DIFFEOLOGICAL, FRÖLICHER, AND DIFFERENTIAL SPACES

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ABSTRACT. Differential calculus on cartesian spaces has many generalisations. In particular, on a set X, a diffeological structure is given by maps from open subsets of cartesian spaces to X, a differential structure is given by maps from X to $\mathbb{R}$, and a Frölicher structure is given by maps from $\mathbb{R}$ to X as well as maps from X to $\mathbb{R}$. We illustrate the relations between these structures through examples.

1. INTRODUCTION

There are many structures in the mathematical literature that generalise differential calculus beyond manifolds. In this paper we focus on the simplest such structures: diffeology (as defined by Souriau), differential structures (in the sense of Sikorski), and Frölicher structures. A diffeology on a set X is given by a set of maps from open subsets of cartesian spaces to X; see Definition 2.1. A differential structure on a set X is given by a set of maps from X to $\mathbb{R}$; see Definition 2.2. A Frölicher structure on a set X is given by a set of maps from $\mathbb{R}$ to X and a set of maps from X to $\mathbb{R}$; see Definition 2.12. These structures are motivated by the following characterisations of smooth maps between manifolds.

Let M and N be open subsets of cartesian spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ and $\psi\colon M \to N$ a function. Smoothness of ψ is equivalent to each of the following conditions.

(1) For each k, each open subset U of $\mathbb{R}^k$, and each smooth map $p\colon U \to M$, the composition $\psi \circ p\colon U \to N$ is smooth.

(2) For each real-valued smooth function $f\colon N \to \mathbb{R}$, the composition $f \circ \psi\colon M \to \mathbb{R}$ is smooth.

(3) For each smooth curve $\gamma\colon \mathbb{R} \to M$, the composition $\psi \circ \gamma\colon \mathbb{R} \to N$ is smooth.

The fact that the third condition implies the smoothness of ψ follows from the following theorem of Jan Boman [11, Theorem 1]: Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}$, and assume that the composition $f \circ u$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ for every $u \in C^\infty(\mathbb{R}, \mathbb{R}^d)$. Then f is in $C^\infty(\mathbb{R}^d, \mathbb{R})$.

In this paper, we illustrate the relation between differential structures, diffeological structures, and Frölicher structures, through examples. The paper should be valuable to researchers and graduate students seeking a quick and effective introduction to these structures. Whereas the goal of the paper is mostly expository, the paper does contain new material. We

identify Fröhlicher spaces with so-called reflexive differential spaces and so-called reflexive diffeological spaces (see Definition 2.6 and Theorems 2.11 and 2.13). This notion of reflexivity that we introduce and its relation to Fröhlicher spaces was in principle known to experts but to our knowledge it has not been made explicit in the literature. Throughout the paper we place particular emphasis on the theme of reflexivity and non-reflexivity. Our examples — many of which are new — illustrate the vast richness of diffeology and differential structures beyond the reflexive ones, while pointing at some that are reflexive for non-trivial reasons. Last but not least, we pose a number of open questions.

We deliberately focus on these structures, which we view as the simplest among the many generalisations of differential calculus. To this end, we do not focus on higher categorical approaches to smoothness such as differentiable stacks, nor algebro-geometric settings such as C^∞-schemes, nor differentiability of finite order. We believe that a good understanding of the simpler structures would be beneficial also for those who wish to work with other generalisations of differential calculus, as different tools capture different subsets of the phenomena that we illustrate. Nevertheless, in response to a referee request, we are including an appendix (Appendix B) in which we comment on relations of these simpler structures to some other structures in the higher categorical and algebro-geometric settings.

The richness of these non-reflexive examples motivates working in the presence of both a diffeology and a differential structure that are compatible but not necessarily reflexive. Such spaces, named "Watts spaces" in [101], have been informally promoted by Jordan Watts for many years.

In Section 2, we identify the category of Fröhlicher spaces with the categories of so-called reflexive differential spaces and so-called reflexive diffeological spaces (see Definition 2.6 and Theorems 2.11 and 2.13) and give some examples of non-reflexive diffeological spaces and non-reflexive differential spaces.

One of the motivations for considering diffeological and differential structures is that they are meaningful for arbitrary subsets and quotients of manifolds. In Section 3, we discuss how diffeological and differential structures relate on these objects; see Propositions 3.2 and 3.5. We include two open questions, one about the diffeology of symplectic reduced spaces, and one about the differential structure of an irrational line in the torus.

In Section 4, we consider orbifolds, quotients by compact Lie group actions, and manifolds with corners. By a result of Gerald Schwarz [86], the *Hilbert map* identifies the quotient of a linear compact Lie group action with a subset of a cartesian space as differential spaces. Consequently, the subspace differential structure on its image is reflexive. As a consequence, manifolds with corners can be defined equivalently as differential spaces or as diffeological spaces; either of these structures is reflexive. See Example 4.6. On the other hand, the quotient diffeology can be non-reflexive, and consequently different from the subset diffeology on the image of the Hilbert map. See Examples 4.1 and 4.4.

In Section 5, we consider finite unions of copies of the real line, as well as finite unions of manifolds. For example, the union of the three coordinate axes in $\mathbb{R}^3$ is diffeomorphic to a union of three concurrent lines in $\mathbb{R}^2$ diffeologically but not as differential spaces. The former differential space is reflexive; the latter is not. See Examples 5.1 and 5.4. For

a generalization to cleanly intersecting submanifolds, see Example 5.3. We conclude this section with a question about the reflexivity of an example that comes from polyfolds.

In Section 6, we consider topological properties of diffeological and differential spaces, and obtain some topological necessary conditions for reflexivity.

Appendix A contains some technical proofs that are deferred from the earlier sections.

In Appendix B, we briefly compare diffeological, differential, and Frölicher spaces to Lie groupoids, stacks, sheaves of sets over the site **Open**, Mostow spaces, subcartesian spaces, differentiable spaces à la Gonzalez-Salas/Spallek, and C^∞-schemes.

This paper evolved from visits of Patrick Iglesias-Zemmour and author Batubenge to the University of Toronto. Batubenge and Iglesias-Zemmour contributed to the initiation and vision of this paper and to Sections 2, 3, 4, 5, and relevant proofs in Appendix A. Many of the details were worked out and written up by Watts as Chapter 2 of his University of Toronto Ph.D. thesis [102], supervised by Yael Karshon. Appendix B was authored mostly by Jordan Watts, in response to our referee's request to clarify how the simple structures on which we are focusing in this paper relate to more complicated structures that occur in the literature.

Some history and notes on the literature.

The development of the various notions of smooth structures discussed in this paper occurred mainly in the 1960s, '70s, and '80s, motivated by the need to push differentiability beyond the confines of finite-dimensional manifolds to the singular subset, singular quotient, and infinite-dimensional settings.

Differential structures (Definition 2.2) were introduced by Sikorski in the late 1960s; see [87, 88]. Many of the properties of the smooth structure on a smooth manifold can be derived from its ring of smooth functions; a differential space is a topological space equipped with a ring of functions that captures these properties. A differential structure determines a sheaf of continuous functions that contains the constants (as considered by Hochschild [44]), which, in turn, is a special case of a ringed space (a topological space equipped with a sheaf of rings; see EGA 1 [39]). Pushing similar notions from algebraic geometry into the realm of differential geometry leads to further developments, C^∞-schemes [52] and differentiable spaces in the sense of Gonzalez-Salas [37] being some resulting theories.

Special cases of differential spaces appear in the literature in various contexts. A subcartesian space, introduced by Aronszajn in the late 1960s and motivated by manifolds with singularities that occur in his study of the Bessel potential in functional analysis, is a Hausdorff differential space that is locally diffeomorphic to (arbitrary) subsets of cartesian spaces; see [2, 3, 4, 91]. In the mid-1970s, interest in equipping singular orbit spaces of compact Lie group actions with a smooth structure (see Bredon [12, Chapter 6]) led to a result of Schwarz [86] showing that while *a priori* quotient spaces, these spaces are in fact subcartesian as well; see also Cushman-Śniatycki [26], who work with orbit spaces of proper Lie group actions in the subcartesian setting. A similar result for symplectic quotients in the early 1990s by Arms-Cushman-Gotay [1] lead to the study of these spaces as subcartesian spaces equipped with Poisson structures; for example, this is used in the treatment of symplectic quotients

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as symplectic stratified spaces by Sjamaar-Lerman [89]. Today, viewing stratified spaces as differential spaces is commonplace; for example, they appear in the book by Pflaum [83] and the work of Śniatycki [91] as subcartesian spaces, and Kreck's stratifolds [61, 98] are a version of stratified differential spaces whose functions satisfy certain conditions at the strata. Differential subspaces of cartesian spaces are special cases of subcartesian spaces: convex subsets are studied in Karshon-Watts [57], and so-called Hölder sets and certain sub-analytic sets are studied in Rainer [85, Theorem 1.13]. Our main reference on the theory of differential spaces is the book by Śniatycki [91].

Diffeology (Definition 2.1) was introduced by Jean-Marie Souriau around 1980; see [92]. An early success of the theory which helped to motivate its further development is the work of Donato-Iglesias on irrational tori [27]; see Example 3.9. Irrational tori, (or "infracircle"s) also appear in the study of geometric quantisation [106], [46], as well as the integration of certain Lie algebroids [25], and so play a role in mathematical physics. Stratified spaces as diffeological spaces appear in [40, 41], where the diffeological language is a natural setting to describe the so-called zero perverse differential forms of the intersection theory of Goresky-MacPherson. Our main reference on the theory of diffeology is the book by Iglesias-Zemmour [47].

Souriau's motivation for developing diffeology came from infinite-dimensional groups appearing in mathematical physics. A similar notion was introduced and studied by Kuo-Tsai Chen already in the 1970s for the purpose of putting differentiability on path spaces used in variational calculus on an equal footing with smooth structures on manifolds; the precise definition went through several revisions [14, 15, 16, 17]. The main difference between diffeological spaces and Chen spaces is that the latter use convex subsets instead of open subsets of cartesian spaces as domains of the so-called plots. In [57], authors Karshon and Watts show that diffeological spaces are isomorphic as a category to a full subcategory of Chen spaces.

Similar motivations in functional analysis lead Frölicher and Kriegl to introduce what are now called Frölicher spaces in their book [35], following the work of Frölicher in the early 1980s [32, 33, 34]. A special case is the "convenient setup" of Frölicher, Kriegl, and Michor [35, 63], which applies to finite and infinite-dimensional vector spaces and manifolds. Vector spaces from the diffeological perspective are studied by Christensen-Wu in [21]. Reflexivity of spaces of smooth maps between diffeological spaces is examined in an upcoming paper [58] by the authors Karshon and Watts. Iglesias-Zemmour and Karshon study Lie groups as diffeological subgroups of diffeomorphism groups in [49], coadjoint orbits of infinite-dimensional groups are studied by Iglesias-Zemmour in [28] and Lee in [65], infinite products and coproducts appear in Karshon's paper on moduli spaces [53], diffeological classifying spaces appear in the work of Magnot-Watts [73] and Christensen-Wu [22], and Magnot studies Frölicher and diffeological Lie groups in [70, 71, 72].

Many of the categories mentioned above are compared in the paper of Andrew Stacey [96]. Along with the diffeological, differential, and Frölicher spaces, he also considers various definitions of Chen spaces, as well as Smith spaces [90] (topological spaces equipped with a set of continuous functions that satisfy a certain "reflexivity" condition), and constructs functors between these categories. Treatments of diffeological and Chen spaces from the

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point-of-view of sheaves on categories is given in Baez-Hoffnung [6] (also see [5]), and a treatment of diffeological spaces from the point-of-view of stacks on manifolds is given in Watts-Wolbert [105].

Quotient spaces of Lie group actions, Lie groupoids, and more generally, singular foliations, form another setting in which the theories of diffeology, Frölicher spaces, and differential spaces are important. Orbifolds are given a diffeological treatment in a paper of Iglesias-Zemmour, Karshon, and Zadka [48], and further studied in an intersection of diffeology with non-commutative geometry in a paper of Iglesias-Zemmour and Laffineur [50]. Quasifolds from a diffeological and groupoid perspective are treated in Karshon-Miyamoto [55], based on the work of Masrour Zoghi [109]. From a differential space perspective, or equivalently in this case, a Frölicher perspective, orbifolds are examined in a paper of Watts [103], and orbit spaces of linear circle actions in a paper of Craig-Downey-Goad-Mahoney-Watts [24]. Differential forms on these quotient spaces from a diffeological perspective are compared with basic forms for compact Lie group actions in Watts' Ph.D. thesis [102], proper Lie group actions in a paper by Karshon-Watts [56], proper Lie groupoids in a paper by Watts [104], and certain singular foliations by Miyamoto [76].

2. Relations Between Structures

Definition 2.1 (Diffeology). Let X be a nonempty set. A *parametrisation* of X is a function $p \colon U \to X$ where U is an open subset of $\mathbb{R}^n$ for some n. A *diffeology* $\mathcal{D}$ on X is a set of parametrisations satisfying the following three conditions.

(1) **(Covering)** For every $x \in X$ and every nonnegative integer $n \in \mathbb{N}$, the constant function $p \colon \mathbb{R}^n \to \{x\} \subseteq X$ is in $\mathcal{D}$.

(2) **(Locality)** Let $p \colon U \to X$ be a parametrisation such that for every $u \in U$ there exists an open neighbourhood V of u in U satisfying $p|_V \in \mathcal{D}$. Then $p \in \mathcal{D}$.

(3) **(Smooth Compatibility)** Let $p \colon U \to X$ be a plot in $\mathcal{D}$. Then for every $n \in \mathbb{N}$, every open subset $V \subseteq \mathbb{R}^n$, and every infinitely-differentiable map $F \colon V \to U$, we have $p \circ F \in \mathcal{D}$.

A set X equipped with a diffeology $\mathcal{D}$ is called a *diffeological space* and is denoted by $(X, \mathcal{D})$. When the diffeology is understood, we may drop the symbol $\mathcal{D}$. The parametrisations in $\mathcal{D}$ are called *plots*. A map $F \colon X \to Y$ between diffeological spaces is *smooth* if for any plot

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$p\colon U \to X$ of X the composition $F \circ p\colon U \to Y$ is a plot of Y. The map is a *diffeomorphism* if it is smooth and has a smooth inverse. To avoid ambiguity, we sometimes say that the map is *diffeologically smooth* or is a *diffeological diffeomorphism.*

The *D-topology* on X is the strongest topology in which every plot is continuous; thus, a subset Y of X is D-open if and only if for each plot $p \in \mathcal{D}$ the preimage $p^{-1}(Y)$ is open in the domain of p. ◇

Given a collection of functions $\mathcal{F}_0$ on a set X, its *initial topology* is the weakest topology on X for which every function in $\mathcal{F}_0$ is continuous. Thus, a sub-basis for the initial topology is given by the pre-images of open intervals by functions in $\mathcal{F}_0$.

Definition 2.2 (Differential space). Let X be a nonempty set. A *differential structure* on X is a nonempty family $\mathcal{F}$ of real-valued functions on X, along with its initial topology, satisfying the following two conditions.

(1) **(Smooth compatibility)** For any positive integer k, functions $f_1, ..., f_k \in \mathcal{F}$, and $F \in C^{\infty}(\mathbb{R}^k)$, the composition $F(f_1, ..., f_k)$ is in $\mathcal{F}$.

(2) **(Locality)** Let $f\colon X \to \mathbb{R}$ be a function such that for any $x \in X$ there exist an open neighbourhood $U \subseteq X$ of x and a function $g \in \mathcal{F}$ satisfying $f|_U = g|_U$. Then $f \in \mathcal{F}$.

A set X equipped with a differential structure $\mathcal{F}$ is called a *differential space* and is denoted by $(X, \mathcal{F})$. When the differential structure is understood, we may drop the symbol $\mathcal{F}$. A map $F\colon X \to Y$ between differential spaces $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ is *smooth* if for every function $f\colon Y \to \mathbb{R}$ in $\mathcal{F}_Y$ the composition $f \circ F$ is in $\mathcal{F}_X$. The map is a *diffeomorphism* if it is smooth and has a smooth inverse. To avoid ambiguity, we sometimes say that the map is *functionally smooth* or is a *functional diffeomorphism.* ◇

Definition 2.3 ("Compatible" and "induces"). Given a set X with a collection $\mathcal{D}_0$ of parametrisations and a collection $\mathcal{F}_0$ of real-valued functions, we say that

(i) $\mathcal{D}_0$ and $\mathcal{F}_0$ are *compatible* if $f \circ p$ is infinitely-differentiable for all $p \in \mathcal{D}_0$ and $f \in \mathcal{F}_0$;

(ii) $\mathcal{D}_0$ *induces* $\mathcal{F}_0$ if $\mathcal{F}_0$ coincides with the set

$$\Phi\mathcal{D}_0 := \{f\colon X \to \mathbb{R} \mid \forall(p\colon U \to X) \in \mathcal{D}_0,\ f \circ p \in C^{\infty}(U)\}$$

of those real-valued functions whose precomposition with each element of $\mathcal{D}_0$ is infinitely-differentiable;

(iii) $\mathcal{F}_0$ *induces* $\mathcal{D}_0$ if $\mathcal{D}_0$ coincides with the set

$$\Pi\mathcal{F}_0 := \{\text{parametrisations } p\colon U \to X \mid \forall f \in \mathcal{F}_0,\ f \circ p \in C^{\infty}(U)\}$$

of those parametrisations whose composition with each element of $\mathcal{F}_0$ is infinitely-differentiable.

Thus, $\mathcal{D}_0$ and $\mathcal{F}_0$ are compatible if and only if $\mathcal{F}_0$ is contained in $\Phi\mathcal{D}_0$, if and only if $\mathcal{D}_0$ is contained in $\Pi\mathcal{F}_0$. ◇

Example 2.4 (Manifolds). On a smooth manifold M, the sets of parametrisations $U \to M$ that are infinitely-differentiable and the set of real-valued functions $M \to \mathbb{R}$ that are infinitely-differentiable are a diffeology and a differential structure that induce each other. This follows from the fact that smoothness is a local property and from the existence of smooth bump functions. ◊

Remark 2.5. We make the following easy observations:

- Each of the operations $\mathcal{D}_0 \mapsto \Phi\mathcal{D}_0$ and $\mathcal{F}_0 \mapsto \Pi\mathcal{F}_0$ is inclusion-reversing.
- We always have $\Pi\Phi\mathcal{D}_0 \supseteq \mathcal{D}_0$ and $\Phi\Pi\mathcal{F}_0 \supseteq \mathcal{F}_0$.

These facts imply that, given a family $\mathcal{D}$ of parametrisations, there exists a family of real-valued functions that induces $\mathcal{D}$ if and only if $\Pi\Phi\mathcal{D} = \mathcal{D}$. Indeed, if $\mathcal{D} = \Pi\mathcal{F}$ then $\Pi\Phi\mathcal{D} \subseteq \mathcal{D}$ amounts to $\Pi\Phi\Pi\mathcal{F} \subseteq \Pi\mathcal{F}$, which follows from $\Phi\Pi\mathcal{F} \supset \mathcal{F}$. Similarly, given a family $\mathcal{F}$ of real-valued functions, there exists a family of parametrisations that induces $\mathcal{F}$ if and only if $\Phi\Pi\mathcal{F} = \mathcal{F}$. ◇

Definition 2.6 (Reflexive). A diffeology $\mathcal{D}$ is *reflexive* if $\Pi\Phi\mathcal{D} = \mathcal{D}$. A differential structure $\mathcal{F}$ is *reflexive* if $\Phi\Pi\mathcal{F} = \mathcal{F}$. ◇

Proposition 2.7 (Reflexive stability). *For any family $\mathcal{F}_0$ of real-valued functions on a set, $\Pi\mathcal{F}_0$ is a reflexive diffeology on the set. For any family $\mathcal{D}_0$ of parametrisations on a set, $\Phi\mathcal{D}_0$ is a reflexive differential structure on the set.*

We prove Proposition 2.7 in §A.1.

Thus, if a diffeology $\mathcal{D}$ and a differential structure $\mathcal{F}$ induce each other, then they are both reflexive. For example, manifolds are reflexive both as diffeological spaces and as differential spaces. Here are examples of diffeological and of differential spaces that are not reflexive:

Example 2.8 (Spaghetti diffeology). The *spaghetti diffeology* (or wire diffeology) on $\mathbb{R}^2$ consists of those parametrisations that locally factor through curves. That is, a parametrisation $p \colon U \to \mathbb{R}^2$ is in the spaghetti diffeology if and only if for every $u \in U$ there is an open neighbourhood V of u in U, a smooth map $F \colon V \to \mathbb{R}$, and a smooth curve $q \colon \mathbb{R} \to \mathbb{R}^2$, such that $p|_V = q \circ F$. See [47, Section 1.10] for more details.

The differential structure that is induced by the spaghetti diffeology consists of those real-valued functions $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ q$ is smooth for every smooth curve $\mathbb{R} \to \mathbb{R}^2$. By Boman's theorem [11, Theorem 1], every such function f is infinitely-differentiable. Thus, this is the standard differential structure on $\mathbb{R}^2$, and the diffeology that it induces is the standard diffeology on $\mathbb{R}^2$.

The spaghetti diffeology and the standard diffeology have the same smooth curves $\mathbb{R} \to \mathbb{R}^2$, but they are different. For example, the identity map on $\mathbb{R}^2$ is in the standard diffeology but not in the spaghetti diffeology. Thus, the spaghetti diffeology is not reflexive. ◊

Example 2.9 (Rational numbers). Consider the set $\mathbb{Q}$ of rational numbers with the differential structure $C^\infty(\mathbb{Q})$ that consists of those functions $f \colon \mathbb{Q} \to \mathbb{R}$ that locally extend to smooth functions on $\mathbb{R}$. This includes, for example, the restriction to $\mathbb{Q}$ of the function $x \mapsto \frac{1}{x - \sqrt{2}}$. All the plots in $\Pi C^\infty(\mathbb{Q})$ are locally constant. (Indeed, since the inclusion

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map $\mathbb{Q} \hookrightarrow \mathbb{R}$ is in $C^\infty(\mathbb{Q})$, every $p \in \Pi C^\infty(\mathbb{Q})$ must be smooth as a function to $\mathbb{R}$. By the intermediate value theorem, such a p must be locally constant.) Consequently, the differential space $(\mathbb{Q}, C^\infty(\mathbb{Q}))$ is not reflexive. $\lozenge$

Example 2.10 ($\mathbf{C^k}(\mathbb{R})$). Fix an integer $k \geq 0$. Consider the real line $\mathbb{R}$ with the differential structure $C^k(\mathbb{R})$ consisting of those real-valued functions that are k-times continuously differentiable. All the plots in $\Pi C^k(\mathbb{R})$ are locally constant. (Indeed, take any parametrisation $p\colon U \to \mathbb{R}$. Since the identity map is in $C^k(\mathbb{R})$, if $p \in \Pi C^k(\mathbb{R})$, then p must be infinitely-differentiable. If p is infinitely-differentiable and not locally constant, then there exists $u \in U$ such that $dp|_u \neq 0$; the composition of p with a map $f \in C^k(\mathbb{R})$ that is not smooth at $p(u)$ is not smooth, so $p \notin \Pi C^k(\mathbb{R})$.) Consequently, the differential space $(\mathbb{R}, C^k(\mathbb{R}))$ is not reflexive. $\lozenge$

Diffeological spaces, along with diffeologically smooth maps, form a category; reflexive diffeological spaces form a full subcategory. Differential spaces, along with functionally smooth maps, form a category; reflexive differential spaces form a full subcategory.

If $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ are two diffeological spaces and $F\colon X \to Y$ is a diffeologically smooth map, then F is also a functionally smooth map from $(X, \Phi\mathcal{D}_X)$ to $(Y, \Phi\mathcal{D}_Y)$. Thus, we have a functor $\mathbf{\Phi}$ from diffeological spaces to reflexive differential spaces that sends a diffeological space $(X, \mathcal{D})$ to the reflexive differential space $(X, \Phi\mathcal{D})$ and that sends each map to itself. Similarly, we have a functor $\mathbf{\Pi}$ from differential spaces to reflexive diffeological spaces that sends a differential space $(X, \mathcal{F})$ to the reflexive diffeological space $(X, \Pi\mathcal{F})$ and that sends each map to itself. In §A.2 we prove these facts and obtain the following theorem:

Theorem 2.11 (Isomorphism of categories of reflexive spaces). *The restriction of the functor $\mathbf{\Phi}$ to the subcategory of reflexive diffeological spaces is an isomorphism of categories onto the subcategory of reflexive differential spaces. The restriction of the functor $\mathbf{\Pi}$ to the subcategory of reflexive differential spaces is an isomorphism of categories onto the subcategory of reflexive diffeological spaces. These isomorphisms are inverses of each other.*

Given a set X and a family $\mathcal{F}_0$ of real-valued functions on X, we also consider the set $\Gamma\mathcal{F}_0$ of those maps from $\mathbb{R}$ to X whose composition with each element of $\mathcal{F}_0$ is infinitely-differentiable:

$$\Gamma\mathcal{F}_0 := \{c\colon \mathbb{R} \to X \mid \forall f \in \mathcal{F}_0, \ f \circ c \in C^\infty(\mathbb{R})\}.$$

The operation $\mathcal{F}_0 \mapsto \Gamma\mathcal{F}_0$ is inclusion-reversing. Also, for any family of functions $\mathcal{F}_0$ from X to $\mathbb{R}$ and family of functions $\mathcal{C}_0$ from $\mathbb{R}$ to X, we have $\mathcal{C}_0 \subseteq \Gamma\Phi\mathcal{C}_0$ and $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$. These facts imply that $\Gamma\Phi\Gamma\mathcal{F}_0 = \Gamma\mathcal{F}_0$.

Definition 2.12 (Frölicher spaces). A *Frölicher structure* on a set X is a family $\mathcal{F}$ of real-valued functions $X \to \mathbb{R}$ and a family $\mathcal{C}$ of maps $\mathbb{R} \to X$, such that

$$\Phi\mathcal{C} = \mathcal{F} \quad \text{and} \quad \Gamma\mathcal{F} = \mathcal{C}.$$

Such a triple $(X, \mathcal{C}, \mathcal{F})$ is a *Frölicher space*.

Let $(X, \mathcal{C}_X, \mathcal{F}_X)$ and $(Y, \mathcal{C}_Y, \mathcal{F}_Y)$ be Frölicher spaces. A map $F\colon X \to Y$ is *Frölicher smooth* if it satisfies one, hence all, of the following equivalent conditions:

(i) $f \circ F \in \mathcal{F}_X$ for every $f \in \mathcal{F}_Y$.

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(ii) $f \circ F \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for every $c \in \mathcal{C}_X$ and $f \in \mathcal{F}_Y$.

(iii) $F \circ c \in \mathcal{C}_Y$ for every $c \in \mathcal{C}_X$.

((i) implies (ii) because $\mathcal{F}_X$ and $\mathcal{C}_X$ are compatible, (ii) implies (i) because $\mathcal{F}_X = \Phi\mathcal{C}_X$, (ii) implies (iii) because $\mathcal{C}_Y = \Gamma\mathcal{F}_Y$, (iii) implies (ii) because $\mathcal{F}_Y$ and $\mathcal{C}_Y$ are compatible.) $\diamond$

Frölicher spaces, along with Frölicher smooth maps, form a category. There is a functor Ξ from the category of Frölicher spaces to the category of reflexive differential spaces that takes $(X, \mathcal{C}, \mathcal{F})$ to $(X, \mathcal{F})$ and takes each map to itself. There is also a functor Γ from the category of differential spaces that takes $(X, \mathcal{F})$ to $(X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ and takes each map to itself. In §A.3 we prove these facts and obtain the following theorem:

Theorem 2.13 (Frölicher spaces as reflexive spaces). *The functor Ξ is an isomorphism from the category of Frölicher spaces to the category of reflexive differential spaces. The functor Γ restricts to an isomorphism from the category of reflexive differential spaces to the category of Frölicher spaces. These isomorphisms are inverses of each other.*

To summarise, we have isomorphisms between the categories of Frölicher spaces $\{(X, \mathcal{C}, \mathcal{F})\}$, reflexive differential spaces $\{(X, \mathcal{F})\}$, and reflexive diffeological spaces $\{(X, \mathcal{D})\}$, where the functors send every map to itself and their actions on objects are given by the following commuting diagram.

$$
\begin{array}{ccc}
& \mathcal{C}=\text{1-dim'l plots, } \mathcal{F}=\Phi\mathcal{D} & \\
\{(X, \mathcal{D})\} & \rightleftarrows & \{(X, \mathcal{C}, \mathcal{F})\} \\
& \mathcal{D}=\Pi\mathcal{F} & \\
& \mathcal{F}=\Phi\mathcal{D} \qquad \mathcal{C}=\Gamma\mathcal{F} & \\
\mathcal{D}=\Pi\mathcal{F} & & \text{same } \mathcal{F} \\
& \{(X, \mathcal{F})\} &
\end{array}
\tag{2.14}
$$

Notes.

(1) In the literature, what we call *differential structure*, *differential space*, *functionally smooth map*, and *functional diffeomorphism*, are sometimes called *Sikorski structure*, *Sikorski space*, *Sikorski smooth map*, and *Sikorski diffeomorphism*.

(2) In the literature, the adjective "reflexive" often refers to a Banach space E and means that the natural inclusion of E into $(E^*)^*$ is an isomorphism. Many Banach spaces (for example $C([0,1])$) are not reflexive as Banach spaces, but the diffeology and differential structure on a Banach (or Fréchet) space that consist of those parametrisations and those real-valued functions that are smooth in the usual sense are always reflexive; see [32, 42]; also see [58]. Also, the analogue of reflexive stability (Proposition 2.7) for the functor sending a Banach space to its dual is not true: by the Hahn-Banach theorem, a Banach space E is reflexive if and only if its dual space E^* is reflexive [31].

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(3) The behaviour of the functors Φ and Π is that of an *antitone Galois connection* [81]. Other examples of such relationships include sets of polynomials and their zero sets in algebraic geometry, as well as field extensions and their Galois groups.

(4) The functor $\Xi \colon (X, \mathcal{C}, \mathcal{F}) \mapsto (X, \mathcal{F})$ from Frölicher spaces to differential spaces was described in Cherenack's paper [18]. The functor $\boldsymbol{\Gamma} \colon (X, \mathcal{F}) \mapsto (X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ from differential spaces to Frölicher spaces was described in Batubenge's Ph.D. thesis [7, §2.7]. A differential space $(X, \mathcal{F})$ is reflexive if and only if $\Phi\Gamma\mathcal{F} = \mathcal{F}$; these spaces were introduced in [7, §5.2] under the name "pre-Frölicher spaces". Further comparisons between Frölicher and differential spaces appear in [9].

(5) Example 2.10 appears in [102, Example 2.79]. In the context of Smith spaces, $(\mathbb{R}, C^0(\mathbb{R}))$ is discussed in [96, p.100, paragraph on "Smith spaces"]; however, when $\mathbb{R}$ is equipped with its standard topology, $(\mathbb{R}, C^0(\mathbb{R}))$ is not a Smith space.

(6) Some of the results of this section can be rephrased in terms of adjoint functors and reflective subcategories; see Sections 8.4.1 and 8.4.4 of [35]. In particular, Ξ is a left adjoint to Γ, Φ is a left adjoint to Π, and $\Gamma \circ \Phi$ is a left adjoint to $\Pi \circ \Xi$. These facts are also in Stacey's paper [96], noting that $(X, \mathcal{C}, \mathcal{F})$ should be $(X, \mathcal{C}_X, \mathcal{F}_X)$ in the last sentence of the second paragraph of the subsection on Smith and Frölicher spaces (Section 5).

3. Subsets and Quotients

In this section we discuss subsets and quotients from several points of view: differential structures, diffeology, and topology. We omit many of the proofs. The interested reader can fill in the details as an exercise or look them up in Iglesias–Zemmour's book [47, Chapter 1], Watts' thesis [102, Chapter 2], or Śniatycki's book [91, Chapter 2].

Definition 3.1 (Subsets). Let X be a set and $Y \subseteq X$ a subset. Given a diffeology $\mathcal{D}$ on X, the *subset diffeology* on Y consists of those parametrisations $p \colon U \to Y$ whose composition with the inclusion map $Y \hookrightarrow X$ is a plot in $\mathcal{D}$. Given a differential structure $\mathcal{F}$ on X, the *subspace differential structure* on Y consists of those functions $f \colon Y \to \mathbb{R}$ that locally extend to X in the following sense: for every $x \in Y$ there exists an open neighbourhood U of x in X with respect to the initial topology and a function $\tilde{f} \in \mathcal{F}$ such that $f|_{U \cap Y} = \tilde{f}|_{U \cap Y}$. $\diamond$

Differential structures are well adapted to subsets:

Proposition 3.2 (Differential subspaces). *Given a differential space $(X, \mathcal{F})$ and a subset $Y \subset X$, we obtain on Y an unambiguous diffeology and an unambiguous topology. Indeed, we can first take the subspace differential structure on Y and then the diffeology on Y that it induces, or we can first take the diffeology on X that $\mathcal{F}$ induces and then the subset diffeology on Y; these two procedures yield the same diffeology on Y. Also, the initial topology corresponding to the subspace differential structure on Y coincides with the subspace topology on Y induced by the initial topology on X.* $\diamond$

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Diffeologies are not as well adapted to subsets:

Remark 3.3. Given a diffeological space $(X, \mathcal{D})$ and a subset $Y \subset X$, the subset Y might not acquire an unambiguous differential structure nor an unambiguous topology. The two procedures — first passing to the induced differential structure on X and then to the subspace differential structure on Y, or first passing to the subset diffeology on Y and then to the induced differential structure on Y — might yield two different differential structures on Y. Also, the D-topology corresponding to the subset diffeology on Y might differ from the subset topology induced by the D-topology on X. Both of these ambiguities occur with the subset $\mathbb{Q}$ of $\mathbb{R}$ of Example 2.9, as well as with the "pinched topologist's sine curve" of Example 6.8.

$\diamond$

Definition 3.4 (Quotients). Let X be a set, let $\sim$ be an equivalence relation on X, and let $\pi \colon X \to X/\sim$ be the quotient map. Given a differential structure $\mathcal{F}$ on X, the *quotient differential structure* on $X/\sim$ consists of those functions $f \colon X/\sim \to \mathbb{R}$ whose pullback $f \circ \pi \colon X \to \mathbb{R}$ is in $\mathcal{F}$. Given a diffeology $\mathcal{D}$ on X, the *quotient diffeology* on $X/\sim$ consists of those parametrisations $p \colon U \to X/\sim$ that locally lift to X in the following sense: for every $u \in U$ there exist an open neighbourhood V of u in U and a plot $q \colon V \to X$ such that $p|_V = \pi \circ q$. $\diamond$

Diffeologies are well adapted to quotients:

Proposition 3.5 (Diffeological quotients). *Given a diffeological space $(X, \mathcal{D})$ and an equivalence relation $\sim$ on X, we obtain on the quotient $X/\sim$ an unambiguous differential structure and an unambiguous topology. Indeed, we can first take the quotient diffeology on $X/\sim$ and then the differential structure that it induces, or we can first take the differential structure on X that $\mathcal{D}$ induces and then take the quotient differential structure on $X/\sim$. These two procedures yield the same differential structure on $X/\sim$. Also, the D-topology corresponding to the diffeology on $X/\sim$ coincides with the quotient topology on $X/\sim$ induced by the D-topology on X.* $\diamond$

Differential structures are not as well adapted to quotients:

Remark 3.6. Given a differential space $(X, \mathcal{F})$ and an equivalence relation $\sim$ on X, the quotient $X/\sim$ might not acquire an unambiguous diffeology nor an unambiguous topology. The two procedures — first passing to the induced diffeology on X and then to the quotient diffeology on $X/\sim$, or first passing to the quotient differential structure on $X/\sim$ and then to the induced diffeology on $X/\sim$ — might yield two different diffeologies on the quotient $X/\sim$. For example, this occurs with the irrational torus $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ as in Example 3.9, with the quotient $\mathbb{R}/\mathbb{Z}_2$ as in Example 4.4, and with the quotient $\mathbb{R}/(0, 1)$ of the real line $\mathbb{R}$ by the open interval $(0, 1)$ as in Example 6.10. Also, the initial topology corresponding to the quotient differential structure on $X/\sim$ might differ from the quotient topology on $X/\sim$ induced by the initial topology on X; for example, this occurs with the quotient $\mathbb{R}/(0, 1)$. $\diamond$

We conclude this section with a couple of examples and open questions.

We start with an important collection of sub-quotients:

Example 3.7 (Reduced Spaces). For a symplectic manifold (M, ω) with an action of a compact Lie group G and momentum map $\mu\colon M \to \mathfrak{g}^*$, the reduced space $\mu^{-1}(0)/G$ inherits from M an unambiguous diffeology and an unambiguous differential structure, which are compatible. The differential structure on the reduced space $\mu^{-1}(0)/G$ does not always induce the diffeology on the reduced space $\mu^{-1}(0)/G$ (Example 4.4). ◇

Example 3.7 raises an interesting question:

Question 3.8. In the setup of Example 3.7, does the diffeology on the reduced space $\mu^{-1}(0)/G$ necessarily induce the differential structure on the reduced space $\mu^{-1}(0)/G$?

The following example illustrates that diffeology can carry rich information about quotients and that differential structures can carry rich information about subsets.

Example 3.9 (Irrational flow on the torus). Fix an irrational number α. Consider the linear flow with slope α on the torus $\mathbb{R}^2/\mathbb{Z}^2$:

$$[x, y] \mapsto [x + t, y + \alpha t].$$

Let T_α be the quotient of the torus by this linear flow, equipped with the quotient diffeology (which induces the quotient differential structure and the quotient topology; see Proposition 3.5). Let L_α be the orbit through $[0, 0]$ of this linear flow, equipped with the subspace differential structure (which induces the subset diffeology and the subset differential structure; see Proposition 3.2).

The differential structure on T_α is trivial: it consists of the constant functions. In contrast, the diffeology of T_α is non-trivial. For example,

$$t \mapsto \begin{cases} [0, 0] & t < 0 \\ [0, r] & t \geq 0 \end{cases}$$

is not a plot of T_α if $r \notin \mathbb{Z} + \alpha\mathbb{Z}$. Thus, the diffeological space T_α is not reflexive.

The diffeology on L_α is standard: the inclusion map $t \mapsto [t, \alpha t]$ is a diffeomorphism from the real line $\mathbb{R}$ with its standard diffeology to L_α. In contrast, the differential structure on L_α is not standard: for example, its topology is not locally connected. It follows that the differential space L_α is not reflexive. ◊

We now elaborate on Example 3.9, leading to an open question.

An automorphism of the torus (as a Lie group) carries the linear flow with slope α to a linear flow with slope β where β is obtained from α by a fractional linear transformation with integer coefficients:

$$\beta = \frac{a\alpha + b}{c\alpha + d} \quad, \quad a, b, c, d \in \mathbb{Z} \quad, \quad ad - bc = \pm 1.$$

When α and β are related in this way, we say that they are $\mathrm{GL}(2, \mathbb{Z})$-congruent. Thus, if α and β are $\mathrm{GL}(2, \mathbb{Z})$-congruent, then the quotients T_α and T_β are diffeomorphic as diffeological spaces (hence also as differential spaces), and the subsets L_α and L_β are diffeomorphic as differential spaces (hence also as diffeological spaces). Donato and Iglesias [27] proved a

striking result: if T_α and T_β are diffeomorphic as diffeological spaces, then α and β are $\mathrm{GL}(2, \mathbb{Z})$-congruent. See Iglesias's book [47, Exercise 4 with solution at the back of the book].

Question 3.10. Assuming that L_α and L_β are diffeomorphic as differential spaces, can we conclude that α and β are $\mathrm{GL}(2, \mathbb{Z})$-congruent?

4. Orbifolds, Quotients by Compact Group Actions, and Manifolds with Corners

In this section, we study the quotient diffeological and quotient differential structures on the orbit space of a linear action of a compact Lie group on a cartesian space, and apply this to proper Lie group actions, to orbifolds, and to manifolds-with-corners.

Example 4.1 (Orthogonal quotient). Let G be a compact Lie group acting linearly on $\mathbb{R}^n$. By a theorem of Hilbert [107, p. 618], the ring of G-invariant polynomials on $\mathbb{R}^n$ is finitely-generated. A choice of m generators for this ring induces a G-invariant proper map $i \colon \mathbb{R}^n \to \mathbb{R}^m$, which we call a *Hilbert map*. By a theorem of Gerald Schwarz [86], every G-invariant smooth function on $\mathbb{R}^n$ can be expressed as the pullback by i of a smooth function on $\mathbb{R}^m$. This implies that the Hilbert map descends to a diffeomorphism from $\mathbb{R}^n/G$, with the quotient differential structure induced from $\mathbb{R}^n$, to the image of the Hilbert map, with the subspace differential structure induced from $\mathbb{R}^m$.

The quotient differential structure on $\mathbb{R}^n/G$ is induced by the quotient diffeology on $\mathbb{R}^n/G$ by Proposition 3.5, so it is reflexive by Proposition 2.7. Consequently, the subspace differential structure on the image of the Hilbert map is reflexive.

In contrast, the quotient diffeology on $\mathbb{R}^n/G$ might not be reflexive. For example, the map $\mathbb{R}^n/\mathrm{O}(n) \to [0, \infty)$ given by $[x] \mapsto \|x\|^2$ is an isomorphism of differential spaces (by Schwarz's theorem), but the quotient diffeologies on $\mathbb{R}^n/\mathrm{O}(n)$ are non-isomorphic for different values of n (see [47], Exercise 50, with solution at the back of the book). In particular, this Hilbert map does not induce a diffeomorphism of diffeological spaces from $\mathbb{R}^n/G$ to its image in $\mathbb{R}^m$.
$$\lozenge$$

Example 4.2 (Proper Lie group action). Combining Proposition 3.5, Example 4.1, and the slice theorem [60, 82], the differential structure on the quotient of a manifold by a compact (or proper) Lie group action is reflexive and is subcartesian; i.e., locally diffeomorphic to subsets of cartesian spaces.
$$\lozenge$$

Example 4.3 ($\mathbb{Z}_2$- and $(\mathbb{Z}_2)^n$-actions). We note two special cases of Schwarz's theorem [86], which in the case $n = 1$ were proved by Whitney [108].

(1) Let the two-element group $\mathbb{Z}_2$ act on $\mathbb{R}^n$ by $(x_1, \ldots, x_n) \mapsto \pm(x_1, \ldots, x_n)$. Then every invariant smooth function has the form $g((x_i x_j)_{1 \leq i \leq j \leq n})$ where $g \colon \mathbb{R}^{n(n+1)/2} \to \mathbb{R}$ is smooth. Here the Hilbert map $\mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ is given by $(x_1, \ldots, x_n) \mapsto ((x_i x_j)_{1 \leq i \leq j \leq n})$. When $n = 2$, after a linear change of coordinates, the image of the Hilbert map becomes the subset $\{z^2 = x^2 + y^2, \ z \geq 0\}$ of $\mathbb{R}^3$.

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(2) Let $(\mathbb{Z}_2)^n$ act on $\mathbb{R}^n$ by $(x_1, \ldots, x_n) \mapsto (\pm x_1, \ldots, \pm x_n)$. Then every invariant smooth function has the form $g(x_1^2, \ldots, x_n^2)$ where $g\colon \mathbb{R}^n \to \mathbb{R}$ is smooth. Here the Hilbert map $\mathbb{R}^n \to \mathbb{R}^n$ is given by $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$. Its image is the positive orthant, $\mathbb{R}^n_{\geq 0}$. ◊

Example 4.4 (Orbifolds). (Effective) orbifolds can be defined as diffeological spaces that are locally diffeomorphic to quotients of the form $\mathbb{R}^n/\Gamma$, where Γ is a finite subgroup of $\mathrm{O}(n)$ (see [48]). As a differential space, an orbifold is reflexive. However, the diffeology on an orbifold is generally not reflexive, as illustrated in the following two examples.

Let the two-element group $\mathbb{Z}_2$ act on $\mathbb{R}$ by $x \mapsto \pm x$, and let $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}_2$ be the quotient map. The quotient diffeology $\mathcal{D}_{\mathbb{R}/\mathbb{Z}_2}$ induces the quotient differential structure $C^\infty(\mathbb{R}/\mathbb{Z}_2)$ (see Proposition 3.5), but it is not induced by this differential structure: the map $p(u,v) := [\pm\sqrt{u^2 + v^2}]$ from $\mathbb{R}^2$ to $\mathbb{R}/\mathbb{Z}_2$ does not have a smooth lift near the origin, but it is in the diffeology that is induced by $C^\infty(\mathbb{R}/\mathbb{Z}_2)$. Indeed, by Schwarz's theorem (see Example 4.3), if $f \in C^\infty(\mathbb{R}/\mathbb{Z}_2)$, then $\pi^* f(x) = g(x^2)$ for some smooth function $g\colon \mathbb{R} \to \mathbb{R}$, and so $f \circ p$ is equal to $(u,v) \mapsto g(u^2 + v^2)$, which is smooth. This shows that $\mathcal{D}_{\mathbb{R}/\mathbb{Z}_2}$ is not reflexive.

The following example is due to Moshe Zadka. Let the two-element group $\mathbb{Z}_2$ act on $\mathbb{R}^2$ by $(x,y) \mapsto \pm(x,y)$, and let $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}_2$ be the quotient map. The quotient diffeology $\mathcal{D}_{\mathbb{R}^2/\mathbb{Z}_2}$ induces the quotient differential structure $C^\infty(\mathbb{R}^2/\mathbb{Z}_2)$ (see Proposition 3.5), but it is not induced by this differential structure: the map

$$p(r\cos\theta, r\sin\theta) := \begin{cases} [e^{-1/r^2}\cos(\theta/2), e^{-1/r^2}\sin(\theta/2)] & r > 0 \\ [0,0] & r = 0 \end{cases}$$

from $\mathbb{R}^2$ to $\mathbb{R}^2/\mathbb{Z}_2$ does not have a smooth (nor even continuous) lift near the origin, but it is in the diffeology that is induced by $C^\infty(\mathbb{R}^2/\mathbb{Z}_2)$. Indeed, by Schwarz's theorem (see Example 4.3), if $f \in C^\infty(\mathbb{R}^2/\mathbb{Z}_2)$, then $\pi^* f(x,y) = g(x^2, xy, y^2)$ for some smooth function $g\colon \mathbb{R}^3 \to \mathbb{R}$, and so $f \circ p$ is equal to

$$(r\cos\theta, r\sin\theta) \mapsto \begin{cases} g\!\left(e^{-2/r^2}\tfrac{1+\cos\theta}{2},\ e^{-2/r^2}\tfrac{\sin\theta}{2},\ e^{-2/r^2}\tfrac{1-\cos\theta}{2}\right) & r > 0 \\ g(0,0,0) & r = 0, \end{cases}$$

which is smooth. This shows that $\mathcal{D}_{\mathbb{R}^2/\mathbb{Z}_2}$ is not reflexive. ◊

Example 4.5 (The positive orthant). On the positive orthant $\mathbb{R}^n_{\geq 0}$, the subspace differential structure that is induced from $\mathbb{R}^n$ is reflexive. Indeed, by Example 4.3, the positive orthant is the image of a Hilbert embedding, and by Example 4.1, this implies that the differential structure $\mathcal{F}$ is reflexive. ◊

Example 4.6 (Manifolds-with-corners). We recall the definition of a manifold-with-corners. An n-dimensional *chart-with-corners* on a topological space M is a homeomorphism $\varphi\colon U \to \Omega$ from an open subset U of M to a relatively open subset Ω of the positive orthant $\mathbb{R}^n_{\geq 0}$. Charts-with-corners φ_1, φ_2 are *compatible* if $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$, which are homeomorphisms between relatively open subsets of $\mathbb{R}^n_{\geq 0}$, are smooth in the sense that they locally extend to smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. An *atlas-with-corners* on M is a set of pairwise compatible charts with corners whose domains cover M. A *manifold-with-corners*

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is a Hausdorff, second-countable topological space M equipped with a maximal atlas with corners.

An equivalent definition of manifold-with-corners is as a Hausdorff, second countable differential space that is locally functionally diffeomorphic to open subsets of $\mathbb{R}^n_{\geq 0}$. A map between manifolds-with-corners is smooth in the classical sense if and only if it is functionally smooth. The differential structure on a manifold-with-corners is reflexive; this follows from Example 4.5 (the positive orthant) and the existence of smooth bump functions.

Manifolds-with-corners can also be viewed as diffeological spaces. The D-topology on the positive orthant $\mathbb{R}^n_{\geq 0}$ coincides with the subspace topology induced from $\mathbb{R}^n$; this follows from the fact that the plot $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$ restricts to a homeomorphism from the positive orthant to itself with respect to the subspace topology. By Proposition 3.2 and Example 4.5, the subset diffeology and the subspace differential structure on the positive orthant $\mathbb{R}^n_{\geq 0}$ induce each other. It follows that a map between relatively open subsets of the positive orthant is a diffeological diffeomorphism if and only if it is a functional diffeomorphism, which is equivalent to being a diffeomorphism in the classical sense. It further follows that a manifold-with-corners can be equivalently defined as a diffeological space that is locally diffeomorphic to open subsets of $\mathbb{R}^n_{\geq 0}$. ◊

Notes.

(1) The argument in Example 4.5 is a generalisation of the same statement for half-spaces $\mathbb{R}^{n-1} \times [0, \infty)$ that was given by Iglesias-Zemmour in [47, ch. 4] to show that the classical notion of a manifold-with-boundary is the same as the diffeological notion. This generalisation appeared in [68]. More generally, the subspace differential structure on any locally closed convex set is reflexive; see [57].

(2) Manifolds-with-corners were introduced in 1961 by Jean Cerf and by Adrien Douady [13, 29] and are now included in standard textbooks such as John Lee's [66, Chapter 16]. Our definitions of a manifold-with-corners are equivalent to theirs. These definitions are local.

Manifolds-with-faces were introduced in 1968 by Klaus Jänich [51]; also see [100, Chap. 4]. The codimension-k *strata* of a manifold-with-corners are the connected components of the set of those points that, in a chart-with-corners, have exactly k coordinates that vanish. Manifolds-with-faces are manifolds-with-corners in which every codimension-k stratum is in the closure of k distinct codimension one strata. This condition is global. The disc-with-one-corner in the plane, given in polar coordinates by $r \leq \sin 2\theta$ for $0 \leq \theta \leq \pi/2$, is a manifold-with-corners but is not a manifold-with-faces. Some authors use the term "t-manifold" for a manifold-with-corners and the term "manifold-with-corners" for a manifold-with-faces; see [74, Definition 1.8.5]; also see [83, Article 1.1.19].

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5. Intersecting submanifolds

In this section we consider the diffeological and differential structures on some unions of lines in the plane, and more generally, on some unions of submanifolds in an ambient manifold.

Example 5.1 (Two coordinate axes). Consider the wedge sum of two copies of $\mathbb{R}$ attached at their origins, which we write as $X = (\mathbb{R}_1 \amalg \mathbb{R}_2)/(0_1 \sim 0_2)$; denote the quotient diffeology by $\mathcal{D}_X$. Let $E \subset \mathbb{R}^2$ be the union of the two coordinate axes in the cartesian plane, equipped with its subspace differential structure. Let

$$\varphi \colon X \to E$$

be the bijection whose pullback to $\mathbb{R}_1$ is $x \mapsto (x, 0)$ and whose pullback to $\mathbb{R}_2$ is $y \mapsto (0, y)$. Then

  (1) The map φ is a functional diffeomorphism from the differential space $(X, \Phi\mathcal{D}_X)$ to the differential subspace $(E, C^\infty(E))$ of $\mathbb{R}^2$. Moreover, the differential structure $\Phi\mathcal{D}_X$ consists of those real-valued functions on X whose pullbacks to $\mathbb{R}_1$ and to $\mathbb{R}_2$ are smooth.

  (2) The differential space $(E, C^\infty(E))$ is reflexive.

  (3) The diffeological space $(X, \mathcal{D}_X)$ is not reflexive. ◊

Proof. We prove Item (1) in §A.4. By Item (1), $C^\infty(E)$ is a differential structure that is induced by some diffeology; Proposition 2.7 ("reflexive stability") then gives Item (2). For Item (3) we need to show that $\Pi\Phi\mathcal{D}_X \supsetneq \mathcal{D}_X$. Consider the parametrisation $p \colon \mathbb{R} \to X$ whose composition with φ is

$$t \mapsto \begin{cases} (e^{-1/t^2}, 0) & \text{if } t < 0 \\ (0, 0) & \text{if } t = 0 \\ (0, e^{-1/t^2}) & \text{if } t > 0. \end{cases}$$

Because this composition is a smooth map with image in E, it is a plot of E; by Proposition 3.2 it is in $\Pi C^\infty(E)$; Item (1) implies that p is in $\Pi\Phi\mathcal{D}_X$. On the other hand, p does not lift to a smooth (nor even continuous) map to $\mathbb{R}_1 \amalg \mathbb{R}_2$ on any neighbourhood of $t = 0$, so p is not in the quotient diffeology $\mathcal{D}_X$ on X. This proves (3). □

Remark 5.2.

  (1) Example 5.1 generalises to any finite number of copies of $\mathbb{R}$. In particular, the subspace differential structure on the union of the three coordinate axes in $\mathbb{R}^3$ is reflexive.

  (2) Example 5.1 generalises to arbitrary pointed manifolds $(N_1, *_1), \ldots, (N_k, *_k)$, of possibly different dimensions, with $X := (N_1 \amalg \ldots \amalg N_k)/*_i \sim *_j$ for all i, j, and with $E \subset N_1 \times \ldots \times N_k$. See [102, Example 2.67].

  (3) Example 5.1 also generalises to transversally intersecting submanifolds; see [102, Example 2.70]. Here, we take N_1 and N_2 to be (embedded) submanifolds of an ambient manifold M. We let $i \colon N_1 \amalg N_2 \to M$ be the map whose restriction to each N_i is the

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inclusion map, we take $X := N_1 \amalg N_2/\sim$ where $x \sim y$ if and only if $i(x) = i(y)$, and we take $E := N_1 \cup N_2 \subset M$. $\diamond$

The generalisations mentioned in Remark 5.2 are special cases of the following more general example:

Example 5.3 (Cleanly-intersecting submanifolds). Let M be a manifold, and let $\{i_\tau \colon N_\tau \hookrightarrow M\}$ be a family of submanifolds whose intersections are jointly clean in the following sense. Each point of M is in the domain of some coordinate chart $\varphi \colon U \to \Omega \subseteq \mathbb{R}^n$ such that, for each τ, if the submanifold N_τ meets U, then $\varphi(U \cap N_\tau)$ is the intersection of Ω with a coordinate subspace of $\mathbb{R}^n$. Let $X := \left(\coprod_\tau N_\tau \right) / \sim$ where, for $x \in N_\tau$ and $y \in N_{\tau'}$, we have $x \sim y$ if and only if $i_\tau(x) = i_{\tau'}(y)$; denote the quotient diffeology by $\mathcal{D}_X$. Consider the subset $E := \bigcup_\tau i_\tau(N_\tau)$ of M, let $C^\infty(E)$ be the subspace differential structure, and let $\varphi \colon X \to E$ be the bijection whose pullback to N_τ is i_τ. Then, as in Example 5.1,

(1) The map φ is a functional diffeomorphism from the differential space $(X, \Phi\mathcal{D}_X)$ to the differential subspace $(E, C^\infty(E))$ of M. Moreover, the differential structure $\Phi\mathcal{D}_X$ consists of those real-valued functions on X whose pullback to each N_τ is smooth.

(2) The differential space $(E, C^\infty(E))$ is reflexive.

(3) The diffeological space $(X, \mathcal{D}_X)$ is not reflexive, unless the components of E are submanifolds of M.

For details, see §A.4.

Example 5.4 (Three lines in $\mathbb{R}^2$). Let S be the subset of $\mathbb{R}^2$ given by the union of the x-axis, the y-axis, and the line $y = x$, with the subspace differential structure $C^\infty(S)$ and the subset diffeology $\mathcal{D}_S$ that are induced from $\mathbb{R}^2$. Let $E \subseteq \mathbb{R}^3$ be the union of the three coordinate axes, with the subspace differential structure $C^\infty(E)$ and the subset diffeology $\mathcal{D}_E$ that are induced from $\mathbb{R}^3$. Consider the bijection $\varphi \colon E \to S$ given by $(t,0,0) \mapsto (t,0)$, $(0,t,0) \mapsto (0,t)$, and $(0,0,t) \mapsto (t,t)$. Then

(1) The map φ is a diffeological diffeomorphism from $(E, \mathcal{D}_E)$, where $\mathcal{D}_E$ is the subset diffeology on E that is induced from $\mathbb{R}^3$, to $(S, \mathcal{D}_S)$, where $\mathcal{D}_S$ is the subset diffeology on S that is induced from $\mathbb{R}^2$.

(2) The differential space $(S, C^\infty(S))$ is not reflexive. $\lozenge$

Proof. Because φ extends to a smooth map between the ambient spaces $\mathbb{R}^3 \to \mathbb{R}^2$, (for example, take $(x,y,z) \mapsto (x+z, y+z)$,) the map $\varphi \colon E \to S$ is functionally and diffeologically smooth, so $\varphi \circ \mathcal{D}_E \subseteq \mathcal{D}_S$. We prove the opposite inclusion, $\mathcal{D}_S \subseteq \varphi \circ \mathcal{D}_E$, in §A.4.

By Item 1 of Remark 5.2, the differential structure $C^\infty(E)$ on E is reflexive. We then have

$$
\begin{aligned}
(S, \Phi\Pi C^\infty(S)) \ &= (S, \Phi\mathcal{D}_S) && \text{since } \mathcal{D}_S = \Pi C^\infty(S) \text{ (see Proposition 3.2),} \\
&\cong (E, \Phi\mathcal{D}_E) && \text{by Part (1),} \\
&= (E, C^\infty(E)) && \text{since } \mathcal{D}_E = \Pi C^\infty(E) \text{ and } C^\infty(E) \text{ is reflexive.}
\end{aligned}
$$

The dimension of the Zariski tangent space at a point in a differential space is invariant under functional diffeomorphisms (see [69]). Since the dimension of the Zariski tangent space at

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the origin in S is 2, and that at the origin in $(E, C^\infty(E))$, hence in $(S, \Phi\Pi C^\infty(S))$, is 3, the differential space $(S, C^\infty(S))$ is not reflexive. □

Example 5.5 (Many lines in $\mathbb{R}^2$). For any integer $k \geq 3$, Example 5.4 generalises to the union of any k distinct lines through the origin in $\mathbb{R}^2$. In particular, every two such unions are diffeomorphic as diffeological spaces. In contrast, two such unions are diffeomorphic as differential spaces if and only if they differ by a linear transformation of $\mathbb{R}^2$. (Given a diffeomorphism between them, take its differential at the origin.) Thus, for each $k \geq 4$, such unions produce a continuum of non-isomorphic differential spaces.

The following example was communicated to us by Katrin Wehrheim as a topological space that can arise in the context of polyfolds. We're interested in its diffeology:

Example 5.6 (Axis and half-plane). Take the space X that is obtained by gluing the x-axis $\{(x, 0) \mid x \in \mathbb{R}\}$ with the open right half-plane $\{(x, y) \mid x > 0, \ y \in \mathbb{R}\}$ along their intersection in $\mathbb{R}^2$. By Proposition 3.5, its quotient diffeology $\mathcal{D}_X$ induces its quotient differential structure $\mathcal{F}_X$ and its quotient topology. These are not induced by the natural inclusion map $X \hookrightarrow \mathbb{R}^2$: the function

$$f(x, y) := \begin{cases} 0 & y = 0 \\ \frac{e^{-1/|y|}}{x} & y \neq 0, \ x > 0 \end{cases}$$

is in $\mathcal{F}_X$ but it is not continuous with respect to the subset topology induced from $\mathbb{R}^2$.

Question 5.7. In Example 5.6, is the diffeology $\mathcal{D}_X$ reflexive?

Notes. From the point of view of Frölicher spaces, wedge products are also analyzed in Batubenge and Ntumba's paper [8, pages 76–78]. Other aspects of the above examples also appeared in Watts' Ph.D. thesis [102, Examples 2.67 and 2.70] and in Christensen and Wu's paper [20, Examples 3.17, 3.19, and 3.20].

6. Topological Considerations

Recall that the D-topology of a diffeological space is the strongest topology making all of its plots continuous, and the initial topology of a differential space is the weakest topology making all of the functions of its differential structure continuous. The purpose of this section is to point out some properties of these topologies that are necessary for reflexivity.

We begin with a simple observation:

Lemma 6.1 (Compatible topologies). *Let X be a set, and let $\mathcal{D}$ and $\mathcal{F}$ be a diffeology and a differential structure on X, respectively. Suppose that $\mathcal{D}$ and $\mathcal{F}$ are compatible. Then the initial topology induced by $\mathcal{F}$ is contained in the D-topology induced by $\mathcal{D}$.*

Proof. Let $f \in \mathcal{F}$ and $p \in \mathcal{D}$. Because $f \circ p$ is (smooth, hence) continuous, for any open interval I the preimage $p^{-1}(f^{-1}(I))$ is open in the domain of p. Since $p \in \mathcal{D}$ is arbitrary, $f^{-1}(I)$ is D-open. Since f is arbitrary, the D-topology contains the initial topology. □

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Propositions 6.2 and 6.3 below are known [43, 64]; for completeness, we include their proofs.

Proposition 6.2 (Diffeological spaces are locally path-connected). *The D-topology of a diffeological space is locally path-connected. (Consequently, the connected components coincide with the path-connected components, and these components are both open and closed.)*

Proof. Let X be a diffeological space, let x be a point of X, and let V be a D-open neighbourhood of x. We need to show that V contains a path-connected neighbourhood of x. Let X' be the smooth path component of x in V; we will show that X' is D-open. Let $p\colon U \to V$ be a plot. For each connected component U' of U, the image $p(U')$ is smoothly path-connected, so it is either contained in X' or disjoint from X'. So the preimage $p^{-1}(X')$, being a union of connected components of U, is open in U. Varying the plot p, this shows that the preimage X' is D-open. Since X' is smoothly path-connected, it is path-connected. $\qquad\square$

Recall that a topological space X is **completely regular** if for every closed set C and point $x \in X \smallsetminus C$ there exists a continuous function $f\colon X \to [0,1]$ that vanishes on C and is equal to 1 on x. Spaces that are T_0 (points are distinguishable by open sets) and completely regular are called $T_{3\frac{1}{2}}$. Such spaces are automatically T_3 (regular and T_0), hence T_2 (Hausdorff), hence T_1 (points are closed).

Proposition 6.3 (Differential spaces are completely regular). *The initial topology of a differential space is completely regular. Consequently, any T_0 differential space is Hausdorff.*

Proof. Let $(X, \mathcal{F})$ be a differential space, equipped with the initial topology. Let C be a closed subset of X, and let x be a point in $X \smallsetminus C$. By the definition of the initial topology, there exist functions $h_1, \ldots, h_k \in \mathcal{F}$ and open intervals $I_1, \ldots, I_k$ such that $x \in \cap_{i=1}^k h_i^{-1}(I_i) \subset X \smallsetminus C$. Take $f := b \circ (h_1, \ldots, h_k)$ where $b\colon \mathbb{R}^k \to [0,1]$ is a smooth function whose support is contained in $I_1 \times \ldots \times I_k$ and such that $f(h_1(x), \ldots, h_k(x)) = 1$. $\qquad\square$

Lemma 6.1 and Propositions 6.2 and 6.3 imply the following topological necessary conditions for the compatibility of a diffeology and a differential structure.

Corollary 6.4. *Let X be a set, and let $\mathcal{D}$ and $\mathcal{F}$ be a diffeology and differential structure on X, respectively. Suppose that $\mathcal{D}$ and $\mathcal{F}$ are compatible.*

    (1) If the initial topology induced by $\mathcal{F}$ is T_0, then the D-topology induced by $\mathcal{D}$ is Hausdorff.

    (2) If the D-topology induced by $\mathcal{D}$ is connected, then the initial topology induced by $\mathcal{F}$ is path-connected.

Using the ideas developed above, we show some necessary conditions for reflexivity of diffeological spaces and of differential spaces.

Proposition 6.5 (T_0 reflexive diffeological spaces). *Every T_0 reflexive diffeological space is Hausdorff.*

Proof. Let $(X, \mathcal{D})$ be a diffeological space whose D-topology is T_0. Let $x, y \in X$ be distinct points such that for any open neighbourhoods U of x and V of y we have $U \cap V \neq \emptyset$. Since the D-topology is T_0, without loss of generality there exists an open neighbourhood W of y so that $x \notin W$. Define $p \colon \mathbb{R} \to \{x, y\}$ by

$$p(t) := \begin{cases} x & \text{if } t < 0, \\ y & \text{if } t \geq 0. \end{cases}$$

Then $p^{-1}(W) = [0, \infty)$. Since p is not continuous, $p \notin \mathcal{D}$.

Let $f \in \Phi\mathcal{D}$. Then $f(x) = f(y)$. (Otherwise, setting $a = f(x)$, $b = f(y)$, and $0 < \epsilon < \frac{|b-a|}{2}$, we have $x \in U := f^{-1}((a - \epsilon, a + \epsilon))$ and $y \in V := f^{-1}((b - \epsilon, b + \epsilon))$, and the intersection $U \cap V$ is empty. By Lemma 6.1, U and V are D-open. This contradicts the choice of x and y.) So the composition $f \circ p$ is (constant, hence) smooth. This shows that $p \in \Pi\Phi\mathcal{D}$. So $\mathcal{D}$ is not reflexive. $\qquad \square$

Proposition 6.6 (Locally smoothly path-connected differential spaces). *On every reflexive differential space, the initial topology is locally smoothly path-connected.*

Proof. Let $(X, \mathcal{F})$ be a reflexive differential space, equipped with the initial topology, let $x \in X$ be any point, and let U be an open neighbourhood of x in X. Let C be the smooth path component of x in U. It is enough to show that C is a neighbourhood of x.

Let $b \in \mathcal{F}$ be a smooth function such that $b(x) = 1$ and whose support $\mathrm{supp}(b)$ is contained in U (cf. the proof of Proposition 6.3). Define $g \colon X \to \mathbb{R}$ to be equal to b on C and zero outside C. Since $g^{-1}((0, \infty))$ contains x and is contained in C, it is enough to show that $g \in \mathcal{F}$.

Let $p \in \Pi\mathcal{F}$. Since the D-topology induced by $\Pi\mathcal{F}$ contains the initial topology induced by $\mathcal{F}$, the connected components of $p^{-1}(U)$, as well as $p^{-1}(X \smallsetminus \mathrm{supp}(b))$, are open in the domain of p. Let q be the restriction of p to one of the connected components of $p^{-1}(U)$. If the image of q does not meet C, then $g \circ q$ is identically 0. Suppose the image of q does meet C. Since C is a smooth path component of U, the image of q is contained in C, and so $g \circ q$ is equal to $b \circ q$, which is smooth. Since $g \circ p$ is identically zero on the $p^{-1}(X \smallsetminus \mathrm{supp}(b))$, it is smooth. It follows that $g \in \Phi\Pi\mathcal{F}$. Since $\mathcal{F}$ is reflexive, $g \in \mathcal{F}$. $\qquad \square$

Here are a couple of applications of the necessary conditions for reflexivity that we gave in Propositions 6.5 and 6.6.

Example 6.7 (Line with double origin). Glue two copies of the real line along the complement of the origin; write the quotient space as $X := (\mathbb{R}_1 \amalg \mathbb{R}_2)/\sim$. Consider its quotient diffeology. Its D-topology, which coincides with the quotient topology (see Proposition 3.5), is T_0 but not Hausdorff. So the diffeological space X is not reflexive; see Proposition 6.5.

Example 6.8 (Pinched Topologist's Sine Curve). Let $Y \subset \mathbb{R}^2$ be the image of the curve $\gamma \colon [0, 1] \to \mathbb{R}^2$ that is given by

$$\gamma(t) = \begin{cases} (0, 0) & \text{if } t = 0 \\ (t, t\sin(1/t)) & \text{if } 0 < t \leq 1, \end{cases}$$

equipped with the subspace topology and differential structure $\mathcal{F}_Y$ induced from $\mathbb{R}^2$ (see Proposition 3.2). The point $(0,0)$ does not have any neighbourhood that is smoothly path connected; this follows from the fact that the curve γ is not rectifiable. It follows from Proposition 6.6 that $\mathcal{F}_Y$ is not reflexive. $\lozenge$

Examples 6.9 and 6.10 below originally appeared in Jordan Watts' thesis [102, Examples 2.74 and 2.76]. In both of these examples, a quotient space is obtained from $\mathbb{R}$ by collapsing an interval to a point.

Example 6.9 ($\mathbb{R}$ modulo a closed interval). Let $X := \mathbb{R}/[0,1]$, equipped with the quotient diffeology $\mathcal{D}_X$, the quotient differential structure $\mathcal{F}_X$, and the quotient topology. The initial topology induced by $\mathcal{F}_X$ coincides with the quotient topology. This follows from the fact that the function $f\colon X \to \mathbb{R}$ whose pullback to $\mathbb{R}$ is

$$x \mapsto \begin{cases} -e^{-\frac{1}{|x|}} & \text{if } x < 0 \\ 0 & \text{if } x \in [0,1] \\ e^{-\frac{1}{|x-1|}} & \text{if } x > 1 \end{cases}$$

is in $\mathcal{F}_X$ and is a homeomorphism with respect to the quotient topology on X. The map $\varphi\colon X \to \mathbb{R}$ whose pullback to $\mathbb{R}$ is

$$x \mapsto \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \in [0,1] \\ x-1 & \text{if } x > 1 \end{cases}$$

is a diffeomorphism between the differential spaces $(X, \mathcal{F}_X)$ and $(\mathbb{R}, \mathcal{F})$, where $\mathcal{F}$ is the set of those smooth functions on $\mathbb{R}$ whose derivatives of all positive orders vanish at 0. The quotient diffeology $\mathcal{D}_X$ is not reflexive: $\varphi^{-1}\colon \mathbb{R} \to X$ is in $(\Pi\mathcal{F}_X$, which by Proposition 3.5 is) $\Pi\Phi\mathcal{D}_X$, but it does not have any smooth (or even continuous) lift to $\mathbb{R}$ in any neighbourhood of the origin. $\lozenge$

Example 6.10 ($\mathbb{R}$ modulo an open interval). Let $Y := \mathbb{R}/(0,1)$, equipped with the quotient diffeology $\mathcal{D}_Y$. Let $\pi_Y\colon \mathbb{R} \to Y$ be the quotient map. The one-point set $\pi_Y((0,1))$ is open with respect to (the quotient topology, hence) the D-topology. In fact, this topology is T_0. But this topology is not Hausdorff: the points $\pi_Y(0)$ and $\pi_Y(1)$ do not have disjoint neighbourhoods. By Proposition 6.5, $\mathcal{D}_Y$ is not reflexive.

Recall that the quotient diffeology $\mathcal{D}_Y$ induces the quotient differential structure $\mathcal{F}_Y$. In contrast with the previous example, the initial topology induced by $\mathcal{F}_Y$ is strictly smaller than the quotient topology: in the initial topology, the points $\pi_Y(0)$, $\pi_Y((0,1))$, and $\pi_Y(1)$ are topologically indistinguishable. The corresponding Kolmogorov quotient (see Remark 6.11 below) can be identified with $\mathbb{R}/[0,1]$. $\lozenge$

Examples 6.9 and 6.10 motivate the following general remark on indistinguishable points.

Remark 6.11 (Indistinguishable points). Given a differential space X, one can create another differential space Y by creating "clones" of points of X, which are not distinguishable by smooth functions. In fact, up to isomorphism, such "cloning" is the only way of obtaining

a differential space whose initial topology is not Hausdorff. More precisely, let $(X, \mathcal{F})$ be a differential space. Let X_K be the quotient of X by the equivalence relation where $x \sim x'$ iff $f(x) = f(x')$ for all $f \in \mathcal{F}$. Equip X_K with the quotient differential structure (see Definition 3.4). Up to isomorphism, X is obtained from X_K by "cloning". The initial topology of X_K induced by $\mathcal{F}_K$ coincides with its quotient topology (contrast with Remark 3.6). It is T_0 (hence, by Proposition 6.3, it is $T_{3\frac{1}{2}}$). The T_0 differential space X_K, with the map $\pi \colon X \to X_K$, satisfies the following universal property:

> If Y is a T_0 differential space and $\varphi \colon X \to Y$ is a functionally smooth map, then there exists a unique functionally smooth map $\varphi_K \colon X_K \to Y$ such that $\varphi = \varphi_K \circ \pi$. $\hfill$ (6.12)

Topologically, X_K coincides with the *Kolmogorov quotient* of X, which is the quotient by the equivalence relation where $x \sim x'$ if and only if each open neighbourhood of x contains x' and vice versa. It also coincides with the *Hausdorffification* of X (whose construction for more general topological spaces may require iterated quotients, and possibly transfinite recursion; see [99]). These satisfy universal properties similar to (6.12) but with respect to continuous maps to T_0 spaces and to Hausdorff spaces, respectively. $\hfill \diamond$

Example 6.13 ($\mathbb{R}$ Modulo $(x \sim 2x)$). Take $X = \mathbb{R}/{\sim}$ where $x \sim y$ if and only if $y = 2^m x$ for some integer m. Its quotient diffeology is non-trivial, but its differential structure consists of the constant functions, so its quotient diffeology is not reflexive. Another way to see this is to note that the D-topology coincides with the quotient topology (Proposition 3.5), which is T_0, but not Hausdorff, and to apply Proposition 6.5. $\hfill \lozenge$

We have seen several non-reflexive quotients of reflexive diffeological spaces: the irrational torus $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ (Example 3.9), the orbifold $\mathbb{R}^2/\mathbb{Z}_2$ (Example 4.4), the quotients $\mathbb{R}/[0, 1]$ and $\mathbb{R}/(0, 1)$ (Examples 6.9 and 6.10), and the quotient $\mathbb{R}/(x \sim 2x)$ (Example 6.13). In the first three of these, the initial topology coincides with the quotient topology; in the last two, the initial topology is different from the quotient topology. The second and third of these are Hausdorff; the others are non-Hausdorff.

These examples raise the question of whether the initial topology coincides with the quotient topology on a quotient differential space that is Hausdorff. The answer is no: the following example, inspired by the Moore-Niemytzki plane (see [97, Example 82], where it is called the Niemytzki tangent disk topology), exhibits a quotient differential space whose initial topology is Hausdorff but is strictly smaller than its quotient topology.

Example 6.14 (A Moore-Niemytzki-like topology). Let H be the open upper half plane in $\mathbb{R}^2$, and let $\overline{H}$ be its closure in $\mathbb{R}^2$. Equip H, $\overline{H}$, and the sets

$$C_x := H \cup \{(x, 0)\},$$

for $x \in \mathbb{R}$, with the subspace differential structures and subset diffeologies that are induced from $\mathbb{R}^2$. Equip

$$X := \coprod_{x \in \mathbb{R}} C_x$$

with the coproduct differential structure, denoted $\mathcal{F}_X$, and with the coproduct diffeology, denoted $\mathcal{D}_X$. Let Y be the gluing of the components of X along H, equipped with the

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quotient differential structure $\mathcal{F}_Y$ and the quotient diffeology $\mathcal{D}_Y$. Denote the inclusion maps of the C_x into X and the quotient map from X to Y as

$$i_x \colon C_x \to X \quad \text{and} \quad \pi \colon X \to Y.$$

There exists a unique bijection

$$\varphi \colon Y \to \overline{H}$$

such that, for each $x \in \mathbb{R}$, the composition $\varphi \circ \pi \circ i_x \colon C_x \to \overline{H}$ is the inclusion map of C_x into $\overline{H}$. Because this inclusion map is smooth, and because the components C_x form an open covering of X, it follows that $\varphi \circ \pi \colon X \to \overline{H}$ is smooth. By the definition of the quotient differential structure, it follows that $\varphi \colon Y \to \overline{H}$ is smooth.

We claim:

> For any function $g \colon \overline{H} \to \mathbb{R}$,
> if $g|_{C_x} \colon C_x \to \mathbb{R}$ is functionally smooth for all x, $\hspace{2em}$ (6.15)
> then g is functionally smooth.

Indeed, let $g \colon \overline{H} \to \mathbb{R}$, and suppose that $g|_{C_x} \colon C_x \to \mathbb{R}$ is smooth for all x. Then $g|_H \colon H \to \mathbb{R}$ is smooth. Let $x \in \mathbb{R}$. Because $g|_{C_x}$ is smooth, there is an open neighbourhood U_x of $(x,0)$ in $\mathbb{R}^2$ and a smooth function $h_x \in C^\infty(U_x)$ that coincides with g on the subset $C_x \cap U_x$. Let $(x',0) \in U_x$. For all sufficiently large n, we have $(x', \frac{1}{n}) \in U_x$, and so $h_x(x', \frac{1}{n}) = g(x', \frac{1}{n})$. But $h_x(x', \frac{1}{n}) \xrightarrow[n\to\infty]{} h_x(x', 0)$ because h_x is smooth on U_x, and $g(x', \frac{1}{n}) \xrightarrow[n\to\infty]{} g(x', 0)$ because g is smooth on $C_{x'}$, so $h_x(x', 0) = g(x', 0)$. Because x' was arbitrary, h_x coincides with g on all of $\overline{H} \cap U_x$. Because x was arbitrary, it follows that g is smooth.

Because the inverse $\varphi^{-1} \colon \overline{H} \to Y$ satisfies

$$\varphi^{-1}|_{C_x} = \pi \circ i_x \colon C_x \to Y$$

for all x, it follows from (6.15) that this inverse is functionally smooth. Thus, φ is a functional diffeomorphism.

By Christensen-Sinnamon-Wu [19, Lemma 3.17], since each C_x is a convex subset of $\mathbb{R}^2$, its D-topology is equal to its subspace topology induced from $\mathbb{R}^2$. Because the D-topology induced by the quotient diffeology $\mathcal{D}_Y$ on Y is equal to the quotient topology τ_Y on Y induced from X (Proposition 3.5), a subset A of Y is closed if and only if $i_x^{-1}(\pi^{-1}(A))$ is closed in C_x for all x. Consequently, every subset of $\varphi^{-1}(\partial H)$ is closed. Thus, while φ is a continuous bijection from (Y, τ_Y) to $\overline{H}$, it is not a homeomorphism.

It follows that the initial topology on Y is strictly smaller than the quotient topology τ_Y. Note, though, that both of these topologies are Haudorff.

Finally, note that the diffeology $\mathcal{D}_Y$ is not reflexive. Indeed, because $\varphi \colon Y \to \overline{H}$ is a functional diffeomorphism, the map $x \mapsto \varphi^{-1}(x, 0)$ from $\mathbb{R}$ to Y is functionally smooth. But this map is not diffeologically smooth. $\lozenge$

Notes.

- In a diffeological space, the path components (with respect to the D-topology) coincide with the (diffeologically) smooth path components [47, Article 5.7].

- By the proof of Proposition 6.3, every differential space is *smoothly regular*, in the following sense [63]: for every closed set C and point $x \in X \smallsetminus C$ there exists a smooth function $f \colon X \to [0,1]$ that vanishes on C and is equal to 1 on x.

- By [101, Theorem 3.10]: given a diffeology $\mathcal{D}$ and a differential structure $\mathcal{F}$ that are compatible, the D-topology coincides with the initial topology if and only if the D-topology is smoothly regular.

- In Example 6.14, we showed that the D-topology of Y is Hausdorff. However, it is not completely regular. Indeed, since any subset of $\varphi^{-1}(\partial H)$ is closed, there is no continuous function that separates $\pi \circ i_x((x,0))$ from its complement in $\varphi^{-1}(\partial H)$. It follows that the D-topology of Y is not smoothly regular.

- In Corollary 6.4 we can obtain a stronger statement: if the initial topology is T_0, then the D-topology is completely Hausdorff, *i.e.*, points are separated by continuous functions $X \to [0,1]$ (in fact, by diffeologically smooth functions contained in $\mathcal{F}$).

A. Proofs

A.1. **Reflexive stability.** Recall that, given a set X with a collection $\mathcal{D}_0$ of parametrizations and a collection $\mathcal{F}_0$ of real-valued functions, $\Phi\mathcal{D}_0$ denotes the set of those real-valued functions $f \colon X \to \mathbb{R}$ whose precomposition with each element of $\mathcal{D}_0$ is infinitely-differentiable, and $\Pi\mathcal{F}_0$ denotes the set of those parametrisations $p \colon U \to X$ whose composition with each element of $\mathcal{F}_0$ is infinitely-differentiable.

Lemma A.1. *Fix a set X, and let $\mathcal{D}_0$ be a family of parametrisations into X. Then $\Phi\mathcal{D}_0$ is a differential structure on X.*

Proof. We first show smooth compatibility. Let $f_1, ..., f_k \in \Phi\mathcal{D}_0$ and let $F \in C^\infty(\mathbb{R}^k)$. Let $p \in \mathcal{D}_0$. Because the components of $(f_1, \ldots, f_k) \circ p$ are infinitely-differentiable, the composition $F \circ (f_1, \ldots, f_k) \circ p$ is infinitely-differentiable. Because p is arbitrary, $F \circ (f_1, \ldots, f_k)$ is in $\Phi\mathcal{D}_0$.

We now show locality. Equip X with the initial topology of $\Phi\mathcal{D}_0$. Let $f \colon X \to \mathbb{R}$ be a function satisfying: for every $x \in X$ there is an open neighbourhood V of x in X and a function $g \in \Phi\mathcal{D}_0$ such that $f|_V = g|_V$. We want to show that $f \in \Phi\mathcal{D}_0$. Fix $(p \colon U \to X) \in \mathcal{D}_0$. Let $V \subseteq X$ be an open subset, and let $g \in \Phi\mathcal{D}_0$ be a function such that $f|_V = g|_V$. Then $f \circ p|_{p^{-1}(V)} = g \circ p|_{p^{-1}(V)}$. The pre-image $p^{-1}(V)$ is open in U. (Indeed, V is a union of pre-images $h^{-1}((a,b))$ of open intervals (a,b) under functions h in $\Phi\mathcal{D}_0$, so $p^{-1}(V)$ is a union of the pre-images $(h \circ p)^{-1}((a,b))$, and $h \circ p \colon U \to \mathbb{R}$ is infinitely-differentiable, hence continuous, because $p \in \mathcal{D}_0$ and $h \in \Phi\mathcal{D}_0$.) Since each such $g \circ p$ is smooth in U and is covered by such open sets $p^{-1}(V)$, and since smoothness is a local condition, $f \circ p \colon U \to \mathbb{R}$ is smooth. Since $p \in \mathcal{D}_0$ is arbitrary, $f \in \Phi\mathcal{D}_0$. $\qquad\square$

Lemma A.2. *Fix a set X, and let $\mathcal{F}_0$ be a set of real-valued functions on X. Then $\Pi\mathcal{F}_0$ is a diffeology on X.*

Proof. To see that $\Pi\mathcal{F}_0$ contains all the constant maps into X, note that if $p\colon U \to X$ is constant then for any $f \in \mathcal{F}_0$ the composition $f \circ p\colon U \to X$ is constant, hence infinitely-differentiable.

Next, we show locality. Let $p\colon U \to X$ be a parametrisation such that for every $u \in U$ there is an open neighbourhood V of u in U such that $p|_V \in \Pi\mathcal{F}_0$; we want to show that $p \in \Pi\mathcal{F}_0$. Let $f \in \mathcal{F}_0$. For any $u \in U$, there is an open neighbourhood V of u in U such that $f \circ p|_V$ is smooth. Since smoothness on U is a local condition, $f \circ p\colon U \to \mathbb{R}$ is smooth. Since $f \in \mathcal{F}_0$ is arbitrary, $p \in \Pi\mathcal{F}_0$.

Finally, we show smooth compatibility. Let U and V be open subsets of cartesian spaces, and let $F\colon V \to U$ be a smooth map. Let $(p\colon U \to X) \in \Pi\mathcal{F}_0$. For any $f \in \mathcal{F}_0$, we have that $f \circ p$ is smooth, so $f \circ p \circ F$ is smooth. Because $f \in \mathcal{F}_0$ is arbitrary, $p \circ F \in \Pi\mathcal{F}_0$. $\qquad\square$

Proof of Reflexive Stability (Proposition 2.7). By Lemma A.1, $\mathcal{F} := \Phi\mathcal{D}_0$ is a differential structure; by Remark 2.5, it is reflexive. By Lemma A.2, $\mathcal{D} := \Pi\mathcal{F}_0$ is a diffeology; by Remark 2.5, it is reflexive. $\qquad\square$

A.2. Isomorphism of categories of reflexive spaces.
Recall that $\boldsymbol{\Phi}(X, \mathcal{D}) = (X, \Phi\mathcal{D})$ on objects and $\boldsymbol{\Phi}(F) = F$ on morphisms.

Proof that $\boldsymbol{\Phi}$ is a functor from the category of diffeological spaces to the category of reflexive differential spaces. By Proposition 2.7, if $(X, \mathcal{D})$ is a diffeological space then $(X, \Phi\mathcal{D})$ is a reflexive differential space. We need to show that if $F\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is diffeologically smooth then F is also functionally smooth as a map between the reflexive differential spaces $(X, \Phi\mathcal{D}_X)$ and $(Y, \Phi\mathcal{D}_Y)$. Let $f \in \Phi\mathcal{D}_Y$. Let $p \in \mathcal{D}_X$. Because F is diffeologically smooth, $F \circ p \in \mathcal{D}_Y$. This and the fact that $f \in \Phi\mathcal{D}_Y$ imply that $f \circ F \circ p$ is infinitely-differentiable. Since $p \in \mathcal{D}_X$ is arbitrary, this shows that $f \circ F \in \Phi\mathcal{D}_X$. Since $f \in \Phi\mathcal{D}_Y$ is arbitrary, this shows that F is functionally smooth. $\qquad\square$

Recall that $\boldsymbol{\Pi}(X, \mathcal{F}) = (X, \Pi\mathcal{F})$ on objects and $\boldsymbol{\Pi}(F) = F$ on morphisms.

Proof that $\boldsymbol{\Pi}$ is a functor from differential spaces to reflexive diffeological spaces. By Proposition 2.7, if $(X, \mathcal{F})$ is a differential space, then $(X, \Pi\mathcal{F})$ is a reflexive diffeological space. We need to show that if $F\colon (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is functionally smooth then F is also diffeologically smooth as a map between the reflexive diffeological spaces $(X, \Pi\mathcal{F}_X)$ and $(Y, \Pi\mathcal{F}_Y)$. Let $p \in \Pi\mathcal{F}_X$. Let $f \in \mathcal{F}_Y$. Because F is functionally smooth, $f \circ F \in \mathcal{F}_X$. This and the fact that $p \in \Pi\mathcal{F}_X$ imply that $f \circ F \circ p$ is smooth. Since $f \in \mathcal{F}_Y$ is arbitrary, this shows that $F \circ p \in \Pi\mathcal{F}_Y$. Because $p \in \Pi\mathcal{F}_X$ is arbitrary, this shows that F is diffeologically smooth. $\qquad\square$

Proof of isomorphism of categories of reflexive spaces (Theorem 2.11). If $(X, \mathcal{F})$ is a reflexive differential space, then $\boldsymbol{\Phi} \circ \boldsymbol{\Pi}(X, \mathcal{F}) = (X, \Phi\Pi\mathcal{F}) = (X, \mathcal{F})$. If $(X, \mathcal{D})$ is a reflexive diffeological space, then $\boldsymbol{\Pi} \circ \boldsymbol{\Phi}(X, \mathcal{D}) = (X, \Pi\Phi\mathcal{D}) = (X, \mathcal{D})$. This and the fact that $\boldsymbol{\Pi}$ and $\boldsymbol{\Phi}$ send every map to itself shows that the restriction of the functor $\boldsymbol{\Phi}$ to the subcategory of reflexive diffeological spaces and the restriction of the functor $\boldsymbol{\Pi}$ to the subcategory of reflexive differential spaces are inverses of each other and give an isomorphism of categories. $\qquad\square$

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A.3. **Frölicher spaces as reflexive spaces.** Recall that $\Xi(X, \mathcal{C}, \mathcal{F}) = (X, \mathcal{F})$ on objects and $\Xi(F) = F$ on morphisms.

Proof that Ξ is a functor from the category of Frölicher spaces to the category of reflexive diffeological spaces. Let $(X, \mathcal{C}, \mathcal{F})$ be a Frölicher space. In particular, $\mathcal{F} = \Phi\mathcal{C}$. By Proposition 2.7, $\mathcal{F}$ is a reflexive differential structure. Thus, Ξ takes Frölicher spaces to reflexive differential spaces. As noted in Definition 2.12, if a map of Frölicher spaces is Frölicher smooth, then it is also functionally smooth. $\qquad\square$

Recall that $\boldsymbol{\Gamma}(X, \mathcal{F}) = (X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ on objects and $\boldsymbol{\Gamma}(F) = F$ on morphisms.

Proof that $\boldsymbol{\Gamma}$ is a functor from the category of differential spaces to the category of Frölicher spaces. Let $(X, \mathcal{F})$ be a differential space. The equality $\Gamma\Phi\Gamma\mathcal{F} = \Gamma\mathcal{F}$ shows that $(X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ is a Frölicher space. As noted in Definition 2.12, if a map of differential spaces is functionally smooth, then it is also Frölicher smooth. $\qquad\square$

Proof of "Frölicher spaces as reflexive spaces" (Theorem 2.13). If f is a real-valued function on a diffeological space $(X, \mathcal{D})$ and $f \circ c$ is infinitely-differentiable for every plot c in $\mathcal{D}$ with domain $\mathbb{R}$, then $f \circ p$ is infinitely-differentiable for every plot $p \colon U \to X$ in $\mathcal{D}$. Indeed, by Boman's theorem [11, Theorem 1] it is enough to show that the composition $f \circ p \circ \gamma$ is infinitely-differentiable for every infinitely-differentiable curve $\gamma \colon \mathbb{R} \to U$, and this is true because $p \circ \gamma$ is a plot in $\mathcal{D}$ with domain $\mathbb{R}$.

If $(X, \mathcal{C}, \mathcal{F})$ is a Frölicher space, then $\boldsymbol{\Gamma} \circ \Xi(X, \mathcal{C}, \mathcal{F}) = \boldsymbol{\Gamma}(X, \mathcal{F}) = (X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F}) = (X, \mathcal{C}, \mathcal{F})$. If $(X, \mathcal{F})$ is a reflexive differential space, then $\Xi \circ \boldsymbol{\Gamma}(X, \mathcal{F}) = \Xi(X, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F}) = (X, \Phi\Gamma\mathcal{F}) = (X, \Phi\Pi\mathcal{F}) = (X, \mathcal{F})$. Here, the equality $\Phi\Gamma\mathcal{F} = \Phi\Pi\mathcal{F}$ is obtained from the previous paragraph by setting $\mathcal{D} = \Pi\mathcal{F}$. This and the fact that Π and $\boldsymbol{\Gamma}$ send every map to itself shows that the functor Ξ and the restriction of the functor Π to the category of reflexive differential spaces are inverses of each other and give an isomorphism of categories. $\qquad\square$

A.4. **Intersecting submanifolds.**

Proof of Part (1) of Example 5.1. Recall that $E \subset \mathbb{R}^2$ is the union of the two coordinate axes and $C^\infty(E)$ is its subspace differential structure, that $\mathcal{D}_X$ is the quotient diffeology on $X := (\mathbb{R}_1 \amalg \mathbb{R}_2)/(0_1 \sim 0_2)$, and that $\varphi \colon X \to E$ is the bijection whose pullback to $\mathbb{R}_1$ is $x \mapsto (x, 0)$ and whose pullback to $\mathbb{R}_2$ is $y \mapsto (0, y)$. Fix a real-valued function $f \colon E \to \mathbb{R}$. Define $f_i \colon \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, by $f_1(x) = f(x, 0)$ and $f_2(y) = f(0, y)$. We need to show that each of the conditions $f \in C^\infty(E)$ and $\varphi^* f \in \Phi\mathcal{D}_X$ is equivalent to f_1 and f_2 being smooth.

First, suppose that $f \in C^\infty(E)$. Then f_1 and f_2, being the compositions of the smooth maps $x \mapsto (x, 0)$ and $y \mapsto (0, y)$ with a smooth extension of f to $\mathbb{R}^2$, are smooth.

Now, suppose that $\varphi^* f \in \Phi\mathcal{D}_X$. Let $i = 1$ or $i = 2$. The inclusion map of the ith copy of $\mathbb{R}$ in X, which we denote $I_i \colon \mathbb{R} \to X$, is in the quotient diffeology $\mathcal{D}_X$. By the definition of $\Phi\mathcal{D}_X$, the composition $(\varphi^* f) \circ I_i$ is smooth. This composition is f_i, so f_i is smooth.

Now, suppose that f_1 and f_2 are smooth. Then $(x, y) \mapsto f_1(x) + f_2(y) - f(0, 0)$ is a smooth extension of f to $\mathbb{R}^2$. This shows that $f \in C^\infty(E)$.

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Still assuming that f_1 and f_2 are smooth, let $p \colon U \to X$ be a plot in the quotient diffeology $\mathcal{D}_X$. Let $u \in U$ be any point. Let V be a connected neighbourhood of u in U and $\tilde{p} \colon V \to \mathbb{R}_1 \amalg \mathbb{R}_2$ a smooth lifting of $p|_V$; these exist by the definition of the quotient diffeology. By continuity, the image of $\tilde{p}$ is contained in $\mathbb{R}_i$ for some $i \in \{1,2\}$. The map $(\varphi^* f) \circ p|_V \colon V \to \mathbb{R}$, being the composition of the smooth maps $\tilde{p}$ and f_i, is smooth. Since smoothness is a local condition and $u \in U$ is arbitrary, $(\varphi^* f) \circ p \colon U \to \mathbb{R}$ is smooth. Since $p \in \mathcal{D}_X$ is arbitrary, $\varphi^* f \in \Phi \mathcal{D}_X$. $\qquad\square$

Sketch of proof of Example 5.3. The proof is similar to that of Example 5.1 once we make the following observation. Let $\mathcal{I}$ be a set of subsets of $\{1, \ldots, n\}$, and let $E_{\mathcal{I}} := \bigcup_{I \in \mathcal{I}} \mathbb{R}^I$ where $\mathbb{R}^I$ is the span of the x_i-axes for $i \in I$. For every $I \in \mathcal{I}$, let $\mathrm{pr}_I \colon \mathbb{R}^n \to \mathbb{R}^I$ denote the natural projection map. Then, for every function $f \colon E_{\mathcal{I}} \to \mathbb{R}$ that is smooth on $\mathbb{R}^I$ for each $I \in \mathcal{I}$, the function

$$\sum_{\substack{A \subseteq \mathcal{I} \\ A \neq \emptyset}} (-1)^{1+|A|} f \circ \mathrm{pr}_{\cap A} \colon \mathbb{R}^n \to \mathbb{R}$$

is a a smooth extension of f to $\mathbb{R}^n$. Indeed, to see that this function coincides with f on $E_{\mathcal{I}}$, we argue as follows. Fix $x \in E_{\mathcal{I}}$. When A is the empty set, then $\mathrm{pr}_{\cap A}(x) = x$ (this is not the same as when $\bigcap A = \emptyset$, in which case $\mathrm{pr}_{\emptyset} = 0$). Hence $(-1)^{1+|A|} f \circ \mathrm{pr}_{\cap A}(x) = -f(x)$. So it is enough to show that the sum

$$\sum_{A \subset \mathcal{I}} (-1)^{1+|A|} f(\mathrm{pr}_{\cap A}(x))$$

vanishes. We can write this sum as

$$\sum_{\substack{A \subseteq \{I_1, \ldots, I_m\} \\ B \subseteq \{J_1, \ldots, J_s\}}} (-1)^{1+|A|+|B|} f(\mathrm{pr}_{(\cap A) \cap (\cap B)}(x)),$$

where $I_1, \ldots, I_m$ is an enumeration of the set $\{I \in \mathcal{I} \mid x \in \mathbb{R}^I\}$ and $J_1, \ldots, J_s$ is an enumeration of the set $\mathcal{I} \setminus \{I_1, \ldots, I_m\}$. Because $x \in \mathbb{R}^{I_i}$ for all $i = 1, \ldots, m$, we have $\mathrm{pr}_{(\cap A) \cap (\cap B)}(x) = \mathrm{pr}_{\cap B}(x)$. So we can write the above sum as

$$\sum_{B \subseteq \{J_1, \ldots, J_s\}} (-1)^{1+|B|} f(\mathrm{pr}_{\cap B}(x)) \sum_{A \subseteq \{I_1, \ldots, I_m\}} (-1)^{|A|}.$$

Because $x \in E_{\mathcal{I}}$, we have $m \geq 1$, and so $\displaystyle\sum_{A \subseteq \{I_1, \ldots, I_m\}} (-1)^{|A|} = 0$, so the above sum vanishes.

$\qquad\square$

Completion of the proof of Part (1) of Example 5.4. Recall that $E \subseteq \mathbb{R}^3$ is the union of the three coordinate axes; $S \subseteq \mathbb{R}^2$ is the union $l_1 \cup l_2 \cup l_3$ where l_1 is the x-axis, l_2 is the y-axis, and l_3 is the line given by $y = x$; and $\varphi \colon E \to S$ is the map $(t,0,0) \mapsto (t,0)$, $(0,t,0) \mapsto (0,t)$, $(0,0,t) \mapsto (t,t)$. We need to prove that $\mathcal{D}_S \subseteq \varphi \circ \mathcal{D}_E$. For this, we fix an open subset U of $\mathbb{R}^k$ for some k and a plot

$$p \colon U \to S$$

of S, and we need to prove that $\varphi^{-1} \circ p \colon U \to E$ is a plot of E. Let $(p_1, p_2) \colon U \to \mathbb{R}^2$ be the composition of $p \colon U \to S$ with the inclusion map $S \to \mathbb{R}^2$, and let $q \colon U \to \mathbb{R}^3$ be the

composition of $\varphi^{-1} \circ p \colon U \to E$ with the inclusion map $E \to \mathbb{R}^3$. On each subset $p^{-1}(l_i)$, the map q coincides with the map g_i, where

$$g_1(u) = (p_1(u), 0, 0), \quad g_2(u) = (0, p_2(u), 0), \quad \text{and} \quad g_3(u) = (0, 0, p_1(u)).$$

The maps $g_i \colon U \to \mathbb{R}$ are smooth (because p is a plot), and we need to prove that the map $q \colon U \to \mathbb{R}^3$ is smooth.

Let

$$U_i = \text{interior}(p^{-1}(l_i)) \quad \text{for } i = 1, 2, 3, \quad \text{and let} \quad W = \bigcup_{j \neq k} \overline{U_j} \cap \overline{U_k}.$$

We claim that

$$U = U_1 \cup U_2 \cup U_3 \cup W, \quad \text{and} \quad \overline{U_i} \subseteq U_i \cup W \quad \text{for } i = 1, 2, 3. \tag{A.3}$$

Indeed, let $u \in U \smallsetminus (U_1 \cup U_2 \cup U_3)$. Then $p(u) = 0$ and each neighbourhood of u contains points from at least two of the sets $p^{-1}(l_i \smallsetminus \{0\})$ for $i = 1, 2, 3$. So there exist $j \neq k$ such that every neighbourhood of u contains points of $p^{-1}(l_j \smallsetminus \{0\})$ and points of $p^{-1}(l_k \smallsetminus \{0\})$. Then $u \in \overline{U_j} \cap \overline{U_k}$, and so $u \in W$. This proves the first part of (A.3). Now suppose that $u \in \overline{U_i}$. By the first part of (A.3), either $u \in U_i$, or $u \in W$, or $u \in U_j$ for $j \neq i$. In the first or second case, $u \in U_i \cup W$. In the third case, $u \in \overline{U_i} \cap U_j \subseteq \overline{U_i} \cap \overline{U_j} \subseteq W \subseteq U_i \cup W$. This proves the second part of (A.3).

Let $t_1, \ldots, t_k$ be the coordinates on $U \subseteq \mathbb{R}^k$. Consider the differentiation operators

$$D_m = \frac{\partial^{m_1 + \ldots + m_k}}{\partial t_1^{m_1} \cdots \partial t_k^{m_k}} \quad \text{for } m = (m_1, \ldots, m_k) \in \mathbb{Z}_{\geq 0}^k.$$

For each $i \in \{1, 2, 3\}$ the restriction $D_m(p_1, p_2)|_{U_i}$ takes values in the linear subspace l_i of $\mathbb{R}^2$. By continuity, $(D_m(p_1, p_2))|_{\overline{U_i}}$ also takes values in l_i. If $j \neq k$, then, because $l_j \cap l_k = \{0\}$, the derivatives $D_m(p_1, p_2)$ vanish on $\overline{U_j} \cap \overline{U_k}$. So

$$\text{if } u \in W, \text{ then } D_m g_i(u) = 0 \quad \text{for all } m \in \mathbb{Z}_{\geq 0}^k \text{ and } i = 1, 2, 3. \tag{A.4}$$

Consider the following statements.

(I_m) $D_m q \colon U \to \mathbb{R}^3$ exists throughout U and vanishes on W.
(II_m) For each $i = 1, 2, 3$, $D_m q$ (exists and) coincides with $D_m g_i$ on $\overline{U_i}$.
(III_m) $D_m q \colon U \to \mathbb{R}^3$ (exists and) is continuous.

(I_m) implies (II_m). This follows by the second part of (A.3) from the facts that q and g_i coincide on the open set U_i and that, assuming (I_m), $D_m q$ and $D_m g_i$ both vanish at the points of W ($D_m q$ by hypothesis and $D_m g_i$ by (A.4)).

(I_m) and (II_m) imply (III_m). This is because $D_m q$ coincides with continuous maps on the closed sets $\overline{U_1}, \overline{U_2}, \overline{U_3}, W$, whose union is U (by the first part of (A.3)).

We will now show that (I_m) is true for all m. For $m = 0$, this follows from (A.4). Arguing by induction, assume that ($\text{I}_{m'}$), and hence ($\text{II}_{m'}$) and ($\text{III}_{m'}$), are true, and let m be obtained from m' by increasing one of its coordinates by one, say, the ℓth coordinate. Because q coincides with the smooth map g_i on the open set U_i, the derivative $D_m q$ exists

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on the U_is. Denote the ℓth standard basis element of $\mathbb{R}^k$ by e_ℓ. Fix a point $u \in W$. For any h such that $u + he_\ell \in U$, we claim that

$$\frac{D_{m'}q(u + he_\ell) - D_{m'}q(u)}{h} = \begin{cases} \dfrac{D_{m'}g_i(u + he_\ell) - D_{m'}g_i(u)}{h} & \text{if } u + he_\ell \in U_i \\[2ex] 0 & \text{if } u + he_\ell \notin U_1 \cup U_2 \cup U_3. \end{cases} \tag{A.5}$$

The first case is because q and g_i, and hence their derivatives, coincide on the open subset U_i, and because $D_{m'}q(u) = 0$ (by $(\mathrm{I}_{m'})$) and $D_{m'}g_i(u) = 0$ (by (A.4) for m'). In the second case $u + he_\ell \in W$ (by the first part of (A.3)) and $u \in W$ (by assumption), so $D_{m'}q(u + he_\ell) = D_{m'}q(u) = 0$ (by $(\mathrm{I}_{m'})$). Since each term on the right hand side of (A.5) converges to zero as $h \to 0$ (by (A.4) for m), we conclude that the left hand side converges to zero, so $D_m q(u)$ exists and is equal to zero. Because $u \in W$ is arbitrary, we obtain (I_m).

Thus, (I_m), (II_m), and (III_m) are true for all m. In particular, q is smooth, as required. $\square$

B. Comparisons with Other Structures

In this appendix, we compare diffeological and differential spaces (and hence Frölicher spaces by Theorem 2.13) with some of the other generalisations of differential calculus that appear in the literature. We refer to Andrew Stacey's paper [96] for a more extensive comparative study of Chen spaces, Smith spaces, diffeological spaces, Frölicher spaces, and differential spaces; we do not address Chen spaces nor Smith spaces here. For a direct comparison of diffeological and Chen spaces, see [57]. We refer to Joao Nuno Mestre's Ph.D. thesis [75, Chapter 2] for a comparative study of differential spaces, Mostow spaces, subcartesian spaces, differentiable spaces (not the same as "differential" spaces), and C^∞-schemes.

For the sake of brevity, we do not give definitions of the structures discussed below. Instead, we refer the reader to the following sources that are more focused on these subjects (we do not claim that this is an exhaustive list):

- Lie groupoids and stacks [10, 67, 77, 84];
- sheaves of sets over a site [6];
- synthetic differential geometry and C^∞-schemes [30, 52, 59, 78];
- Mostow spaces [79];
- subcartesian spaces [2, 3, 4, 91];
- differentiable spaces [37, 93, 94, 95].

B.1. **From Lie groupoids to diffeological spaces.** There is a functor from the category of Lie groupoids to the category of diffeological spaces, sending a Lie groupoid to its orbit space equipped with the quotient diffeology, and smooth morphisms between Lie groupoids to diffeologically smooth maps between the orbit spaces. This functor is neither faithful nor full, even when restricted to effective étale proper Lie groupoids (*i.e.* effective orbifolds); see the examples of Moshe Zadka [48, Examples 24 and 25].

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However, this functor does factor through Morita equivalence. In fact, the bicategory of Lie groupoids with bibundles between them and isomorphisms of bibundles as 2-arrows (see [67, 77] for definitions) has a pseudofunctor to diffeological spaces, in which bibundles are sent to diffeologically smooth maps, and 2-arrows are sent to trivial 2-arrows (diffeological spaces form an honest 1-category); this is proven by Watts in [104, Theorem 3.8]. Moreover, when this pseudofunctor is restricted to effective orbifolds with "locally invertible" bibundles between them, then this is an equivalence of categories onto diffeological orbifolds with locally invertible smooth maps between them. In fact, Karshon and Miyamoto in [55] (following an earlier preprint by Karshon and Zoghi that was announced in [109]) prove that this restriction works in the more general setting of so-called effective quasifold groupoids and diffeological quasifolds.

An important feature of this pseudofunctor is that much of the isotropic information is generally lost. Indeed, consider $\mathrm{U}(n)$ and $\mathrm{SO}(2n)$ acting on $\mathbb{R}^{2n}$ by rotations. The resulting diffeological quotients are diffeologically diffeomorphic, but the groupoids are not even Morita equivalent: the stabilisers at the origin are not isomorphic. However, in certain circumstances these stabilisers can be recovered: it follows from [48] that the restricted pseudofunctor from effective étale proper Lie groupoids to diffeological orbifolds is injective on objects. It follows from [24] that the same pseudofunctor restricted to action groupoids of faithful linear representations of the circle is also injective on objects. See Example 4.1 for more such examples. It would be interesting to pin down precisely which isotropic information is lost and which is retained, even in the case of linear compact group actions. There are also examples of Lie groupoids whose orbit spaces are diffeologically diffeomorphic, the isotropic information is the same, but the Lie groupoids are not Morita equivalent; see [55, Section 7].

B.2. **From stacks to diffeological spaces.** There is an equivalence of bicategories between the bicategory of Lie groupoids and the 2-category of differentiable stacks, and thus by B.1 there is a pseudofunctor from differentiable stacks to diffeological spaces (viewing diffeological spaces as a bicategory with trivial 2-arrows). The image of a differentiable stack via this pseudofunctor is the orbit space of a Lie groupoid representing the stack, but different choices of representative Lie groupoid yield only diffeomorphic orbit spaces.

In Watts-Wolbert [105], the authors show that this pseudofunctor can be described more strictly as a 2-functor: given a differentiable stack, this 2-functor sends it to a diffeological space that only depends on the stack (although it is diffeomorphic to the orbit space of any representative Lie groupoid). Moreover, this 2-functor extends to all stacks over the site of smooth manifolds, sending a stack to its underlying diffeological "coarse moduli space". Up to an application of the comparison lemma of sheaves on sites, the 2-functor factors through the so-called concretization functor of Baez and Hoffnung [6], which sends a sheaf of sets over the site of smooth manifolds to its underlying diffeological space. In fact, the 2-functor is adjoint to the inclusion functor from diffeological spaces into stacks, which again factors through sheaves of sets over the site of manifolds; see Subsection B.3 for more details. In other words, stacks form a language that unifies Lie groupoids up to Morita equivalence, sheaves of sets over manifolds, and diffeological spaces.

Differential spaces do not fit into this setting at all. For instance, $\mathbb{R}$ with its standard diffeology is diffeologically diffeomorphic to the cusp given by $x^2 = y^3$ in $\mathbb{R}^2$; see Karshon-Miyamoto-Watts [54]. However, the standard differential structure on $\mathbb{R}$ is not functionally diffeomorphic to the subspace differential structure on the cusp; indeed, the structural dimension of the cusp is 2, whereas at each non-cuspoidal point, it is 1. It is also equal to 1 at all points of $\mathbb{R}$. Since structural dimension is an invariant of such differential spaces (see [69]), these two spaces are not functionally diffeomorphic.

B.3. **Diffeological spaces as sheaves over Open.** A diffeology on a set X cannot be obtained from a "structure sheaf" on X as a topological space; for example, the irrational torus (Example 3.9) has an interesting diffeology but a trivial topology. Instead, diffeology can be viewed as a sheaf of sets over a site.

Namely, let **Open** denote the category whose objects are the open subsets of cartesian spaces ($\mathbb{R}^n$, $n \geq 0$) and whose arrows are smooth maps. This is a site whose coverages are exactly the standard open covers of open subsets of cartesian spaces. A diffeology $\mathcal{D}_X$ on a set X determines a contravariant functor $\mathcal{D}_X\colon \mathbf{Open} \to \mathbf{Set}^{\mathrm{op}}$ (*i.e.* a presheaf), sending U to the set of plots with domain U. The locality axiom of diffeology guarantees that this presheaf is in fact a sheaf. Furthermore, if we let $\underline{X}$ be the sheaf assigning to each object U of **Open** *all* functions $U \to X$, then $\mathcal{D}_X$ is a subsheaf of $\underline{X}$ that satisfies $\mathcal{D}_X(\mathbb{R}^0) = \underline{X}(\mathbb{R}^0)$. This definition is due to Lerman [23, Definition A.13]. A similar approach to diffeology as sheaves over a category already appears in the work of Iglesias-Zemmour in the appendix to his 1986 paper [45].

In the language of Baez and Hoffnung, diffeologies are exactly the sheaves over **Open** that are "concrete" [6]. Not all sheaves over **Open** are concrete, however. For instance, for $k > 0$, consider the sheaf $\Omega^k(\cdot)$ assigning to the object U in **Open** all differential k-forms $\Omega^k(U)$. Then $\Omega^k(\mathbb{R}^0)$ is trivial. Note, however, that this does not correspond to the diffeological space $\mathbb{R}^0$, as $\mathcal{D}_{\mathbb{R}^0}$ sends U to the singleton consisting of the constant plot $U \to \mathbb{R}^0$, whereas $\Omega^k(U)$ is not a singleton for general U.

B.4. **Differential structures and ringed spaces.** Let $(X, \mathcal{F})$ be a differential space. Then there is a naturally induced reduced ringed space $(X, \widehat{\mathcal{F}})$ where for each open $U \subseteq X$ (with respect to the initial topology induced by $\mathcal{F}$), the ring $\mathcal{F}(U)$ is the subspace differential structure on U. (A "reduced" ringed space is a ringed space whose sheaf is a sheaf of continuous real-valued functions.) Moreover, any smooth map of differential spaces $f\colon (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ induces a morphism of reduced ringed spaces $(f, f^\sharp)\colon (X, \widehat{\mathcal{F}}_X) \to (Y, \widehat{\mathcal{F}}_Y)$. One can recover $(X, \mathcal{F})$ by taking $\mathcal{F}$ to be the ring of global sections.

The opposite operation does not work: starting with an appropriate reduced ringed space, the sheafification of the global sections as above does not necessarily return the original ringed space. For example, consider the non-Hausdorff manifold "the real line with two origins"; see Example 6.7. This is the quotient of $\mathbb{R} \amalg \mathbb{R}$ by the relation $x \sim y$ if x and y are non-zero and copies of the same real number. Equip the resulting quotient topological space X with the sheaf sending an open set U (in the quotient topology) to the subspace differential structure on U induced by the quotient differential structure on X. This sheaf

cannot be obtained from a differential space, as the topology on X is not initial with respect to the global sections of the sheaf.

B.5. From differential spaces to C^∞-schemes.

A C^∞-ringed space is a topological space equipped with a sheaf of C^∞-rings. An affine C^∞-scheme is a locally C^∞-ringed space isomorphic to the real spectrum of a C^∞-ring. A C^∞-scheme is a locally C^∞-ringed space $(X, \mathcal{O}_X)$ such that X admits an open cover $\{U_\alpha\}$ in which $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine C^∞-scheme.

Proposition 2.77 of [75] states that any reduced affine C^∞-scheme can be considered to be a differential space, and Corollary of 2.78 of [75] states that any reduced C^∞-scheme can be considered to be a Mostow space. In general, however, a reduced C^∞-scheme is not the sheafification of a differential space; see the example in B.4. On the other hand, differential spaces embed into (reduced) C^∞-schemes; see [75, Corollary 2.76].

We must keep the adjective "reduced" above. Indeed, even for affine C^∞-schemes, there may be, for instance, nilpotent elements in the C^∞-ring of global sections that cannot be realised as real-valued functions on a set. A simple example of this is given by the so-called "dual numbers": $\mathbb{R}[x]/(x^2)$ has a real spectrum given by a point with corresponding sheaf of functions exactly those of $\mathbb{R}^0$. All elements generated by $x(x^2)$ are forgotten by this sheaf. See [78, page 19] for more details on this example.

B.6. From differential spaces to Mostow spaces.

A Mostow space is a reduced ringed space $(X, \mathcal{F})$ such that for any open set $U \subseteq X$, the ring $\mathcal{F}(U)$ satisfies the smooth compatibility condition of differential spaces: for any $f_1, \ldots, f_k \in \mathcal{F}(U)$ and any $g \in C^\infty(\mathbb{R}^k)$, the composition $g(f_1, \ldots, f_k)$ is in $\mathcal{F}(U)$. It follows that there is a full embedding of differential spaces into Mostow spaces by converting a differential space into a ringed space as above. Thus, one may view a Mostow space as a differential space in which the topology on the space is allowed to be finer than that of the initial topology. For example, given a diffeological space $(X, \mathcal{D})$ where X is equipped with the D-topology, the corresponding ringed space induced from $(X, \Phi\mathcal{D})$ is naturally a Mostow space. This becomes a differential space if we replace the D-topology with the initial topology. In fact, replacing the topology of the Mostow space with the initial topology induced by its global sections is adjoint to the embedding of differential spaces into Mostow spaces; see [75, Proposition 2.66].

B.7. From subcartesian spaces to differentiable spaces.

Recall from the introduction that a subcartesian space is a differential space that is locally functionally diffeomorphic to subsets of cartesian spaces.

Differentiable spaces are a special case (the "∞-standard" case) of spaces introduced by Spallek [93, 94, 95]; a standard reference on these is the book by Gonzalez-Salas [37]. A differentiable algebra is an $\mathbb{R}$-algebra isomorphic to $C^\infty(\mathbb{R}^n)/\mathfrak{a}$ for some n and ideal $\mathfrak{a}$ closed with respect to the Fréchet topology. An affine differentiable space is a locally ringed space isomorphic to the real spectrum of a differentiable algebra. A differentiable space $(X, \mathcal{O}_X)$ is a locally ringed space admitting an open cover $\{U_\alpha\}$ such that $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine differentiable space for each α. Warning: again, a *differential* space and a *differentiable*

space are two different things; we will prepend "Sikorski" to the former in this subsection to avoid confusion.

Proposition 2.81 of [75] states that any reduced affine differentiable space is a subcartesian space, and consequently, any reduced differentiable space is a Mostow space. These facts follow from the discussion on C^∞-schemes mentioned above, and the fact that differentiable spaces form a full subcategory of C^∞-schemes [75, Corollary 2.76]. In fact, one can say more. It follows from [37, Proposition 5.6, Corollary 5.7] that any reduced affine differentiable space is a closed Sikorski differential subspace of $\mathbb{R}^n$ for some n, which is stronger than the subcartesian condition.

B.8. **Conclusion.** It is the opinion of the author Watts that these various theories described above should not be viewed as competing with each other, but instead, that each theory individually focuses on specific attributes desired in what one calls a "smooth space". For example, neither diffeological, Frölicher, nor differential spaces start with a topological space; they all start with structures consisting of functions mapping into and/or out of a set. (While a differential structure uses the initial topology to define the locality axiom, this topology is induced by the differential structure). On the contrary, ringed spaces such as C^∞-schemes, differentiable spaces, and Mostow spaces, start with a topological space upon which a sheaf is defined.

As another example, Lie groupoids and stacks may encode more information than diffeological spaces. In some situations, one can get away with using the simpler language of diffeology, however, in other situations, one needs the language of Lie groupoids and stacks.

A more unifying setting that keeps to concrete 1-categories is to consider sets equipped with a diffeology *and* a differential structure that are compatible; see Definition 2.3. This setting is used in [101] and [57], and has the advantage of containing as full subcategories the categories of diffeological spaces, differential spaces, and Frölicher spaces. It also allows one to simultaneously utilise invariants and techniques specifically designed for diffeological or differential spaces.

REFERENCES

[1] Judith M. Arms, Richard H. Cushman, and Mark J. Gotay, "A universal reduction procedure for Hamiltonian group actions", In: *Geometry of Hamiltonian Systems* (Berkeley, 1989), Math. Sci. Res. Inst. Publ., **22** (1991), Springer, New York, 33–51.

[2] Nachman Aronszajn, "Subcartesian and subriemannian spaces", *Notices Amer. Math. Soc.* **14** (1967), 111.

[3] Nachman Aronszajn and Pawel Szeptycki, "The theory of Bessel potentials, part IV", *Ann. Inst. Fourier (Grenoble)* **25-3/4** (1975), 27–69.

[4] ———, "Subcartesian spaces", *J. Diff. Geom.* **15** (1980), 393–416.

[5] John C. Baez, "The n-category Café" blog, http://golem.ph.utexas.edu/category/.

[6] John C. Baez and Alexander E. Hoffnung, "Convenient categories of smooth spaces", *Trans. Amer. Math. Soc.*, **363** (2011), no. 11, 5789–5825.

[7] Augustin T. Batubenge, "Symplectic Frölicher spaces of constant dimension", Ph.D. Thesis, University of Cape Town, 2004, http://hdl.handle.net/11427/4949.

33

[8] Augustin T. Batubenge and Patrice P. Ntumba, "Characterisation of vector fields on the Frölicher standard n-simplex, imbedded in $\mathbb{R}^n$, and Hamiltonian formalism on differential spaces", *Quaestiones Mathematicae* **32** (2009), 71–90.

[9] Augustin T. Batubenge and M. Herme Tshilombo, "Notes on Pre-Frölicher Spaces", *Quaestiones Mathematicae* **39** (2016), 1115–1129.

[10] Kai Behrend and Ping Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom. **9** (2011), no. 3, 285–341.

[11] Jan Boman, "Differentiability of a function and of its compositions with functions of one variable", *Math. Scand.*, **20** (1967), 249–268.

[12] Glen E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics volume **46**, Academic Press, New York and London, 1972.

[13] Jean Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. France, **98**, 1961, 227–380.

[14] Kuo-Tsai Chen, "Iterated integrals of differential forms and loop space homology", *Ann. of Math.* **97** (1973), no. 2, 217–246.

[15] _____, "Iterated integrals, fundamental groups and covering spaces", *Trans. Amer. Math. Soc.* **206** (1975), 83–98.

[16] _____, "Iterated path integral", *Bull. of Amer. Math. Soc.* **83** (1977), no. 5, 831–879.

[17] _____, "On differentiable spaces", In: *Categories in continuum physics* (Buffalo, N.Y. 1982), Lecture Notes in Math., **1174** (1986), Springer-Verlag, Berlin, 138–142.

[18] Paul Cherenack, "Frölicher versus differential spaces: A prelude to cosmology", *Papers in honour of Bernhard Banaschweski (Cape Town, 1996)*, Kluwer Acad. Publ., Dordrecht (2000), 391–413.

[19] Daniel J. Christensen, Gordon Sinnamon, and Enxin Wu, "The D-topology for diffeological spaces", *Pacific J. Math.* **272** (2014), no. 1, 87–110.

[20] Daniel J. Christensen and Enxin Wu, "Tangent Spaces and Tangent Bundles for Diffeological Spaces", Cah. Topol. Géom. Différ. Catég. **57** (2016), no. 1, 3–50.

[21] _____, "Diffeological vector spaces", *Pacific J. Math.* **303** (2019), no. 1, 73–92.

[22] _____, "Smooth classifying spaces", *Israel J. Math.* **241** (2021), no. 2, 911–954.

[23] Brian Collier, Eugene Lerman, and Seth Wolbert, *Parallel transport on principal bundles over stacks*, J. Geom. Phys. **107** (2016), 187–213.

[24] Suzanne Craig, Naiche Downey, Lucas Goad, Michael J. Mahoney, and Jordan Watts, "Orbit spaces of linear circle actions" *Involve*, **12** (2019), no. 6, 941–959.

[25] Marius Crainic, "Prequantization and Lie brackets", *J. Symplectic Geom.* **2** (2004), 579-602.

[26] Richard Cushman and Jędrzej Śniatycki, "Differential structure of orbit spaces", *Canadian J. Math.* **53** (2001), no. 4, 715–755.

[27] Paul Donato and Patrick Iglesias, "Exemples de groupes difféologiques: flots irrationnels sur le tore", Compte Rendu de l'Académie de sciences Paris Ser. I Math. **301** (1985), no. 4, 127–130.

[28] Paul Donato and Patrick Iglesias-Zemmour "Every symplectic manifold is a (linear) coadjoint orbit", *Canad. Math. Bull.* **65** (2022), no. 2, 345–360.

[29] Adrien Douady, *Variétés à bords anguleux et voisinages tubulaires; Théorèmes d'isotopie et de recollement*, Séminaire Henri Cartan **14**, (1961/62).

[30] Eduardo J. Dubuc, "C^∞-schemes", *Amer. J. Math.* **103** (1981), no. 4, 683–690.

[31] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications, 2nd Ed.*, Wiley-Interscience, 1999.

[32] Alfred Frölicher, "Applications lisses entre espaces et variétés de Fréchet", *C.R. Acad. Sc. Paris*, **293**, no. 1 (1981), 125–127.

[33] _____, "Smooth structures", In: *Category Theory (Gummersbach, 1981)*, Lecture Notes in Math., **962** (1982), Springer-Verlag, New York, 69–81.

[34] _____, "Cartesian closed categories and analysis of smooth maps", In: *Categories in continuum physics (Buffalo, NY, 1982)*, Lecture Notes in Math., **1174** (1986), Springer, Berlin, 43–51.

[35] Alfred Frölicher and Andreas Kriegl, *Linear Spaces and Differentiation Theory*, Wiley-Interscience, 1988.

[36] Christopher G. Gibson, *Singular Points of Smooth Mappings, Research Notes in Mathematics, 25*, Pitman (Advanced Publishing Program), Boston Mass.-London, 1979.

[37] J.A. Navarro Gonzalez, J.B. Sancho de Salas, C^∞-Differentiable Spaces, Springer Lecture Notes in Math. 1824, 2003.

[38] Alexander Grothendieck, "Technique de descente et théorèmes d'existence en géoméétrie algébrique. I. Généralitiés. Descente par morphismes fidèlement plats", *Séminaire Bourbaki* **5** (Exposé 190), 1959.

[39] Alexander Grothendieck and Jean Dieudonné, "Eléments de géométrie algébrique: I. Le language des schémas", Publications Mathématiques de l'IHÉS **4**, 1960.

[40] Serap Gürer and Patrick Iglesias-Zemmour, "Differential forms on stratified spaces", *Bull. Aust. Math. Soc.* **98** (2018), 319-330.

[41] _____, "Differential forms on stratified spaces II", *Bull. Aust. Math. Soc.* **99** (2019), 311-318.

[42] Richard M. Hain, "A characterization of smooth functions defined on a Banach space", *Proc. Amer. Math. Soc.* **77** no. 1 (1979).

[43] Gilbert Hector, "Géométrie et topologie des espaces difféologiques", In: *Analysis and Geometry in Foliated Manifolds (Santiago de Compostela, 1994)*, World Sci. Publishing, 1995.

[44] Gerhard Paul Hochschild, *The Structure of Lie Groups*, Holden-Day, San-Francisco, California, 1965.

[45] Patrick Iglesias, *Connexions et difféologie*, Aspects dynamiques et topologiques des groupes infinis de transformation de la mécanique (Lyon, 1986), 61–78, Travaux en Cours, 25, Hermann, Paris, 1987.

[46] _____, "La trilogie du moment", *Ann. Inst. Fourier (Grenoble)* **45** (1995), 825–857.

[47] Patrick Iglesias-Zemmour, *Diffeology*, Math. Surveys and Monographs, Amer. Math. Soc., 2012.

[48] Patrick Iglesias-Zemmour, Yael Karshon, and Moshe Zadka, "Orbifolds as diffeologies", *Trans. Amer. Math. Soc.*, **362** (2010), no. 6, 2811–2831.

[49] Patrick Iglesias-Zemmour and Yael Karshon, "Smooth Lie group actions are parametrized diffeological subgroups", *Proc. Amer. Math. Soc.* **140** (2012), no. 2, 731–739.

[50] Patrick Iglesias-Zemmour and Jean-Pierre Laffineur, *Noncommutative geometry and diffeology: the case of orbifolds.* J. Noncommut. Geom. **12** (2018), no. 4, 1551–1572.

[51] Klaus Jänich, "On the classification of $O(n)$-manifolds", *Math. Ann.* **176** (1968), 53–76.

[52] Dominic Joyce, "Algebraic Geometry over C^∞-Rings", *Mem. Amer. Math. Soc.* **260** (2019), no. 1256.

[53] Yael Karshon, "An algebraic proof for the symplectic structure of moduli space", Proc. Amer. Math. Soc. **116** (1992), no. 3, 591–605.

[54] Yael Karshon, David Miyamoto, and Jordan Watts, "Diffeological submanifolds and their friends" (preprint)
https://arxiv.org/abs/2204.10381.

[55] Yael Karshon and David Miyamoto, "Quasifold groupoids and diffeological quasifolds" (preprint)
https://arxiv.org/abs/2206.14776.

[56] Yael Karshon and Jordan Watts, "Basic forms and orbit spaces: a diffeological approach", SIGMA **12** (2016), 026, 19 pp.

[57] _____, "Smooth maps on convex sets", *Contemporary Mathematics*, American Mathematical Society (to appear)
https://arxiv.org/abs/2212.06917.

[58] _____, "Reflexivity of infinite-dimensional diffeological and differential spaces" (tentative title, in preparation).

[59] Anders Kock, *Synthetic Differential Geometry, 2nd Ed., London Math. Soc. Lecture Note Series, 333*, Cambridge University Press, 2006.

[60] Jean-Louis Koszul, "Sur certains groupes de transformations de Lie", Colloque international du Centre National de la Recherche Scientifique **52** (1953), 137–141.

[61] Matthias Kreck, *Differential Algebraic Topology: From Stratifolds to Exotic Spheres*, Graduate Studies in Mathematics, Vol. 110, Amer. Math. Soc., Providence, Rhode Island, 2010.

[62] Andreas Kriegl, "Remarks on germs in infinite dimensions", *Acta Math. Univ. Comenianae*, **66** (1997), 1–18.

[63] Andreas Kriegl and Peter W. Michor, *The Convenient Setting of Global Analysis*, Amer. Math. Soc., Providence, Rhode Island, 1997.

[64] Martin Laubinger, "Diffeological spaces", *Proyecciones* **25** (2006), 151–178.

[65] Brian Lee, "Geometric structures on spaces of weighted submanifolds", SIGMA **5** (2009), 099, 46 pp.

[66] John M. Lee, *Introduction to Smooth Manifolds, 2nd Ed. Graduate Texts in Mathematics, 218*, Springer, New York, 2013.

[67] Eugene Lerman, "Orbifolds as Stacks?", *Enseign. Math. (2)* **56** (2010), 315–363.

[68] Jiayong Li and Jordan Watts, "The orientation-preserving diffeomorphism group of S^2 deforms to $SO(3)$ smoothly", *Transformation Groups*, Birkhäuser Boston, **16** (2011), no. 2, 537–553.

[69] Tsasa Lusala, Jędrzej Śniatycki, and Jordan Watts, "Regular points of a subcartesian space", *Canad. Math. Bull.*, **53** (2010), no. 2, 340–346.

[70] Jean-Pierre Magnot, "Ambrose-Singer theorem on diffeological bundles and complete integrability of the KP equation", *Int. J. Geom. Methods Mod. Phys.* **10**, no. 9 (2013), 1350043, 31 pp.

[71] Jean-Pierre Magnot, "The group of diffeomorphisms of a non-compact manifold is not regular", *Demonstr. Math.* **51** (2018), no. 1, 8–16.

[72] Jean-Pierre Magnot and Enrique G. Reyes, "Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups", *Ann. Henri Poincaré* **21**, no. 6 (2020), 1893–1945.

[73] Jean-Pierre Magnot and Jordan Watts, "The diffeology of Milnor's classifying space", *Topology Appl.* **232** (2017), 189–213.

[74] Richard Melrose, *Differential Analysis on Manifolds*, 1996 (unpublished manuscript). https://math.mit.edu/~rbm/book.html

[75] João Nuno Mestre, *Differentiable stacks: stratifications, measures and deformations*, Ph.D. thesis, Utrecht University, 2016.

[76] David Miyamoto, "The basic de Rham complex of a singular foliation", *Int. Math. Res. Not. IMRN* (2022), rnac044

[77] Ieke Moerdijk and Janez Mrčun, "Lie groupoids, sheaves and cohomology" In: Poisson Geometry, Deformation Quantisation and Group Representations, 145–272, *London Math. Soc. Lecture Note Ser.* **323**, Cambridge University Press, 2005.

[78] Ieke Moerdijk and Gonzalo E. Reyes, *Models for smooth infinitesimal analysis*, Springer-Verlag, New York, 1991.

[79] Mark A. Mostow, "The differentiable space structures of Milnor classifying spaces, simplicial complexes, and geometric realizations", *J. Differential Geometry*, **14** (1979), 255–293.

[80] James R. Munkres, *Topology, 2nd Ed.*, Prentice Hall, 2000.

[81] Oystein Ore, "Galois connexions", *Trans. Amer. Math. Soc.* **55** (1944), 493–513

[82] Richard Palais, "On the existence of slices for actions of non-compact Lie groups", *Ann. of Math.*, **73** (1961), 295–323.

[83] Markus Pflaum, *Analytic and Geometric Study of Stratified Spaces, Lecture Notes in Mathematics 1768*, Springer, 2001.

[84] Dorette Pronk, "Étendues and stacks as bicategories of fractions", *Compositio Math.* **102** (1996), no. 3, 243–303.

[85] Armin Rainer, "Arc-smooth functions on closed sets", *Compos. Math.* **155** (2019), no. 4, 645–680.

[86] Gerald Schwarz, "Smooth functions invariant under the action of a compact Lie group", *Topology*, United Kingdom, **14** (1975), 63–68.

[87] Roman Sikorski, "Abstract covariant derivative", *Colloq. Math.*, **18** (1967), 251–272.

[88] _____, "Differential modules", *Colloq. Math.*, **24** (1971), 45–79.

[89] Reyer Sjamaar and Eugene Lerman, "Stratified symplectic spaces and reduction", Ann. of Math. (2) **134** (1991), no. 2, 375–422.

[90] J. Wolfgang Smith, "The De Rham Theorem for General Spaces", Tôhoku Math. Journ. vol. **18**, no. 2, 1966.

[91] Jędrzej Śniatycki, *Differential Geometry of Singular Spaces and Reduction of Symmetry*, New Mathematical Monographs (no. 23), Cambridge University Press, 2013.

[92] Jean-Marie Souriau, "Groupes différentiels", In: *Differential geometric methods in mathematical physics (Proceedings of conference held in Aix-en-Provence Sept. 3-7, 1979 and Salamanca, Sept. 10-14, 1979)* (P. L. Garcia, A. P'erez-Rendón, and J.-M. Souriau, eds.), Lecture Notes in Mathematics, **836** (1980), Springer, New York, 91–128.

[93] Karlheinz Spallek, "Differenzierbarer Räume", *Math. Ann.* **180** (1969), 269–296.

[94] _____, "Differential forms on differentiable spaces", *Rend. Mat. (6)* **4** (1971), 231–258.

[95] _____, "Differential forms on differentiable spaces II", *Rend. Mat. (6)* **5** (1972), 375–389.

[96] Andrew Stacey, "Comparative smootheology", *Theory Appl. Categ.* **25** (2011), no. 4, 64–117.

[97] Lynn Arthur Steen and J. Arthur Seebach Jr., *Counterexamples in Topology*, reprint of 2nd edition, Dover Publications, Mineola, New York, 1995.

[98] Haggai Tene, *Stratifolds and Equivariant Cohomology Theories*, Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2010.

[99] Bart van Munster, "The Hausdorff Quotient", Bachelor Thesis (2014), Mathematisch Instituut, Universiteit Leiden.
https://www.math.leidenuniv.nl/scripties/BachVanMunster.pdf

[100] Andrei Verona, *Stratified Mappings – Structure and Triangulability, Lecture Notes in Mathematics, 1102*, Springer-Verlag, Berlin, 1984.

[101] Bryce Virgin, *Watts spaces and smooth maps*, JUMP Journal of Undergraduate Research **4** (2021), 63–85.

[102] Jordan Watts, *Diffeologies, Differential Spaces, and Symplectic Geometry*, Ph.D. Thesis (2012), University of Toronto, Canada.

[103] _____, "The differential structure of an orbifold", *Rocky Mountain J. Math.*, **47** (2017), 289–327.

[104] _____, "The orbit space and basic forms of a proper Lie groupoid", in: *Trends in Mathematics, Research Perspectives: Proceedings of the 12th ISAAC Congress*, Aveiro, Portugal, 2019, Birkh"auser (to appear).

[105] Jordan Watts and Seth Wolbert, "Diffeological coarse moduli spaces of stacks over manifolds", (submitted).

[106] Alan Weinstein, "Cohomology of symplectomorphism groups and critical values of Hamiltonians", *Math. Z.* **201** (1989), 75–82.

[107] Hermann Weyl, *David Hilbert and his mathematical works*, Bull. Amer. Math. Soc. **50** (1944), 612–654.

[108] Hassler Whitney, "Differentiable even functions", *Duke Math. J.* **10** (1943), 159–160.

[109] Masrour Zoghi, "Orbifolds", Chapter 5 of Ph.D. Thesis, University of Toronto, 2010.

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