. Furthermore, $\iota_v(\phi) = \phi\circ\partial_n^v$ gives rise to $\iota\colon \mathrm{Hom}(\mathrm{Im}\partial_n, R) \cong \mathrm{Im}\delta^{n-1}$. We have $\theta^n = \iota\circ\theta^{n-1}\circ\iota_v^{-1}$ in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathrm{Im}\partial_n^v, R) & \xrightarrow{\theta^{n-1}} & \mathrm{Hom}(\mathrm{Im}\partial_n, R) & \longrightarrow & \bigoplus_{\mu_n\in\mathscr{M}_n} R/(v(\mu_n)) & \longrightarrow & 0 \\
 & & \downarrow \iota_v & & \downarrow \iota & & & & \\
0 & \longrightarrow & \mathfrak{T}_v^n & \xrightarrow{\theta^n} & \mathrm{Im}\delta^{n-1} & \longrightarrow & \bigoplus_{\kappa_-^n\in\mathscr{K}_-^n} R/(v(\kappa_-^n)) & \longrightarrow & 0.
\end{array}$$

Indeed, for $\iota \circ \theta^{n-1}(\phi) \in \mathrm{Im}\delta^{n-1}$,

$$\iota \circ \theta^{n-1}(\phi) = \theta^{n-1}(\phi) \circ \partial_n = \phi \circ (\theta_{n-1} \circ \partial_n) = (\phi \circ \partial_n^v) \circ \theta_n = \theta^n \circ \iota_v(\phi).$$

Finally, by Lemma 5 we can replace $\mathscr{K}_-^n = \mathscr{M}_n$ in the above diagram, whence the Theorem. □

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References

[1] Gregory Bell, Austin Lawson, Joshua Martin, James Rudzinski, and Clifford Smyth. Weighted persistent homology. *Involve, a Journal of Mathematics*, 12(5):823–837, 2019.

[2] Andrei Bura, Qijun He, and Christian Reidys. Weighted Homology of Bi-Structures over Certain Discrete Valuation Rings. *Mathematics*, 9(7):744, 2021.

[3] Andrei C. Bura, Neelav S. Dutta, Thomas J. X. Li, and Christian M. Reidys. A computational framework for weighted simplicial homology. 2022.

[4] Andrei C Bura, Qijun He, and Christian M Reidys. Loop homology of bi-secondary structures. *Discrete Mathematics*, 344(6):112371, 2019.

[5] Andrei C. Bura, Qijun He, and Christian M Reidys. Loop homology of bi-secondary structures II. *Journal of Algebraic Combinatorics*, page 785–798, 2022.

[6] Gunnar Carlsson. Topology and data. *Bull. Amer. Math. Soc.*, 46:255–308, 2009.

[7] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng. The quantum orbifold cohomology of weighted projective spaces. 2009.

[8] Michael W Davis, Jan Dymara, Tadeusz Januszkiewicz, and Boris Okun. Weighted l2–cohomology of coxeter groups. *Geometry & Topology*, 11(1):47–138, 2007.

[9] Robert J MacG Dawson. A simplification of the eilenberg-steenrod axioms for finite simplicial complexes. *Journal of Pure and Applied Algebra*, 53(3):257–265, 1988.

[10] Robert J. MacG Dawson. Homology of weighted simplicial complexes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 31(3):229–243, 1990. Publisher: Dunodéditeur, publié avec le concours du CNRS.

[11] Stefania Ebli, Michaël Defferrard, and Gard Spreemann. Simplicial neural networks. 2020.

[12] Samuel Eilenberg and Norman E. Steenrod. Axiomatic approach to homology theory. *Proceedings of the National Academy of Sciences of the United States of America*, 31(4):117–120, 1945.

[13] Chad Giusti, Robert Ghrist, and Danielle S. Bassett. Two's company, three (or more) is a simplex. *Journal of Computational Neuroscience*, 41(1):1–14, 2016.

[14] Mark Goresky, Günter Harder, and Robert MacPherson. Weighted cohomology. *Inventiones mathematicae*, 116(1):139–213, 1994.

[15] Thomas J. X. Li and Christian M. Reidys. On the Loop Homology of a Certain Complex of RNA Structures. *Mathematics*, 9(15):1749, 2021.

[16] Thomas J. X. Li and Christian M. Reidys. On Weighted Simplicial Homology. 2022.

[17] Tsau Young Lin and I-Jen Chiang. A simplicial complex, a hypergraph, structure in the latent semantic space of document clustering. *International Journal of Approximate Reasoning*, 40(1):55–80, 2005.

[18] Terrence J Moore, Robert J Drost, Prithwish Basu, Ram Ramanathan, and Ananthram Swami. Analyzing collaboration networks using simplicial complexes: A case study. In *2012 Proceedings IEEE INFOCOM Workshops*, pages 238–243. IEEE, 2012.

[19] R. Ramanathan, A. Bar-Noy, P. Basu, M. Johnson, W. Ren, A. Swami, and Q. Zhao. Beyond graphs: Capturing groups in networks. In *2011 IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, pages 870–875, 2011.

[20] Shiquan Ren, Chengyuan Wu, and Jie Wu. Weighted persistent homology. *Rocky Mountain Journal of Mathematics*, 48(8):2661–2687, 2018.

[21] David I. Spivak. Higher-dimensional models of networks. *arXiv:0909.4314 [cs]*, 2009.

[22] L. Vietoris. Über die Homologiegruppen der Vereinigung zweier Komplexe. *Monatshefte für Mathematik und Physik*, 37(1):159–162, 1930.

[23] Chengyuan Wu, Shiquan Ren, Jie Wu, and Kelin Xia. Weighted (Co)homology and Weighted Laplacian. 2019.

[24] Afra Zomorodian and Gunnar Carlsson. Computing persistent homology. In *Proceedings of the twentieth annual symposium on Computational geometry*, pages 347–356, 2004.

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