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UNIQUE MILD SOLUTION FOR FRACTIONAL PARTIAL AND NEUTRAL EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY

NARDJIS LACHACHI-MERAD, SELMA BAGHLI-BENDIMERAD, MOUFFAK BENCHOHRA, AND DJILLALI AOUAD

ABSTRACT. In this paper, the uniqueness of mild solutions for two classes of partial functional and neutral functional evolution equations with finite state-dependent delay where fractional Caputo derivatives is investigated for $\alpha \in (0,1)$. The study is based on Banach's contraction theorem combined with semigroup theory in a real Banach space.

1. Introduction

In this paper, we establish the existence and the uniqueness of mild solutions for the two following class of semilinear partial functional and neutral functional evolution equations with a finite state dependent-delay involving Caputo's fractional order derivative using the Banach contraction theorem in the real Banach space $(E, |\cdot|)$ combined with semigroup theory.

The first considered problem, studied in Section 3, is as follows

$$ {}^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a. e. } t \in J := [0,b],\ 0 < \alpha < 1, \tag{1} $$

$$ y(t) = \varphi(t), \qquad t \in H := [-r, 0], \tag{2} $$

where $b, r > 0$ are given constants; $f : J \times C(H,E) \longrightarrow E$; $\rho : J \times C(H,E) \longrightarrow [-r,b]$ and $\varphi \in C(H,E)$ are given functions; ${}_cD_0^\alpha$ is the standard Caputo's fractional order derivative for $\alpha \in (0,1)$ and $\{A(t)\}_{t\in J}$ is a family of linear closed operators (not necessarily bounded) from E into E.

For any continuous function y and any $t \in J$, we denotes by y_t the element of $C(H;E)$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in H$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

Next, the second considered problem, studied in Section 4, is as follows

$$ {}^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a. e. } t \in J, \tag{3} $$

$$ y(t) = \varphi(t), \qquad t \in H, \tag{4} $$

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where $A(\cdot)$, f and φ are as in problem (1)-(2) and $g : J \times C(H,E) \longrightarrow E$ is a given function. Finally, two examples are given in Section 5 to clarified the obtained results.

Integer-order functional and neutral functional differential equations appear in many fields of applied mathematics, and these equations have attracted much attention in recent years. The first occurrence of out-of-order derivations appears in 1695, the famous letter send by Leibniz to De l'Hopital. Then the derivative of the disorder has evolved from Euler, Fourier, Liouville, and Riemann to the present. The type of derivatives employed varies between the integer version and fractional order; the former uses exponents with integers while the latter uses exponents with fractions. The existence results of differential equations are significantly influenced by the fractional order. It is possible for fractional-order equations to have no solutions, numerous solutions, or singular solutions. The asymptotic behavior of solutions is also influenced by fractional order and can result in power-law decay, oscillatory decay, or algebraic growth. These variations result from fractional derivatives' non-locality and memory dependence. Fractional-order differential equations demand specialized methods for analysis and solution. Recently, many phenomena in various fields of science and engineering can be modeled with fractional differential equations: in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc... Fractional Caputo derivation has important biological implications because it captures memory effects, nonlocal interactions, and anomalous transport observed in biological systems. It is used to model memory- and history-related processes such as biological diffusion, neuronal signalling, and growth. It accurately represents sub diffusion or super diffusion behavior and can be used to analyze biological signals.

Overall, Caputo order derivation improves our understanding of biological phenomena and allows for more accurate modeling and analysis in different biological domains: for details, including some applications and recent results, see the work of Hilfer [14], Kilbas *et al.* [15], Lakshmikantham *et al.* [16], Podlubny [20] and the references contained therein. In recent years, significant developments in fractional order ordinary differential equations and partial evolutionary differential equations are obtained by for Benchohra and his collaborators in [3], El Borai in [8], El-Sayed in [9], Vijayakumar *et al.* in [22]-[24] and Zhou and his team in [25, 26].

Recently, state-dependent delay equations in modeling have been proposed. Existence results are obtained from functional differential equations, where the hysteresis depends on the solution under study. Existing results have been derived for functional differential equations when, among other things, the solution depends on delays on bounded intervals. We refer the reader to the work of Benchohra *et al.* [1] on the bounded interval J. Mesri and Benchohra use non-compactness measures to study fractional-order nonautonomous evolution equations in Fréchet spaces in [18].

After a study of several first-order evolution problems with independent and state-dependent delays by Baghli *et al.* in [2], [4]-[7] and [17], we seek in this article to extend our research to consider these evolution equations with state-dependent delay when the derivative is fractional in the sense of Caputo in this time. Therefore, this work studies the existence and uniqueness of mild solutions for Caputo fractional partial functional and neutral functional evolution equations with state-dependent delay using the Banach's contraction theorem combined with semigroup theory.

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2. Preliminaries

We introduce notations, definitions, propositions, lemmas, and theorems which are used throughout this paper.

Let $C(J;E)$ be the space of continuous functions from J into E with the norm $|\cdot|$ and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremos norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

Let $L^1(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner-integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| \, dt.$$

A measurable function $y : J \to E$ is Bochner-integrable if and only if $|y|$ is Lebesgue-integrable.

Definition 2.1. *A function $f : J \times E \to E$ is said to be an L^1-Carathéodory function if it satisfies :*

(i) *for each $t \in J$, the function $f(t,\cdot) : E \to E$ is continuous ;*
(ii) *for each $y \in E$, the function $f(\cdot,y) : J \to E$ is measurable ;*
(iii) *for every positive integer k, there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that*

$$|f(t,y)| \leq h_k(t) \quad \text{for all } \ |y| \leq k \quad \text{and almost every } \ t \in J.$$

We give some state-dependent delay properties.
Assume that $\rho : J \times C(H;E) \to [-r,b]$ is continuous. Additionally, we introduce the following hypothesis:

$$\mathscr{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H;E), \ \rho(s,\varphi) \leq 0\}.$$

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathscr{R}(\rho^-)$ into $C(H;E)$ and there exists a bounded and continuous function $\mathscr{L}^\varphi : \mathscr{R}(\rho^-) \to (0,\infty)$ such that

$$\|\varphi_t\| \leq \mathscr{L}^\varphi(t)\|\varphi\| \qquad \text{for every } t \in \mathscr{R}(\rho^-).$$

Remark 2.1. *The condition (H_φ), is frequently verified by functions that are continuous and bounded. For more details, see for instance* [1, 12].

Lemma 2.1. *([12], Lemma 2.4) If $y : [-r,b] \to E$ is a function such that $y_0 = \varphi$, then*

$$\|y_s\| \leq \mathscr{L}^\varphi \|\varphi\| + \sup_{0\leq\theta\leq\hat{s}} \{|y(\theta)| \ s \in \mathscr{R}(\rho^-) \cup J\}, \ \hat{s} := \max(0;s),$$

where $\mathscr{L}^\varphi = \sup_{t\in\mathscr{R}(\rho^-)} \mathscr{L}^\varphi(t).$

Proposition 2.1. [2] *The function y in above lemma satisfy for every $t \in J$ and ρ the inequality*

$$\|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathscr{L}^\varphi \|\varphi\|.$$

We give here fractional order derivative definitions.

Definition 2.2. [20] *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined as*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

provided the right hand side when $\Gamma(\cdot)$ is the Euler gamma function.

For instance, $I^\alpha f$ exists for all $\alpha > 0$ when $f \in C(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$. Note also that when $f \in C(\mathbb{R}^+)$ then $I^\alpha f \in C(\mathbb{R}^+)$ and moreover $I^\alpha f(0) = 0$.

Definition 2.3. [20] *The fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ in the Caputo sense is given by*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{m-\alpha-1} f(s)ds = \frac{d}{dt} I_0^{1-\alpha} f(t).$$

Remark 2.2. *Caputo fractional derivative is often applicable to control theory.*

Let us talk about evolution operator. In what follows, we assume that $\{A(t)\}_{t\in J}$ is a family of closed densely defined linear operators not necessarily bounded on the Banach space E with domain $D(A(t))$ independent of t. Additionally, throughout this paper, we assume that the linear operator $A(t)$ satisfies the following conditions [8]

(A_1)**:** For any λ with $Re(\lambda) \geq 0$, the operator $\lambda I - A(t)$ exists a bounded inverse operator $(\lambda I - A(t))^{-1}$ in $B(E)$ and

$$\left\| (\lambda I - A(t))^{-1} \right\| \leq \frac{C}{|\lambda| + 1}$$

where C is a positive constant independent of both t and λ.

(A_2)**:** For any $t, \tau, s \in I$, there exists a constant $\gamma \in (0,1]$ such that

$$\left\| [A(t) - A(\tau)] A^{-1}(s) \right\| \leq C|t-\tau|^\gamma$$

where the constants γ and $C > 0$ are independent of both t, τ and s.

Remark. From Henry [13], Pazy [19] and Temam [21], we know that the assumption (A_1) means that for each $s \in I$, the operator $A(s)$ generates an analytic semigroup $e^{-tA(s)}$ $(t > 0)$, and there exists a positive constant C independent of both t and s such that

$$\left\| -A(s)e^{tA(s)} \right\| \leq \frac{C}{t}$$

where $t > 0$ and $s \in J$.

Definition 2.4. [11] *Define the operators $\Psi(t,s)$, $\phi(t,s)$ and $U(t)$ by*

$$\Psi(t,s) = \alpha \int_0^{+\infty} \theta t^{\alpha-1} \xi_\alpha(\theta) e^{t^\alpha \theta A(s)} d\theta,$$

$$\phi(t,s) = \sum_{k=1}^{+\infty} \phi_k(t,s)$$

and

$$U(t) = A(t)A^{-1}(0) - \int_0^t \phi(t,s)A(s)A^{-1}(0)ds,$$

where ξ_α *is a probability density function defined on* $[0,+\infty)$ *such that its Laplace transform is given by*

$$\int_0^{+\infty} \xi_\alpha(\theta)e^{\theta x}d\theta = \sum_{i=1}^{+\infty} \frac{(-x)^i}{\Gamma(1+\alpha i)} \quad 0<\alpha\le 1,\ x>0,$$

$$\phi_1(t,s) = [-A(t)+A(s)]\Psi(t-s,s),$$

and

$$\phi_{k+1}(t,s) = \int_s^t \phi_k(t,\tau)\phi_1(\tau,s)d\tau, \quad k=1,2,\dots.$$

For more details about the definition and property of the probability density function, one can see the paper [11].

A recall of contraction definition:

Definition 2.5. [10] *A function* $f: X \to X$ *is said to be a contraction if there exists* $k \in [0,1)$ *such that*

$$\|f(x)-f(y)\| \le k\,\|x-y\| \quad \textit{for all } x,y\in X.$$

The used fixed point theorem is as follows:

Theorem 2.1. *(Banach contraction principle* [10]*)*
Let C *be a non-empty closed subset of a Banach space* X*, then any contraction mapping* T *of* C *into itself has a unique fixed point.*

3. Semilinear evolution equations

We give now from [8] the definition of mild solution for fractional partial evolution problem with finite state-dependent delay $(1)-(2)$.

Definition 3.1. *We say that the function* $y(t):[-r,b]\to E$ *is a mild solution of* $(1)-(2)$ *if* $y(t)=\varphi(t)$ *for* $t\in H$ *and* y *satisfies the integral equation*

$$\begin{aligned} y(t) &= U(t)\varphi(0) - \int_0^t \Psi(t-s,s)U(s)A(0)\varphi(0)ds \\ &+ \int_0^t \Psi(t-s,s)f\left(s,y_{\rho(s,y_s)}\right)ds \\ &+ \int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)d\tau ds. \end{aligned} \tag{5}$$

The following properties about the operators Ψ, ϕ and U will be needed in our argument.

Lemma 3.1. [8] *The operator-valued functions* $\Psi(t-s,s)$ *and* $A(t)\Psi(t-s,s)$ *are continuous in uniform topology about the variables* t *and* s*, where* $t\in J$*,* $0\le s\le t-\varepsilon$ *for any* $\varepsilon>0$*, and*

$$\|\Psi(t-s,s)\| \le C(t-s)^{\alpha-1}, \tag{6}$$

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where C is a positive constant independent of both t and s. Furthermore,

(7) $$\|\phi(t,s)\| \leq C(t-s)^{\gamma-1}$$

and

(8) $$\|U(t)\| \leq C\left(1+t^{\gamma}\right).$$

We will need to introduce the following hypotheses which are assumed thereafter :

$(H1)$ The function f is Carathéodory.

$(H2)$ For all $R>0$, there exists $l_R \in L^{\infty}(J;\mathbb{R}_+)$ such that

$$|f(t,u)-f(t,v)| \leq l_R(t)\ \|u-v\|$$

for all $u,v \in C(H,E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.
Set $l_R^* := \operatorname{ess\,sup}_{t\in J} l_R(t)$.

Denote by $\beta(\alpha,\gamma)=\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt$ the β Euler's function. Then we can give now our main result.

Theorem 3.1. *Assume that* (H_φ), $(H1)$ *and* $(H2)$ *are satisfied, and moreover if*

(9) $$Cl_R^* b^{\alpha}\left[\alpha^{-1}+C\gamma^{-1}b^{\gamma}\beta(\alpha,\gamma+1)\right]<1,$$

then the problem $(1)-(2)$ *has a unique mild solution on* $[-r,b]$.

Proof. Transform the problem $(1)-(2)$ into a fixed-point problem.
Consider $\Omega := C([-r,b];E)$ and let the operator $N:\Omega\to\Omega$ is defined by :

$$N(y)(t)=\begin{cases}\varphi(t), & \text{if } t\in H;\\ U(t)\varphi(0)-\displaystyle\int_0^t \Psi(t-s,s)U(s)A(0)\varphi(0)ds & \\ +\displaystyle\int_0^t \Psi(t-s,s)f\left(s,y_{\rho(s,y_s)}\right)ds & \\ +\displaystyle\int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f\left(\tau,y_{\rho(\tau,y_\tau)}\right)d\tau ds, & \text{if } t\in J.\end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(1)-(2)$.

We proof that the operator N is a contraction. For $y,\overline{y}\in J$ we have

$$\begin{aligned}|(Ny)(t)-(N\overline{y})(t)| \leq & \int_0^t \left|\Psi(t-s,s)\left[f\left(s,y_{\rho(s,y_s)}\right)-f\left(s,\overline{y}_{\rho(s,\overline{y}_s)}\right)\right]\right|ds\\ & +\int_0^t\int_0^s \left|\Psi(t-s,s)\phi(s,\tau)\left[f\left(\tau,y_{\rho(\tau,y_\tau)}\right)-f\left(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)}\right)\right]\right|d\tau ds.\end{aligned}$$

By the hypothesis $(H2)$ and Lemma 3.1, we have

$$|(Ny)(t)-(N\overline{y})(t)| \leq C\int_0^t (t-s)^{\alpha-1} l_R(s)\left\|y_{\rho(s,y_s)}-\overline{y}_{\rho(s,\overline{y}_s)}\right\| ds$$
$$+C^2\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1} l_R(\tau)\left\|y_{\rho(\tau,y_\tau)}-\overline{y}_{\rho(\tau,\overline{y}_\tau)}\right\| d\tau ds$$

By Proposition 2.1, we get

$$\begin{aligned}|(Ny)(t)-(N\overline{y})(t)| &\leq Cl_R^*\int_0^t (t-s)^{\alpha-1}\left[|y(s)|-|\overline{y}(s)|\right] ds\\ &+ C^2 l_R^*\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1}\left[|y(\tau)|-|\overline{y}(\tau)|\right] d\tau ds\\ &\leq Cl_R^*\int_0^t (t-s)^{\alpha-1} ds\|y-\overline{y}\|\\ &+ C^2 l_R^*\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1} d\tau ds\|y-\overline{y}\|\\ &\leq Cl_R^*\left[\int_0^t (t-s)^{\alpha-1}\left[1+C\int_0^s (s-\tau)^{\gamma-1} d\tau\right] ds\right]\|y-\overline{y}\|.\end{aligned}$$

Since

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha}$$

and

$$\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\gamma-1} d\tau ds = \frac{t^{\alpha+\gamma}}{\gamma}\beta(\alpha,\gamma+1),$$

we obtain for $t\in J$

$$\begin{aligned}|(Ny)(t)-(N\overline{y})(t)| &\leq Cl_R^*\left[\frac{t^\alpha}{\alpha}+C\frac{t^{\alpha+\gamma}}{\gamma}\beta(\alpha,\gamma+1)\right]\|y-\overline{y}\|\\ &\leq Cl_R^* b^\alpha\left[\frac{1}{\alpha}+C\frac{b^\gamma}{\gamma}\beta(\alpha,\gamma+1)\right]\|y-\overline{y}\|.\end{aligned}$$

Consequently,

$$\|N(y)-N(\overline{y})\| \leq Cl_R^* b^\alpha\left[\frac{1}{\alpha}+C\frac{b^\gamma}{\gamma}\beta(\alpha,\gamma+1)\right]\|y-\overline{y}\|.$$

So by the condition (9), we deduce that the operator N is a contraction. By the Banach contraction principle, the operator N has a unique fixed point which is the unique mild solution of the partial fractional evolution system with state-dependent delay $(1)-(2)$ on $[-r,b]$.

4. Semilinear neutral evolution equations

In this section, we give our second main result for a unique mild solution of the neutral fractional evolution equation with state-dependent delay $(3)-(4)$ by the Banach contraction principle [10]. Firstly, we define such a mild solution.

Definition 4.1. *We say that the function* $y(t) : [-r,b] \to E$ *is a mild solution of* $(3)-(4)$ *if* $y(t) = \varphi(t)$ *for* $t \in H$ *and* y *satisfies the following integral equation*

$$
\begin{aligned}
y(t) &= U(t)\left[\varphi(0) - g(0,\varphi)\right] + g(t, y_{\rho(t,y_t)}) \\
&- \int_0^t \Psi(t-s,s)U(s)A(0)\left[\varphi(0) - g(0,\varphi)\right] ds \\
&- \int_0^t \Psi(t-s,s)A(0)g\left(s, y_{\rho(s,y_s)}\right) ds + \int_0^t \Psi(t-s,s) f\left(s, y_{\rho(s,y_s)}\right) ds \\
&+ \int_0^t \int_0^s \Psi(t-s,s)\phi(s,\tau) f\left(\tau, y_{\rho(\tau,y_\tau)}\right) d\tau ds. \quad \text{for each } t \in J.
\end{aligned} \tag{10}
$$

We will need the following hypothesis on g

$(H3)$ For all $R > 0$, there exists $\chi(s) \in L^{+\infty}(J;\mathbb{R}_+)$ such that

$$
|g(t,u) - g(t,v)| \leq \chi(t)\, \|u - v\|
$$

for all $u, v \in C(H,E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.
Set $\chi^* := ess \sup_{t \in J} \chi(t)$.

Theorem 4.1. *Assume that* (H_φ), $(H1)-(H3)$ *hold. Then, if we have*

$$
\chi^*\left(1 + C^2|A(0)|b^\alpha \Theta_\gamma\right) + Cl_R^* b^\alpha \Upsilon < 1 \tag{11}
$$

where $\Theta_\gamma = \alpha^{-1} + b^\gamma \beta(\alpha, \gamma+1)$ *and* $\Upsilon = \alpha^{-1} + C\gamma^{-1} b^\gamma \beta(\alpha, \gamma+1)$, *then the neutral problem* $(3)-(4)$ *has a unique mild solution on* $[-r,b]$.

Proof. Transform the problem $(3)-(4)$ into a fixed-point problem.
Consider the operator $\tilde{N} : \Omega \to \Omega$ defined by :

$$
\tilde{N}(y)(t) = \begin{cases}
\varphi(t), & \text{if } t \in H; \\
U(t)\left[\varphi(0) - g(0,\varphi)\right] + g(t, y_{\rho(t,y_t)}) & \\
- \int_0^t \Psi(t-s,s)U(s)A(0)\left[\varphi(0) - g(0,\varphi)\right] ds & \\
- \int_0^t \Psi(t-s,s)A(0)g\left(s, y_{\rho(s,y_s)}\right) ds + \int_0^t \Psi(t-s,s) f\left(s, y_{\rho(s,y_s)}\right) ds & \\
+ \int_0^t \int_0^s \Psi(t-s,s)\phi(s,\tau) f\left(\tau, y_{\rho(\tau,y_\tau)}\right) d\tau ds, & \text{if } t \in J.
\end{cases}
$$

Clearly, fixed points of the operator $\tilde{N}$ are mild solutions of the problem $(3)-(4)$.

We will proof that the operator $\tilde{N}$ is a contraction.

For $t\in J$, we have for $y,\overline{y}\in\Omega$

$$\begin{aligned}\left|\left(\tilde{N}y\right)(t)-\left(\tilde{N}\overline{y}\right)(t)\right| &\leq \left|g\left(t,y_{\rho(t,y_t)}\right)-g\left(t,\overline{y}_{\rho(t,\overline{y}_t)}\right)\right| \\ &+\int_0^t\left|\Psi(t-s,s)U(s)A(0)[g\left(s,y_{\rho(s,y_s)}\right)-g\left(s,\overline{y}_{\rho(s,\overline{y}_s)}\right)]\right|ds \\ &+\int_0^t\left|\Psi(t-s,s)\left[f\left(s,y_{\rho(s,y_s)}\right)-f\left(s,\overline{y}_{\rho(s,\overline{y}_s)}\right)\right]\right|ds \\ &+\int_0^t\int_0^s\left|\Psi(t-s,s)\phi(s,\tau)\left[f\left(\tau,y_{\rho(\tau,y_\tau)}\right)-f\left(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)}\right)\right]\right|d\tau ds.\end{aligned}$$

By Proposition 2.1, Lemma 3.1 and the hypotheses $(H2)$ and $(H3)$, we have

$$\begin{aligned}\left|\left(\tilde{N}y\right)(t)-\left(\tilde{N}\overline{y}\right)(t)\right| &\leq \chi(t)\left[|y(t)|-|\overline{y}(t)|\right] \\ &+ C^2|A(0)|\int_0^t(t-s)^{\alpha-1}(1+s^\gamma)\chi(s)\left[|y(s)|-|\overline{y}(s)|\right]ds \\ &+ C\int_0^t(t-s)^{\alpha-1}l_R(s)\left[|y(s)|-|\overline{y}(s)|\right]ds \\ &+ C^2\int_0^t(t-s)^{\alpha-1}\int_0^s(s-\tau)^{\gamma-1}l_R(\tau)\left[|y(\tau)|-|\overline{y}(\tau)|\right]d\tau ds.\end{aligned}$$

Hence

$$\begin{aligned}\left|\left(\tilde{N}y\right)(t)-\left(\tilde{N}\overline{y}\right)(t)\right| &\leq \chi^*\left[1+C^2|A(0)|\int_0^t(t-s)^{\alpha-1}(1+s^\gamma)ds\right]\|y-\overline{y}\| \\ &+Cl_R^*\left[\int_0^t(t-s)^{\alpha-1}ds+C\int_0^t(t-s)^{\alpha-1}\int_0^s(s-\tau)^{\gamma-1}d\tau ds\right]\|y-\overline{y}\|.\end{aligned}$$

Since

$$\int_0^t(t-s)^{\alpha-1}(1+s^\gamma)ds=t^\alpha\left(\frac{1}{\alpha}+t^\gamma\beta(\alpha,\gamma+1)\right),$$

hence we have

$$\begin{aligned}\left|\left(\tilde{N}y\right)(t)-\left(\tilde{N}\overline{y}\right)(t)\right| &\leq \chi^*\left[1+C^2|A(0)|t^\alpha\left(\frac{1}{\alpha}+t^\gamma\beta(\alpha,\gamma+1)\right)\right]\|y-\overline{y}\| \\ &+ Cl_R^*\left[\frac{t^\alpha}{\alpha}+C\frac{t^{\alpha+\gamma}}{\gamma}\beta(\alpha,\gamma+1)\right]\|y-\overline{y}\|.\end{aligned}$$

Set $\Theta_\eta:=\dfrac{1}{\alpha}+b^\eta\beta(\alpha,\gamma+1)$ and $\Upsilon:=\dfrac{1}{\alpha}+C\dfrac{b^\gamma}{\gamma}\beta(\alpha,\gamma+1)$ to get for $t\in J$

$$\left|\left(\tilde{N}y\right)(t)-\left(\tilde{N}\overline{y}\right)(t)\right|\leq\left[\chi^*\left(1+C^2|A(0)|b^\alpha\Theta_\gamma\right)+Cl_R^*b^\alpha\Upsilon\right]\|y-\overline{y}\|.$$

Then,

$$\left\|\tilde{N}(y)-\tilde{N}(\overline{y})\right\|\leq\left[\chi^*\left(1+C^2|A(0)|b^\alpha\Theta_\gamma\right)+Cl_R^*b^\alpha\Upsilon\right]\|y-\overline{y}\|.$$

By (11), $\tilde{N}$ is a contraction operator and by the Banach contraction principle, $\tilde{N}$ has a unique fixed point which is the unique mild solution of neutral fractional evolution equation with state dependent-delay (3)-(4).

5. Examples

To illustrate the previous results, we give in this section two examples.

Example 5.1. *Consider the partial functional differential equation of the form :*

$$(12)\quad \begin{cases} {}^cD_0^{\alpha}u(t,\xi) = a_0(t,\xi)\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds, \\ \qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases}$$

where $0 < \alpha < 1$, $a_0(\xi,\cdot)$ is a continuous function for $\xi \in [0,\pi]$ and $a_0(\cdot,t)$ is uniformly Hölder continuous in $t \in [0,b]$; $a_1 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $\rho_1 : [0,b] \to \mathbb{R}$; $\rho_2 : \mathbb{R} \to \mathbb{R}$ and $u_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ are given continuous functions.

To study this system, we consider the space $E = L^2([0,\pi];\mathbb{R})$. And we define the operator $A : D(A) \subset E \to E$ given by $A(t)w = a_0(t,w)w''$ with domain

$$D(A) := \{\, w \in E\ : w'' \in E,\ w(0) = w(\pi) = 0 \,\}.$$

Then $A(s)$ generates an analytic infinitesimal generator of semigroup $e^{tA(s)}$ on E which satisfies the assumptions (A_1) and (A_2).

Then we can use the theoretical study below and enounce the following theorem:

Theorem 5.1. *Let $\varphi \in C(H;E)$ be continuous and bounded. Assume that the condition (H_φ) holds and the functions $a_1 : [-r,0] \to \mathbb{R}$, $a_2 : [0,\pi] \to \mathbb{R}^+$, $\rho_1 : [0,b] \to \mathbb{R}$, $\rho_2 : \mathbb{R}^+ \to \mathbb{R}$ and $u_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of (12).*

Proof. *From the assumptions, we have that*

$$y(t)(\xi) = u(t,\xi),$$

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right)$$

and

$$\varphi(t)(\xi) = u_0(t,\xi)$$

are well defined functions, which permit to transform system (12) into the abstract system (1) − (2). Now, the existence of mild solutions can be deduced from a direct application of Theorem 3.1.

Example 5.2. *Consider the neutral functional differential equation of the form :*

$$(13)\quad \begin{cases} {}^cD_0^\alpha \left[u(t,\xi) - \int_{-r}^0 a_3(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \int_{-r}^0 a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds, \\ \qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases}$$

where $a_3 : [-r,0] \to \mathbb{R}$ is a given continuous function.

Theorem 5.2. *Let $\varphi \in C(H;E)$ be continuous and bounded. Assume that the condition (H_φ) holds and the functions $a_1, a_3 : [-r,0] \to \mathbb{R}$, $a_2 : [0,\pi] \to \mathbb{R}^+$, $\rho_1 : [0,b] \to \mathbb{R}$, $\rho_2 : \mathbb{R}^+ \to \mathbb{R}$ and $u_0 : [-r,0] \times [0,\pi] \Rightarrow \mathbb{R}$ are continuous. Then there exists a unique mild solution of (13).*

Proof. *From the assumptions, we have that*

$$y(t)(\xi) = u(t,\xi),$$
$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$
$$g(t,\psi)(\xi) = \int_{-r}^0 a_3(s)\psi(s,\xi)ds,$$
$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right)$$

and

$$\varphi(t)(\xi) = u_0(t,\xi)$$

are well defined functions, which permit to transform system (13) into the abstract system (3) – (4). Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.1.

6. Conclusion

In this study, we prove the existence and uniqueness of mild solutions for Caputo fractional partial functional and neutral functional evolution equations with state dependent delay using Banach's contraction theorem with semigroup theory. We will look also to research mild solutions for other fractional problems.

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Mathematics Laboratory, Djillali Liabes University, Po. Box 89 22000 Sidi Bel-Abbes, ALGERIA
E-mail address: nardjis.lachachi@univ-sba.dz

Mathematics Laboratory, Djillali Liabes University, Po. Box 89 22000 Sidi Bel-Abbes, ALGERIA
E-mail address: selmabaghli@gmail.com

Mathematics Laboratory, Djillali Liabes University, Po. Box 89 22000 Sidi Bel-Abbes, ALGERIA
E-mail address: benchohra@yahoo.com

Mathematics Laboratory, Djillali Liabes University, Po. Box 89 22000 Sidi Bel-Abbes, ALGERIA
E-mail address: djillali.aouad@univ-sba.dz