Finite quantum hypergroups

Magnus B. Landstad [(1)] and *Alfons Van Daele* [(2)]

Abstract

A *finite quantum hypergroup* is a finite-dimensional unital algebra A over the field of complex numbers, together with a coproduct on A, satisfying certain natural conditions. For a finite quantum hypergroup, the dual can be constructed. It is again a finite quantum hypergroup.

The more general concept of an *algebraic quantum hypergroup* is studied in [3, 4]. If the underlying algebra of an algebraic quantum hypergroup is finite-dimensional, it is a finite quantum hypergroup in the sense of this paper.

Here we treat finite quantum hypergroups independently, from another point of view (providing also some more results) and with an emphasis on the development of the notion.

We also include several examples to illustrate the theory.

Date: *12 October 2023*

---

(1) Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway. E-mail: `magnus.landstad@ntnu.no`

(2) Department of Mathematics, University of Leuven, Celestijnenlaan 200B, B-3001 Heverlee, Belgium. E-mail: `alfons.vandaele@kuleuven.be`

0. Introduction

Let A and B be associative algebras over the field $\mathbb{C}$ of complex numbers. We assume they are finite-dimensional and that they have a unit. Further we need a pairing of A with B. This is a bilinear map $(a, b) \mapsto \langle a, b\rangle$ from $A \times B$ to $\mathbb{C}$. It is assumed to be non-degenerate.

In [9] we study pairings of possibly infinite-dimensional, non-unital algebras. In this note, we will only work with unital finite-dimensional algebras.

Pairing of finite-dimensional unital algebras

So we assume that A and B are *unital, finite-dimensional* algebras over $\mathbb{C}$ and that we have given a pairing of A with B as above.

The product in B induces a coproduct Δ_A on A and the product in A gives a coproduct Δ_B on B. The coproduct on A, given by $\langle \Delta_A(a), b \otimes b'\rangle = \langle a, bb'\rangle$, is a coassociative linear map from A to $A \otimes A$. Similarly for the coproduct on B. The identity in B induces a counit on A and the one in A gives a counit on B.

Dual pairs of algebras like above appear in various cases, with different extra properties of the coproducts and the counits.

If A is a finite-dimensional *Hopf algebra* and B its dual Hopf algebra, then the coproduct on A is a unital homomorphism and the counit on A is a homomorphism. Similarly for the coproduct and the counit on B.

If A is a *weak Hopf algebra*, we still have that Δ is a homomorphism, but it is no longer assumed to be unital. The counit is no longer a homomorphism.

On the other hand, for a pairing of *finite quantum hypergroups*, as in [3], we no longer have that Δ is a homomorphism, but keep the condition that the coproduct is unital, both on A and B.

Loosely speaking, for a quantum group, the coproduct is a unital homomorphism. For a quantum groupoid, it is no longer required to be unital while for a quantum hypergroup, it is no longer a homomorphism.

For precise definitions, we refer to the item 'Basic References' further in this introduction. The notion of a finite quantum group in this paper is found in Definition 3.9 in Section 3.

The antipode for pairs of algebras

When we have a pairing of *-algebras, there are linear maps S_A on A and S_B on B determined by

$$\langle a, b^*\rangle = \overline{\langle S_A(a)^*, b\rangle} \qquad \text{and} \qquad \langle a^*, b\rangle = \overline{\langle a, S_B(b)^*\rangle}$$

for all $a \in A$ and $b \in B$. In the case of pairing of Hopf *-algebras, these are the antipodes. They are completely determined by the pairing.

For any Hopf algebra, the antipode is unique and hence, also for a pairing of Hopf algebras, the antipodes are determined by the pairing of algebras.

We have the same property for a multiplier Hopf algebra with integrals, see [17]. Also here all information can be recovered from the algebra A, its dual algebra $\widehat{A}$ and the pairing between the two. In fact there are plenty of other examples with the same phenomena.

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

This also applies to the pairing of algebraic quantum hypergroups as studied in [3]. Here, in this paper, we only consider the finite-dimensional case. We give a treatment that also works for the quantum hypergroups we will be studying in [9].

In this paper, we find conditions on a pair of finite-dimensional unital algebras to be a pair of finite quantum hypergroups. The conditions are formulated in terms of the existence of integrals and antipodes. We not only recover the results from [3], but from a different point of view and we provide some slightly more general results.

In [16] the duality of algebraic quantum groupoids is treated from the same point of view. But that case is very different from, and more involved than the one studied here.

Content of the paper

In *Section* 1 we consider a couple of finite-dimensional unital algebras A and B with a non-degenerate pairing. Various extra properties of the induced coproducts and counits are considered. We are particularly interested in a pairing of *-algebras. The main, but natural condition is now that the coproducts are *-maps. As mentioned earlier, in this case there are natural candidates for the antipodes and their behavior serves as a motivation for the introduction of antipodes in this context in the next section.

In *Section* 2 we introduce the notion of left and right integrals. The notion involves the existence of an antipode. We make a distinction between invariant functionals and integrals. Invariant functionals are considered in the next section. For a faithful left integral, the associated antipode is unique. Further, given a faithful left integral on A, there exists a faithful right integral on B. This extends the duality from the algebras A and B to the case where these algebras have integrals. In [8], we do the same but for possibly infinite-dimensional algebras.

In *Section* 3 we work towards the main definition of this note. We first study the relation between integrals and invariant functionals. An important result is the uniqueness of integrals when the coproduct is unital. This eventually leads to the definition of a finite quantum hypergroup. The duality obtained in the previous section applies here and gives a duality for finite quantum hypergroups.

We recover the notion and results about finite quantum hypergroups as they follow from the general theory in [3]. The approach is different and it is preparing for another generalization that we study in [9].

In the subsequent sections we illustrate our theory with examples.

In *Section* 4 we treat the case of a finite group G with a subgroup H and the *-algebra of functions on G with the property that $f(hpk) = f(p)$ for all $p \in G$ and $h, k \in H$. We have a coproduct Δ on A so that (A, Δ) is a finite *-quantum hypergroup. We have an explicit construction of the dual. These are the classical Hecke algebras. This example is the finite-dimensional motivating example for the study of algebraic quantum hypergroups as in [3, 4].

We make it more concrete with the subgroup generated by a single permutation in the group S_3 of permutations of a set with 3 elements in *Section* 5. We also derive some generalizations here.

In *Section* 6 we have a group G, not necessarily a finite group, now with two finite subgroups H and K satisfying $H \cap K = \{e\}$ where e is the identity of G. We associate a dual pair of finite quantum hypergroups to such a pair of subgroups.

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

We finish with some conclusions and suggestions for further research in Section 7.

Notations and conventions

We work with associative algebras over the field $\mathbb{C}$ of complex numbers. They are always assumed to be finite-dimensional and unital.

We use 1 for the identity in any algebra we consider. For the identity in a group, we use e. Finally ι is used for the identity map.

In this note, a coproduct on an algebra A is just a coassociative linear map from A to the tensor product $A \otimes A$. We call it coabelian if $\zeta \circ \Delta = \Delta$ where ζ is the linear map on $A \otimes A$ defined by $\zeta(a \otimes a') = a' \otimes a$ for all $a, a' \in A$.

Often in this paper, we will use the same symbol for objects associated with different algebras. We not only use e.g. 1 to denote the identity in all our algebras, we will do the same for coproducts, counits and antipodes.

We are inspired by the theory of locally compact groups and operator algebras. This has some consequences for the terminology, notations and techniques we use. This also explains why we only work with algebras over the complex numbers and why we also treat *-algebras.

Basic references

For the theory of Hopf algebras, we refer to the basic works by Abe [1] and Sweedler [23]. See also [15] for a more recent treatment. For multiplier Hopf algebras and algebraic quantum groups, the main references are [17] and [18]. The original theory of quantum hypergroups is found in [3] and [4]. There is also the earlier work on compact quantum hypergroups in [2]. The theory of weak Hopf algebras and weak multiplier Hopf algebras is developed in a series of papers. References can e.g. be found in [22]. In particular, the pairing of weak multiplier Hopf algebras is treated in [16].

In this paper, many of the more easy arguments are omitted. The reader can find them if necessary in an extended version of the paper (with the same title), available on the arXiv, see [7].

Acknowledgments

We want to thank NTNU (Norway) and KU Leuven (Belgium) for the opportunities we get to continue our research after our official retirements.

1. Dual pairs of algebras and *-algebras

We start this section with some well-known properties of finite-dimensional unital algebras over the field $\mathbb{C}$ of complex numbers. Further, when we have a coproduct with a counit, the dual space is again a unital algebra. We get a pairing of algebras with special attention to the case of *-algebras. As explained already, this case is used to motivate the main formulas in the non-involutive case.

Finite-dimensional unital algebras, coproducts and counits

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

A linear function ω on A is called *faithful* if given $a \in A$, we have $a = 0$ if either $\omega(aa') = 0$ for all $a' \in A$ or $\omega(a'a) = 0$ for all $a' \in A$. First observe that a faithful functional on an algebra can only exist if the algebra is non-degenerate. We have the following property.

1.1. Proposition Let ω be a faithful functional on an algebra A. For any linear functional f on A, there exist elements c, c' in A so that $f(a) = \omega(ac)$ and $f(a) = \omega(c'a)$ for all a.

Proof: The maps $c \mapsto \omega(\,\cdot\, c)$ and $c' \mapsto \omega(c'\,\cdot\,)$ are injective because ω is faithful. Then they are surjective because A is finite-dimensional. $\square$

1.2. Proposition For any faithful linear functional ω on A there is an automorphism σ of A satisfying $\omega(ac) = \omega(c\sigma(a))$ for all a, c. Moreover $\omega(\sigma(a)) = \omega(a)$ for all a.

Proof: Given $a \in A$, the functional $c \mapsto \omega(ac)$ is of the form $c \mapsto \omega(ca')$ for some a' by the previous result. Because ω is faithful, the element a' is uniquely determined by a. Denote it by $\sigma(a)$. Then we have

$$\omega(c\sigma(aa')) = \omega(aa'c) = \omega(a'c\sigma(a)) = \omega(c\sigma(a)\sigma(a'))$$

for all c and hence, by the faithfulness of ω, we must have $\sigma(aa') = \sigma(a)\sigma(a')$. The map σ is injective and so it is an automorphism of A. If we take $c = 1$ in the defining formula, we find $\omega(a) = \omega(\sigma(a))$. $\square$

In the literature, a finite-dimensional algebra with a faithful functional is called Frobenius and the inverse of the automorphism σ is called the Nakayama automorphism, see [13].

The terminology and notations we use come from the theory of locally compact groups and operator algebras. Then the automorphism σ is called the modular automorphism and linear functionals admitting such a modular automorphism are called KMS-functionals, see e.g. [14].

Next we consider a coproduct Δ on the algebra A. We do not require it to be a homomorphism. And in the first place, we also do not assume that $\Delta(1) = 1 \otimes 1$.

However, we need the existence of a counit. It is defined as a linear map $\varepsilon : A \to \mathbb{C}$ satisfying $(\varepsilon \otimes \iota)\Delta(a) = a$ and $(\iota \otimes \varepsilon)\Delta(a) = a$ for all $a \in A$. Such a counit is unique. We do not require it to be a homomorphism for the moment.

Recall the following simple and well-known property.

1.3. Proposition Denote by B the space of linear functionals on A and use $\langle a, b\rangle$ to denote the value of b in the point $a \in A$. The coproduct makes B into an (associative) algebra by the formula

$$\langle a, bb'\rangle = \langle \Delta(a), b \otimes b'\rangle.$$

We use the obvious pairing of $A \otimes A$ with $B \otimes B$. The counit on A is a unit for this algebra B and $\langle a, 1\rangle = \varepsilon(a)$ for all $a \in A$.

Observe that the bilinear form is non-degenerate by definition.

Conversely, the product on A induces a coproduct on B by the formula

$$\langle a \otimes a', \Delta(b)\rangle = \langle aa', b\rangle.$$

The identity on A gives a counit ε on B by the formula $\langle 1, b\rangle = \varepsilon(b)$.

From the point of view of duality, it is natural to assume that there is a counit for (A, Δ) as this is equivalent with the requirement that the dual algebra B has a unit.

We use Δ both for the coproduct on A and the coproduct on B and similarly for the counit ε.

Dual pairs of algebras

We see that an algebra A with a coproduct Δ as above gives rise to a pairing of algebras as in the following definition.

1.4. Definition Let A and B be algebras. A pairing of A and B is a non-degenerate bilinear form $(a, b) \mapsto \langle a, b\rangle$ from the Cartesian product $A \times B$ to $\mathbb{C}$.

The coproduct on one algebra is induced by the product on the other one. The counit on one algebra is given by the pairing with the identity of the other one.

There is an obvious converse. Given such a pairing of algebras, the product in B induces a coproduct on A and the unit in B gives a counit for this coproduct on A. Then the algebra B is recovered as the dual of A, as described before. This observation is important for the rest of the paper.

Assume now that we have a dual pair of algebras A and B as in Definition 1.4. We consider the induced coproducts and counits as discussed before.

1.5. Proposition The counit on B is a homomorphism if and only if $\Delta(1) = 1 \otimes 1$ in $A \otimes A$. Similarly for the counit on A.

For the further discussion, we do not assume that the coproducts are unital. But the condition will play a crucial role later.

We have *associated actions* of one algebra on the other induced by the pairing as follows.

1.6. Proposition There are left and right actions of A on B and of B on A given by the formulas

$$\langle a, a' \triangleright b\rangle = \langle aa', b\rangle = \langle a', b \triangleleft a\rangle$$
$$\langle a \triangleleft b, b'\rangle = \langle a, bb'\rangle = \langle b' \triangleright a, b\rangle$$

for all $a, a' \in A$ and $b, b' \in B$.

There are some obvious properties of these actions.

1.7. Proposition The four actions associated with a pairing of algebras are faithful, unital and non-degenerate.

Proof: We prove this for the left action of A on B.

Suppose that $a \in A$ and that $a \triangleright b = 0$ for all b. If we pair with the identity of A, we find that $\langle a, b\rangle = 0$. This holds for all b and because the pairing is non-degenerate, we have $a = 0$. This shows that the action is faithful.

We clearly have $1 \triangleright b = b$ for all b so that the action is unital.

Finally, let $b \in B$ and assume that $a \triangleright b = 0$ for all a. With $a = 1$ we have $b = 1 \triangleright b = 0$ and we see that the action is non-degenerate. $\square$

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

This result is trivial but it will not be so in the general case of infinite-dimensional algebras in [9]. That is the reason for formulating it here as a separate result.

Pairing of *-algebras

It is interesting to consider *-algebras (i.e. algebras over $\mathbb{C}$ with an involution), not only because there are plenty of natural examples, but also because of the special role of the antipode in this case. We have mentioned this already in the introduction. Now we give more details.

So assume now that A and B are *-algebras and that we have a non-degenerate pairing of A with B as in Definition 1.4. We can prove the following result.

1.8. Proposition There is a bijective linear map S_A from A to A satisfying

$$\langle S_A(a)^*, b\rangle = \overline{\langle a, b^*\rangle}$$

for all a, b. Similarly, there is a bijective linear map S_B from B to itself satisfying

$$\langle a, S_B(b)^*\rangle = \overline{\langle a^*, b\rangle}.$$

These two maps are each others adjoint, i.e.

$$\langle S_A(a), b\rangle = \langle a, S_B(b)\rangle$$

for all a, b.

Proof:

We can define a linear map S_A on A by $\langle S_A(a)^*, b\rangle = \overline{\langle a, b^*\rangle}$ for all $a \in A$ and $b \in B$. Similarly we have a linear map S_B on B defined by $\langle a, S_B(b)^*\rangle = \overline{\langle a^*, b\rangle}$ for all a, b.

It is clear that the maps $a \mapsto S_A(a)^*$ and $b \mapsto S_B(b)^*$ are conjugate and involutive. We also find

$$\langle S_A(a), b\rangle = \langle S_A(a)^{**}, b\rangle = \overline{\langle S_A(a)^*, S_B(b)^*\rangle}$$

and by symmetry we get $\langle S_A(a), b\rangle = \langle a, S_B(b)\rangle$. $\square$

In the operator algebraic approach there is a slightly different convention. The map S_B is replaced by its inverse and so the last formula becomes $\langle S_A(a), b\rangle = \langle a, S_B^{-1}(b)\rangle$ instead, see e.g. [20]. We stick to the algebraic conventions as set out in the previous proposition.

1.9. Proposition We have $\Delta(S_A(a)^*) = \zeta((S_A \otimes S_A)\Delta(a))^*$ for all a where ζ is the flip map on $A \otimes A$. Similarly $\Delta(S_B(b)^*) = \zeta((S_B \otimes S_B)\Delta(b))^*$.

Recall that $A\otimes A$ and $B\otimes B$ are *-algebras with the natural involution defined by $(a\otimes a')^* = a^* \otimes a'^*$ and similarly for B. The first result follows easily from the fact that $(bb')^* = b'^* b^*$ for all $b, b' \in B$. Simiarly for the second one.

In a similar way, we get the following.

1.10. Proposition Assume that we have a pairing of two *-algebras A and B. The coproduct Δ on A is a *-map, that is $\Delta(a^*) = \Delta(a)^*$ for all a, if and only if $b \mapsto S_B(b)^*$ is an algebra map, i.e. if $S_B(bb')^* = S_B(b)^* S_B(b')^*$ for all $b, b' \in B$. Similarly, the coproduct on B is a *-map if and only if $a \mapsto S_A(a)^*$ is an algebra map.

Again the argument is a straightforward consequence of the definitions.

If Δ is a *-map on B, then S_A is an anti-isomorphism of A. This follows from the above result. On the other hand, if Δ is a *-map on A, then the result of Proposition 1.9 implies that S_A also flips the coproduct.

We are using the symbol S because further it will behave like an antipode. That the maps S_A and S_B are anti-isomorphisms, flipping the coproduct is what we expect of an antipode. However, the reader should be aware of the fact that this will not be enough to characterize the antipode. Indeed, if the algebra A is abelian and Δ on A is coabelian, also the identity map ι will have this property on A.

It is natural, for a pairing of *-algebras, to assume that the coproducts are *-maps as in Proposition 1.10. Hence it is reasonable to include this as conditions for a pairing of *-algebras. This is done with the following definition.

1.11. Definition Let A and B be *-algebras. A non-degenerate bilinear form $(a,b) \mapsto \langle a,b\rangle$ on $A \times B$ is said to be a *pairing of* **-algebras* if the coproducts are *-maps.

So, for such a pairing of *-algebras, we have the linear maps S_A and S_B on A and B resp. They are anti-isomorphisms and they flip the associated coproducts.

In this case, we moreover have the following property of the counits.

1.12. Proposition The counit ε on A satisfies $\varepsilon(a^*) = \overline{\varepsilon(a)}$ and $\varepsilon(S(a)) = \varepsilon(a)$ for all a. Similarly for the counit on B.

The first property follows from the uniqueness of the counit and because Δ is a *-map. The second one is a consequence of the fact that S_B is an anti-isomorphism of B so that $S_B(1) = 1$.

1.13. Remark If we start with a *-algebra with a coproduct Δ satisfying $\Delta(a^*) = \Delta(a)^*$, we have the associated product on the dual B, but there is no canonical way to make B into a *-algebra. We need an anti-isomorphism S_A on A that flips the coproduct. Then we can define the involution on B by the formula $\langle a, b^*\rangle = \langle S_A(a)^*, b\rangle^-$. We can define S_B on B as the adjoint of S_A as before. Then we end up with a pairing of *-algebras as in Definition 1.11.

Again in what follows, we will drop the indices and write S, both for S_A on A as for S_B on B.

We have the following formulas for the behavior of the maps S with respect to actions, associated with the pairing of *-algebras.

1.14. Proposition For all $a \in A$ and $b \in B$ we have

$$S(b \triangleright a)^* = S(a)^* \triangleleft b^* \qquad \text{and} \qquad S(a \triangleright b)^* = S(b)^* \triangleleft a^*.$$

For these properties we use that $a \mapsto a^*$ and $b \mapsto b^*$ are involutions of the algebras. On the other hand, when we use that $S(aa')^* = S(a)^* S(a')^*$ and $S(bb')^* = S(b)^* S(b')^*$, we arrive at another pair of relations.

1.15. Proposition For all a, b we have

$$(a \triangleleft b)^* = a^* \triangleleft S(b)^* \qquad \text{and} \qquad (a \triangleright b)^* = S(a)^* \triangleright b^*.$$

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

The two sets of equalities are fundamentally different from each other.

Using that the antipodes are anti-isomorphisms of the algebras, we get similar forms for the action, not involving the involutions.

1.16. Proposition For all a, b we have

$$S(S(b) \triangleright a) = S(a) \triangleleft b \qquad \text{and} \qquad S(S(a) \triangleright b) = S(b) \triangleleft a.$$

We have similar formulas with S^{-1} because S^{-1} is also an anti-isomorphism. This results in the following analogous formulas

$$a \triangleright b = S(S^{-1}(b) \triangleleft S(a)) \qquad \text{and} \qquad b \triangleright a = S(S^{-1}(a) \triangleleft S(b)).$$

In fact, the result also follows by combining the formulas from the two previous results with the involutions.
These properties lead us naturally to the following section. But first, we introduce a new related concept.

1.17. Definition Suppose that we have a pairing of algebras A and B, not necessarily *-algebras, as in Definition 1.4. Assume that we have anti-isomorphisms S_A on A and S_B on B, satisfying $\langle S_A(a), b\rangle = \langle a, S_B(b)\rangle$ for all a, b. Then we call S_A and S_B a pair of pre-antipodes. In this case, we say that the pair (A, B) is *a pair with pre-antipodes.*

So, for a pairing of *-algebras, as in Definition 1.11, we always have pre-antipodes.

2. Left and right integrals

In this section we will introduce the notion of integrals. This is done in such a way that we get a nice and fairly general duality result. In the next section, we impose one more condition and get a duality of finite quantum hypergroups.

To explain where our definition of integrals comes from, we again look first at a pair of finite-dimensional unital *-algebras as in Definition 1.11 of the previous section.

Integrals and the antipode for a pairing of *-algebras

Consider a non-degenerate pairing of finite-dimensional unital *-algebras A and B as in Definition 1.11. We have the anti-isomorphisms S on A and on B as defined in Proposition 1.8.

There is a left action $(a, b) \mapsto a \triangleleft S^{-1}(b)$ of B on A. We use the pre-antipode S on B as obtained from the pairing (see Proposition 1.8). This gives for each $b \in B$ an operator $\lambda(b)$ defined on A by $\lambda(b)(a) = a \triangleleft S^{-1}(b)$. Because the pre-antipode S is assumed to be an anti-isomorphism, we get a representation of B.

Let φ be a faithful positive linear functional on A. It defines a scalar product on A given by $(a, c) \mapsto \varphi(c^*a)$. For the formulation of the following result, we use the notation $[a, c] = \varphi(c^*a)$. The more common notation $\langle\,\cdot\,,\,\cdot\,\rangle$ is reserved here for the pairing.

2.1. Proposition Given $b \in B$ we have $[\lambda(b)(a), c] = [a, \lambda(b^*)(c)]$ for all a, c in A if and only if

$$\varphi(c(a \triangleleft S^{-1}(b))) = \varphi((c \triangleleft b)a). \tag{2.1}$$

for all $a, c \in A$.

Proof: i) Let $a, c \in A$, $b \in B$ and assume that $[\lambda(b)(a), c] = [a, \lambda(b^*)(c)]$. For the left hand side we find

$$[\lambda(b)(a), c)] = \varphi(c^*(a \triangleleft S^{-1}(b))).$$

For the right hand side we have

$$[a, \lambda(b^*)(c)] = \varphi((c \triangleleft S^{-1}(b^*))^* a).$$

Now we use the first formula in Proposition 1.15. We obtain $(c \triangleleft S^{-1}(b^*))^* = c^* \triangleleft b$. If we insert this we find

$$\varphi(c^*(a \triangleleft S^{-1}(b))) = \varphi((c^* \triangleleft b)a).$$

Finally we replace c by c^* to get the required formula.

ii) It is clear that the argument also works for the other direction. $\square$

In other words, λ is a *-representation of B on A for this scalar product if and only if Equation (2.1) holds for all $a, c \in A$. Equation (2.1) holds for all $b \in B$ if and only if

$$(\iota \otimes \varphi)((1 \otimes c)\Delta(a)) = S((\iota \otimes \varphi)(\Delta(c)(1 \otimes a))) \tag{2.2}$$

for all a, c.

This equality appears in many other theories of this kind. The above result shows why it is a natural property in the *-algebra case.

Equation (2.2) no longer involves the involutive structures. Therefore it makes sense to impose it on the antipode in relation with the functional φ. It leads us to the notion of integrals in this framework.

Left and right integrals

Take a pairing of finite-dimensional unital algebras A and B, not necessarily *-algebras.

Motivated by the previous item, we introduce the main concept, namely left and right integrals.

2.2. Definition Let φ be a linear functional on A. It is called *a left integral* if there is a linear map S from A to itself satisfying

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes c))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(c)) \tag{2.3}$$

for all a, c in A. Similarly a linear functional ψ is called *a right integral* if there is a linear map S' from A to itself satisfying

$$S'((\psi \otimes \iota)((c \otimes 1)\Delta(a))) = (\psi \otimes \iota)(\Delta(c)(a \otimes 1)) \tag{2.4}$$

for all a, c.

We will see from some of the examples we construct in Section 5 that the existence of such a linear map S for a given φ is a fairly strong condition.

We have the following equivalent formulation.

2.3. Proposition A linear functional φ on A is a left integral if and only if there is linear map S on B such that

$$\varphi((a \triangleleft S(b))c) = \varphi(a(c \triangleleft b)) \tag{2.5}$$

for all $a, c \in A$ and $b \in B$. Similarly, a linear functional ψ on A is a right integral if and only if there is a linear map S' on B such that

$$\psi(c(S'(b) \triangleright a)) = \psi((b \triangleright c)a) \tag{2.6}$$

for all $a, c \in A$ and $b \in B$.

We arrive at the formulas by pairing the ones in Equations (2.3) and (2.4) with an element b of B.

We know that these properties are true for integrals on Hopf algebras. There they are proven by using that the coproduct is a homomorphism. It is somewhat remarkable however that the results are still valid for quantum hypergroups, see [3], and also for weak Hopf algebras and weak multiplier Hopf algebras, see [22]. When treating the examples in Sections 4, 5 and 6, this will become more clear.

2.4. Remark In the theory of operator algebras, there is an object called Kac algebras, see [5]. It is an operator algebra with a coproduct. And the existence of an antipode is assumed. It is an anti-isomorphism that flips the coproduct. In that theory, the left integral is defined with reference to the antipode using the same formula as above.

An immediate and important consequence of the definition is the following.

2.5. Proposition For a faithful left integral, the associated map S on A is unique and injective. It flips the coproduct. Similarly for the map S' associated to a faithful right integral.

Proof: i) We claim that elements of the form $(\iota \otimes \varphi)(\Delta(a)(1 \otimes c))$ span all of A when φ is faithful. Indeed, suppose that ω is a linear functional on A that kills all such elements. By the faithfulness of φ it follows that $(\omega \otimes \iota)\Delta(a) = 0$ and if we apply the counit, we get $\omega(a) = 0$ for all a so that $\omega = 0$. This proves the claim.

We have a similar result for the three other types of elements we have in the Equations (2.3) and (2.4) in Definition 2.2

ii) It follows that the map S associated to φ is uniquely defined and surjective. As the space is finite-dimensional, the map is bijective.

iii) To prove that S flips the coproduct, we consider its adjoint on B and show that this is an anti-isomorphism. For the adjoint S on B we have the formula

$$\varphi(a(c \triangleleft b)) = \varphi((a \triangleleft S(b))c). \tag{2.7}$$

for all $a, c \in A$ and $b \in B$. Now take two elements $b, b' \in B$. Then we have

$$\begin{aligned}\varphi((a \triangleleft S(bb'))c) &= \varphi(a(c \triangleleft bb')) \\ &= \varphi((a \triangleleft S(b'))(c \triangleleft b)) \\ &= \varphi((a \triangleleft S(b')S(b))c).\end{aligned}$$

Because this holds for all c and as φ is faithful, we get $a \triangleleft S(bb') = a \triangleleft S(b')S(b)$. This holds for all a and since the action is faithful, we find that $S(bb') = S(b)S(b')$. $\square$

Because of this result, the following definition makes sense.

2.6. Definition Let φ be a faithful left integral. We call S the antipode associated with φ. Similarly we call S' the antipode associated with a faithful right integral ψ.

It is expected that, under certain obvious conditions, a left integral composed with its antipode, is a right integral with the same antipode. Indeed, and more precisely, we have the following result.

2.7. Proposition Assume that φ is a left integral on A with antipode S. Assume that S is an anti-isomorphism of A. Then $\varphi \circ S$ is a right integral on A with the same antipode.

This follows easily from the fact that S flips the coproduct and that it is an anti-isomorphism.

We have a similar result for the composition $\psi \circ S'$ in the case of a right integral. The extra condition, namely that S is an anti-isomorphism of A is also natural as we have seen in the case of involutive algebras, see Proposition 1.10.

The duality theorem

In this subsection, we construct a dual right integral on B for a left integral on A.

Assume that we have a dual pair of algebras A and B as in Definition 1.4. Assume that φ is a faithful left integral on A with antipode S. We consider also the adjoint of S on B and use again S for this adjoint.

2.8. Proposition Define ψ on B by $\psi(b) = \varepsilon(c)$ if $b = \varphi(\,\cdot\,c)$. If S is an anti-isomorphism of A, then ψ is a faithful right integral on B with antipode S.

Proof: i) Because φ is faithful, ψ is well-defined on all of B.

ii) Let $b, d \in B$ and assume that $b = \varphi(\,\cdot\,c)$ for $c \in A$, so $\langle a, b\rangle = \varphi(ac)$. For all $a \in A$ we have

$$\langle a, db\rangle = \langle a \triangleleft d, b\rangle = \varphi((a \triangleleft d)c) = \varphi(a(c \triangleleft S^{-1}(d)))$$

by using the formula in Proposition 2.3. We see that $db = \varphi(\,\cdot\,c')$ where $c' = c \triangleleft S^{-1}(d)$. Then $\psi(db) = \langle c, S^{-1}(d)\rangle$. The faithfulness of ψ is a consequence of the fact that the pairing is non-degenerate.

iii) Let $a \in A$ and $b, d \in B$ with $b = \varphi(\,\cdot\,c)$. Then

$$\psi(d(S(a) \triangleright b)) = \langle S(a)c, S^{-1}(d)\rangle$$

because $S(a) \triangleright b = \varphi(\,\cdot\,S(a)c)$. On the other hand, because $a \triangleright d = S(S^{-1}(d) \triangleleft S(a))$, see the remark after Proposition 1.16, also

$$\begin{aligned}\psi((a \triangleright d)b) &= \langle c, S^{-1}(d) \triangleleft S(a)\rangle \\ &= \langle S(a)c, S^{-1}(d)\rangle\end{aligned}$$

as in item ii) above. We see that $\psi(d(S(a) \triangleright b)) = \psi((a \triangleright d)b)$ so that ψ is a right integral with antipode S. $\square$

We again need that S is an anti-isomorphism of A to prove this result. That is natural because when we have an integral on B, we must have that its antipode is an anti-isomorphism on A, see Proposition 2.5.

We find the following duality result.

2.9. Theorem Let A and B be unital finite-dimensional algebras with a pairing as in Definition 1.4. Assume that there is a faithful left integral φ on A and that its antipode S is an anti-isomorphism of A. Define ψ on B by $\psi(b) = \varepsilon(c)$ when $b = \varphi(\,\cdot\, c)$. Then ψ is a faithful right integral on B with antipode S' on B, given as the adjoint of S on A. It is an anti-isomorphism of B.

If $b = \varphi(\,\cdot\, c)$, then $\psi(\,\cdot\, b) = S^{-1}(c)$, see Item ii) in the proof of Proposition 2.8. Now we can take the dual left integral on A constructed from ψ on B as above. We get back the original left integral on A. We refer to this property as *biduality*.

The map $c \mapsto \varphi(\,\cdot\, S(c))$ can be considered as a generalization of the Fourier transform and then $b \mapsto \psi(\,\cdot\, b)$ is the inverse Fourier transform. We can refer to [19] for a discussion on the Fourier transform in quantum group theory.

We also have the following alternative formulation of this duality result.

2.10. Proposition Let (A, Δ) be a pair of a finite-dimensional unital algebra A with a coproduct Δ. We assume that there is a counit on A. We also assume that there is a faithful left integral on A with an antipode that is an anti-isomorphism. Then the dual pair (B, Δ) is again an unital algebra with a coproduct, admitting a counit. Moreover there is a left integral on B with an antipode that is an anti-isomorphism.

To show this, we use that a right integral composed with its antipode, is a left integral when this antipode is an anti-isomorphism and we apply the theorem.

Recall that the antipodes will flip the coproducts by the very existence of the integrals.

We call (B, Δ) the dual of (A, Δ). The dual of (B, Δ) is the same as the original pair (A, Δ).

In the case of a *-algebra we have the following result.

2.11. Proposition If φ is a left integral, $a \in A$ and $b = \varphi(\,\cdot\, a)$, then $\psi(b^*b) = \overline{\varphi(a^*a)}$. In particular, if φ is positive, then so is ψ and $\psi(b^*b) = \varphi(a^*a)$.

Proof: We know that $\psi(b^*b) = \langle S^{-1}(a), b^*\rangle$ and so

$$\psi(b^*b) = \langle a^*, b\rangle^- = \varphi(a^*a)^-.$$

The second statement is obvious. □

If we think of the map $a \mapsto \varphi(\,\cdot\, a)$ as a Fourier transform, then the above result is like Plancherel's theorem. See again [19].

We have seen that the composition $\varphi \circ S$ is a right integral if S is an anti-isomorphism. If S is a *-map, then this right integral will again be positive when φ is positive. However, it is not expected that this is true in general.

To finish this section, let us make a first comparison with the notion and the results obtained in [3].

2.12. Remark i) Consider Definition 1.10 of [3] and restrict it to the case of a finite-dimensional unital algebra A. The coproduct Δ is assumed to be regular and coassociative as in Definition 1.1 and 1.2 of [3]. As we consider now only finite-dimensional unital algebras, the only requirement is that Δ is coassociative in the common sense. In the setting of this paper, the coproduct on A is obtained from the product in B and so it is coassociative. Regularity is automatic since we are working with unital algebras.

ii) In Definition 1.10 of [3] it is required that there is a counit ε on A. According to Definition 1.3 of [3], a counit is assumed to be a homomorphism of A. Here we encounter a first difference. In the setting of this paper, the counit on A is obtained from the identity in B but is not required to satisfy $\Delta(1) = 1 \otimes 1$ in $B \otimes B$ at this point. Hence, the associated counit on A is not required to be a homomorphism.

iii) Finally, in Definition 1.10 of [3] the existence of a left invariant functional (as in Definition 1.5 of [3]) is required. This again is not a condition for the left integral we consider in the first place here. On the other hand, Definition 1.9 of [3] is as in our Definition 2.2.

To summarize, at this level, we have *two differences*. First it is not required that the counit is a homomorphism. And secondly, we use a different notion of a left integral for a linear functional. In [3] a left integral φ is by definition left invariant in the sense that $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all a. This is not assumed at this level in this note. However, further, the notions of an antipode relative to a linear functional are the same. Indeed compare Equation 2.3 with the formula in Definition 1.9 of [3].

As a consequence the results we have so far about duality in this section do not follow from the results in [3].

3. Finite quantum hypergroups

In what follows, we again take finite-dimensional unital algebras A and B, not necessarily *-algebras, with a non-degenerate pairing $(a, b) \mapsto \langle a, b\rangle$ from $A \times B$ to $\mathbb{C}$ as in Definition 1.4. In the previous section, we obtained a duality of algebras with integrals under the extra condition that the associated antipodes are anti-isomorphisms. The intention here is to impose more conditions on the pairing to get a pair of *finite quantum hypergroups*.

Left and right invariant functionals

We begin with the introduction of left and right invariant functionals for a pair (A, Δ) of a finite-dimensional unital algebra with a coproduct. Recall that we do not assume that Δ is a homomorphism. The condition $\Delta(1) = 1 \otimes 1$ is also not required from the beginning.

3.1. Definition A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. Similarly, a linear functional ψ is called *right invariant* if $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all a.

In the previous section, we have introduced the notion of a left integral and of a right integral. There are cases where left and right integrals exist and no invariant functionals. But it is also possible to have invariant functionals without the existence of integrals. For some examples, we refer to Section 5 of the extended version of this paper [7].

The following result gives a necessary condition to have invariant functionals.

3.2. Proposition If we have a non-zero left invariant linear functional on (A, Δ), then $\Delta(1) = 1 \otimes 1$.

Proof: Let φ be a left invariant functional. Take any $a \in A$ and assume that $\varphi(a) \neq 0$. Then

$$\begin{aligned}\Delta(1)\varphi(a) &= \Delta((\iota \otimes \varphi)(\Delta(a)))\\ &= (\iota \otimes \iota \otimes \varphi)((\Delta \otimes \iota)\Delta(a))\\ &= (\iota \otimes \iota \otimes \varphi)((\iota \otimes \Delta)\Delta(a))\\ &= (\iota \otimes \iota \otimes \varphi)\Delta_{13}(a) = \varphi(a)\, 1 \otimes 1.\end{aligned}$$

We use the *leg numbering notation*. More precisely $\Delta_{13}(a) = (\iota \otimes \zeta)(\Delta(a) \otimes 1)$ here where ζ is the flip map on $A \otimes A$. This proves the result. $\square$

We see that a left invariant functional can only exist when Δ is unital. This is equivalent with ε on B being a homomorphism. In fact, there is the following converse.

3.3. Proposition Assume that the counit ε on B is a homomorphism. If there is a *faithful* linear functional on B, then there is a left and a right invariant functional on A.

Proof: Let ω be a faithful linear functional on B. Because B is finite-dimensional, by Proposition 1.1 there is an element $h \in B$ so that $\varepsilon = \omega(\,\cdot\, h)$. Then, for all $b, b' \in B$, we have

$$\omega(b'bh) = \varepsilon(b'b) = \varepsilon(b')\varepsilon(b) = \varepsilon(b)\omega(b'h).$$

This holds for all b'. Because ω is faithful, we have $bh = \varepsilon(b)h$ for all b. This means that φ, defined on A by $\varphi(a) = \langle a, h\rangle$ is left invariant.

Similarly, if we take $k \in B$ satisfying $\varepsilon = \omega(k\,\cdot\,)$, we will have $kb = \varepsilon(b)k$ for all $b \in B$. Then $a \mapsto \langle a, k\rangle$ is a right integral. $\square$

3.4. Remark If the coproduct is a homomorphism, we can construct an antipode if we have faithful invariant functionals. This follows from the Larson Sweedler theorem. For the case of a unital coproduct, see [12]. For the more general case, see [21]. In this case, a left invariant functional is also a left integral.

As we aim to study quantum hypergroups, we do not require that the coproduct is a homomorphism. Then, unfortunately, we can not say much more at this point. We see from Proposition 3.3 that invariant functionals exist under mild conditions. To have integrals, we need the existence of an antipode as in Definition 2.2 and this is a more stringent condition.

Invariant functionals and integrals

Recall that there is a difference between invariant functionals and integrals. Here we study the relation between the two concepts.

The following is easy to prove.

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

3.5. Proposition Assume that (A, Δ) admits a faithful left integral φ. Denote by S the associated antipode. If Δ is unital and if $S(1) = 1$, then φ is left invariant. Similarly for a faithful right integral.

The result follows by taking $c = 1$ Equation (2.3) in Definition 2.2.

That Δ has to be unital is obviously needed for this property as non-trivial invariant functionals can only exits in that case, see Proposition 3.2. The condition $S(1) = 1$ will follow when we require that S is an anti-isomorphism of the algebra A. We know that also this is a natural condition. We needed it to have the duality in the previous section and we will in the end also need it here.

But first we prove the following *uniqueness property*.

3.6. Proposition Assume that φ is a faithful left integral on A with antipode S and that $S(1) = 1$. Let φ' be any left invariant functional on A. Then φ' is a scalar multiple of φ.

Proof: By proposition 1.1 we have an element c so that $\varphi'(a) = \varphi(ac)$ for all a. Then we have

$$\begin{aligned}\varphi(ac)1 &= \varphi'(a)1 \\ &= (\iota \otimes \varphi')\Delta(a) \\ &= (\iota \otimes \varphi)(\Delta(a)(1 \otimes c)) \\ &= S^{-1}((\iota \otimes \varphi)((1 \otimes a)\Delta(c)))\end{aligned}$$

for all a. By the faithfulness of φ we get $S(1) \otimes c = \Delta(c)$. Now apply the counit and we get $c = \varepsilon(c)S(1)$. As it is assumed that $S(1) = 1$, the result follows. □

This result has some important consequences, see further. We first give another proof, providing also some extra results.

3.7. Proposition Assume that we have a faithful left integral φ on A with an antipode S satisfying $S(1) = 1$. Assume that φ' is a left invariant functional. Then there is an element δ in A so that $(\varphi' \otimes \iota)\Delta(a) = \varphi(a)\delta$. We also have $\varphi'(S(a)) = \varphi(a\delta)$ for all a.

Proof: Assume that φ is a left integral and that φ' is left invariant. As φ is a left integral with antipode S we get

$$(\varphi' \otimes \varphi)((1 \otimes a)\Delta(c)) = (\varphi' \circ S \otimes \varphi)(\Delta(a)(1 \otimes c)).$$

Because S flips the coproduct and $S(1) = 1$ we have that $\varphi' \circ S$ is right invariant. Then the right hand side equals $\varphi'(S(a))\varphi(c)$. For the left hand side we get $\varphi(a\delta_c)$ where $\delta_c = (\varphi' \otimes \iota)\Delta(c)$. So $\varphi'(S(a))\varphi(c) = \varphi(a\delta_c)$ for all a, c.

Now choose c_0 so that $\varphi(c_0) = 1$ and use δ for the associated element. Then we find $\varphi'(S(a)) = \varphi(a\delta)$. If we use this in the original equality we get $\varphi(a\delta)\varphi(c) = \varphi(a\delta)$. This holds for all a and from the faithfulness of φ we find $\varphi(c)\delta = \delta_c$ for all c. This means that $(\varphi' \otimes \iota)\Delta(c) = \varphi(c)\delta$ for all c. □

The element δ is often called the *modular element.*

If we apply the counit we get from $(\varphi'\otimes\iota)\Delta(a)=\varphi(a)\delta$ that $\varphi'(a)=\varphi(a)\varepsilon(\delta)$ and we see that the result also implies uniqueness as in Proposition 3.6.

We can apply Proposition 3.7 with φ in the place of φ' if we include the condition that $\Delta(1)=1\otimes 1$. Indeed, then we know by Proposition 3.5 that φ is left invariant. We obtain the existence of an element δ in A satsfying $(\varphi\otimes\iota)\Delta(a)=\varphi(a)\delta$ and $\varphi(S(a))=\varphi(a\delta)$. This property is also true for algebraic quantum groups, see [18] and for algebraic quantum hypergroups, see [3]. In fact, the proof above is inspired by the arguments used to obtain these results.

3.8. Proposition Assume that Δ is unital and that there is a faithful left integral φ with an antipode S satisfying $S(1)=1$. Then any left invariant functional is a scalar multiple of φ. In particular, any non-zero left invariant functional is a faithful left integral. Moreover, any non-zero left integral is a faithful left invariant functional.

So we get uniqueness of left integrals if Δ is unital and if there is a left integral with an antipode satisfying $S(1)=1$. Because the antipode is determined by the integral, we also have a unique antipode for the pair (A,Δ).

Finite quantum hypergroups

If we want duality for finite quantum hypergroups, we will need to impose the conditions on A what we have on B. In the first place, we need a faithful integral on both so that we have a pair like in Theorem 2.9. In particular, we need antipodes that are anti-isomorphisms. Moreover, and this *distinguishes the finite quantum hypergroups* from the objects we had in the previous section, we need unital coproducts on both sides. Or equivalently, we need that the counits are homomorphisms.

This leads to the following definition.

3.9. Definition Let A be a finite-dimensional unital algebra with a coproduct Δ that admits a counit. Assume that there is a faithful left integral on A. If the coproduct is unital, the counit a homomorphism and the antipode associated with the integral an anti-isomorphism, then we call (A,Δ) a *finite quantum hypergroup.* When A is a *-algebra and Δ a *-map, we call it a *finite *-quantum hypergroup.*

Remember that the existence of a faithful left integral implies that its antipode S also flips the coproduct.

Remark that the counit is unique if it exists. As we assume that the antipode S of the left integral is an anti-isomorphism, we must have $S(1)=1$. Then we also have a unique left integral and the antipode associated with it is also unique. Therefore we do not need to include these objects in the notation.

We now make another reference to the original work on algebraic quantum hypergroups ([3].

3.10. Remark i) Compare Definition 3.9 above with the Definition 1.10 in [3]. In Definition 1.9 of [3] of an antipode relative to a an invariant functional, it is required that S is an anti-homomorphism. It follows from the definition that S is always surjective and so, in the finite-dimensional case, the conditions imply that S is an isomorphism. We therefore get the same notions.

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

From the duality results in the previous section, when applied to a finite quantum hypergroup, we find the following.

3.11. Theorem Let (A, Δ) be a finite quantum hypergroup. Consider the dual space B with product and coproduct Δ adjoint to the coproduct and product of A. Then (B, Δ) is again a finite quantum hypergroup.

Proof: i) The algebra B is unital. We have a coproduct Δ on B. It admits a counit, given by the identity in A. The coproduct on B is unital because the counit on A is assumed to be an homomorphism. On the other hand, the counit on B is a homomorphism because the coproduct on A is unital.

ii) Let φ be a faithful left integral on A and S its antipode. By Proposition 2.8 we have a faithful right integral ψ on B. Its antipode is the adjoint of the antipode on A. It is an anti-isomorphism of B because the antipode on A flips the coproduct (by Proposition 2.5). The antipode on B flips the coproduct as the antipode on A is an anti-isomorphism.

iii) If we compose ψ with its antipode, we get a faithful left integral on B. Hence (B, Δ) is again a finite-quantum hypergroup. □

The biduality result that we have for finite-dimensional unital algebras with a faithful integral (see Theorem 2.9 and Proposition 2.10) is also valid here. The extra assumption, namely that the coproducts are unital, is satisfied for the algebra and its dual.

Once more we refer to the original work on quantum hypergroups.

3.12. Remark The duality and biduality we obtain for finite quantum hypergroups here is a special case of the more general results on algebraic quantum hypergroups that are obtained in [3]. In this paper however, they are obtained from another more general result about pairings of finite-dimensional algebras. These more general results play a role in our forthcoming paper [8] where we develop a new theory, more general than the one we have in [3]

We end this section with a brief look at the involutive case. We have the following version of Theorem 3.11 for *-algebras.

3.13. Theorem Assume that (A, Δ) is a finite *-quantum hypergroup with antipode S. The dual is a *-algebra if the involution is defined by $\langle a, b^* \rangle = \langle S(a)^*, b \rangle^-$. It is again a *-quantum hypergroup. If the left integral on A is positive, then so is the dual right integral on B.

For the last statement, see Proposition 2.11.

4. Classical Hecke algebras

This is the first of three sections with examples of finite quantum hypergroups.

In the first place, we look at the case of a finite group with a subgroup. This gives rise to the classical Hecke algebras. But we go beyond these and derive some other, related examples in the next section.

Most calculations are easy and therefore we leave them as an exercise for the reader.

The basic examples associated with a group G and a subgroup H

Let G be a finite group. First we recall the following well-known fact.

Let A_0 be the *-algebra of all complex functions on G with pointwise operations. It is a Hopf *-algebra for the coproduct Δ_0 on A_0 defined by $\Delta_0(f)(p,q) = f(pq)$ where $p, q \in G$. The counit is given by $\varepsilon(f) = f(e)$ and the antipode by $(S(f))(p) = f(p^{-1})$ for all p.

On the other hand, let B_0 be the group algebra $\mathbb{C}G$. It is also a Hopf *-algebra. If we use $p \mapsto \lambda_p$ for the canonical embedding of G in the group algebra, we have $\lambda_p\lambda_q = \lambda_{pq}$ and $\lambda_p^* = \lambda_{p^{-1}}$ for all $p \in G$. The coproduct is given by $\Delta_0(\lambda_p) = \lambda_p \otimes \lambda_p$ and the counit satisfies $\varepsilon(\lambda_p) = 1$ for all p. For the antipode we have $S(\lambda_p) = \lambda_{p^{-1}}$.

There is a natural pairing between A_0 and B_0 given by $\langle f, \lambda_p\rangle = f(p)$ for all $p \in G$. It is a pairing of Hopf *-algebras in the sense that B_0 is the dual of A_0.

Now let H be a subgroup of G. We will associate a dual pair of finite *-quantum hypergroups A and B. They are derived from the Hopf *-algebras A_0 and B_0 using a similar procedure.

4.1. Definition We define the linear map E from A_0 to itself by

$$(E(f))(p) = \frac{1}{n^2}\sum_{h,k\in H} f(hpk)$$

for $p \in G$. Here n is the number of elements in H. We use A for the range of E_0.

4.2. Proposition The subspace A is the *-subalgebra of A_0 of functions f constant on double cosets. Moreover E is a conditional expectation from A_0 to A, it is unital and self-adjoint.

This is mostly trivial. That E is a conditional expectation means that $E(f) = f$ when $f \in A$ and that $E(ff') = E(f)f'$ and $E(f'f) = f'E(f)$ when $f \in A_0$ and $f' \in A$.

It is clear that $E(S(f)) = S(E(f))$ for all f. On the other hand, for the counit we have $\varepsilon(E(f)) = \frac{1}{n}\sum_{h\in H} f(h)$ and this is not $\varepsilon(f)$ (except when H is the trivial subgroup $\{e\}$).

For the coproduct, we have the following crucial property.

4.3. Proposition If $f \in A$ then

$$(E\otimes E)\Delta_0(f)(p,q) = (E\otimes\iota)\Delta_0(f)(p,q) = (\iota\otimes E)\Delta_0(f)(p,q) = \frac{1}{n}\sum_{k\in H} f(pkq).$$

These elements belong to $A \otimes A$.

Proof: One verifies easily that

$$((E\otimes\iota)\Delta_0(f))(p,q) = ((\iota\otimes E)\Delta_0(f))(p,q) = \frac{1}{n}\sum_{k\in H} f(pkq).$$

for $f \in A$ and for all $p, q \in G$. We see that $(E\otimes\iota)\Delta_0(f) = (\iota\otimes E)\Delta_0(f)$. Hence these functions belong to $A \otimes A$. $\square$

We now have the main property.

4.4. Theorem Define Δ on A by $\Delta(f) = (E\otimes\iota)\Delta_0(f)$. Then (A, Δ) is a finite *-quantum hypergroup.

Proof: i) It follows from the previous proposition that

$$(\Delta \otimes \iota)\Delta(f) = (\iota \otimes \Delta)\Delta(f) = (E \otimes \iota \otimes E)((\Delta_0 \otimes \iota)\Delta_0(f))$$

for $f \in A$. So Δ is coassociative on A because Δ_0 is coasociative on A_0. We also have $\Delta(1) = 1 \otimes 1$ because $E(1) = 1$. Further Δ is a *-map on A because that is so for Δ_0 and E is self-adjoint.

ii) For the counit ε on A we find, when $f \in A$, that

$$(\varepsilon \otimes \iota)\Delta(f) = E((\varepsilon \otimes \iota)\Delta_0(f)) = E(f) = f$$

and similarly for the other side. So the restriction of the counit to A is a counit on A for Δ. It is a homomorphism.

iii) Define $\varphi(f) = \sum_{p\in G} f(p)$. Let S be the antipode of the Hopf algebra (A_0, Δ_0). Then we have

$$S((\iota \otimes \varphi)(\Delta_0(f)(1 \otimes f'))) = (\iota \otimes \varphi)((1 \otimes f)\Delta_0(f'))$$

for all $f, f' \in A_0$. This is a well-known property of the antipode and in this case, easy to verify. Now suppose that $f, f' \in A$. Then apply E on the first factor in this equation and use that $E \circ S = S \circ E$. Then we find

$$S((\iota \otimes \varphi)(\Delta(f)(1 \otimes f'))) = (\iota \otimes \varphi)((1 \otimes f)\Delta(f')).$$

This proves that the restriction of φ to A is a left integral with S on A as its antipode.

iv) The restriction of φ to A is still faithful. In this case we can use that φ is positive and faithful on A_0 and therefore its restriction to the *-subalgebra A_0 is still faithful.

v) The antipode is an anti-homomorphism on A_0 and so also on A. It flips the coproduct Δ_0 and hence

$$\Delta(S(f)) = (E \otimes \iota)\zeta(S \otimes S)\Delta_0(f) = \zeta(S \otimes S)(\iota \otimes E)\Delta_0(f)$$

because E and S commute. We use ζ for the flip map on $A_0 \otimes A_0$. So S also flips the coproduct Δ on A.

This completes the proof. □

One can also prove the main property of the antipode on A by a direct and straightforward calculation.

4.5. Remark By using the conditional expectation E, we can use the knowledge about the original Hopf algebra. Moreover, it turns out to be a general procedure to get a quantum hypergroup from a quantum group, see e.g. [4]. In particular, we see why the basic formula relating the integral with the antipode still holds. We will also use it for the dual quantum hypergroup of this example. Finally, the use of a conditional expectation to construct quantum hypergroups will also be illustrated in Section 6.

The treatment of this case is not very different from what is found already in the earlier papers [3, 4]. However, the following result is not included in these papers.

When H is a normal subgroup of G we get a Hopf *-algebra, but as we see from the next result, only in that case.

4.6. Proposition The coproduct is a homomorphism on A if and only if H is a normal subgroup of G.

Proof: i) Assume that H is a normal subgroup. Then we have, for $f \in A$,

$$\Delta(f)(p,q) = \frac{1}{n}\sum_{h\in H} f(phq) = \frac{1}{n}\sum_{h\in H} f(php^{-1}pq) = f(pq)$$

because $php^{-1} \in H$ for all $p \in G$ and $h \in H$. Therefore in this case we have $\Delta = \Delta_0$ on A. Hence Δ_0 is a homomorphism on A_0.

ii) Conversely assume that Δ is a homomorphism. Take a point $p \in G$ and consider the map $f \mapsto \Delta(f)(p,p^{-1})$. It is a homomorphism on A and so there is an element q in G such that $\Delta(f)(p,p^{-1}) = f(q)$ for all $f \in A$. Now take the function f that is 1 on the double coset HqH containing q and 0 on other cosets. The equality

$$f(q) = \frac{1}{n}\sum_{H} f(php^{-1})$$

can only be satisfied if all terms $f(php^{-1})$ are equal to 1. This means that $php^{-1} \in HqH$ for all h. In particular for $h = e$. This means that $e \in HqH$ and $HqH = H$. So $q \in H$ and $php^{-1} \in H$ for all h. This holds for all p and therefore H is a normal subgroup.

□

The dual *-quantum hypergroup for this example

There are two possible approaches in line with the two options discussed before. We can use the general construction of the dual of A as in the previous section, or we can construct another finite *-quantum hypergroup B, then define the pairing and argue that the new quantum hypergroup can be identified with the dual of the first one. We will use the second method as it is more instructive.

However, still we have to make a choice. We can either take the direct construction as it could be done also for the previous case, or we can construct the dual B from the group algebra B_0, as we did for A. We will follow this path.

For this purpose, we use the following element in the group algebra $\mathbb{C}G$ of G. As before, $\mathbb{C}G$ is denoted by B_0.

4.7. Proposition Define $u \in B_0$ by $u = \frac{1}{n}\sum_{h\in H}\lambda_h$. Then u is a self-adjoint idempotent satisfying

$$\Delta_0(u)(1 \otimes u) = \Delta_0(u)(u \otimes 1) = u \otimes u.$$

So u is a *group-like* projection as defined in [6].

We can now construct the finite *-quantum hypergroup B.

4.8. Theorem Let B be the subalgebra of B_0 of elements ubu with $b \in B_0$. Define Δ on B by $\Delta(b) = (u \otimes u)\Delta_0(b)(u \otimes u)$. Then (B, Δ) is a finite *-quantum hypergroup.

Proof: i) It is clear that Δ is a linear map from B to $B \otimes B$. To show that it is coassociative we use that Δ_0 is coassociative and that

$$\Delta(b) = (u \otimes 1)\Delta_0(b)(u \otimes 1) = (1 \otimes u)\Delta_0(b)(1 \otimes u)$$

when $b \in B$. This follows from the fact that

$$\Delta_0(u)(1 \otimes u) = \Delta_0(u)(u \otimes 1) = u \otimes u$$
$$(1 \otimes u)\Delta_0(u) = (u \otimes 1)\Delta_0(u) = u \otimes u.$$

Also Δ is unital on B because u is the unit in the subalgebra B.

ii) For the counit ε on B_0 we have when $b \in B$,

$$(\varepsilon \otimes \iota)\Delta(b) = u((\varepsilon \otimes \iota)\Delta_0(b))(u) = ubu = b.$$

Similarly on the other side. So the restriction of ε to B is a counit for Δ on B.

iii) For the antipode on B we use the antipode on B_0. Because $S(u) = u$, it leaves B invariant and it is an anti-isomorphism also of B.

iv) We consider a left integral φ on B_0, given by $\varphi(\lambda_e) = 1$ and $\varphi(\lambda_p) = 0$ when $p \neq e$. We restrict it to B. This restriction is still faithful.

v) Just as before it follows from the fact that

$$S((\iota \otimes \varphi)(\Delta_0(b)(1 \otimes d))) = (\iota \otimes \varphi)((1 \otimes b)\Delta_0(d))$$

for all b, d in the group algebra, that the same formula holds for Δ on B. We again use that $\Delta(b) = (u \otimes 1)\Delta_0(b)(u \otimes 1)$ for all $b \in B$ and also that $S(u) = u$. $\square$

4.9. Remark Observe the similarity between the two cases. The map $F : b \mapsto ubu$ is also a conditional expectation from B_0 onto B. Here however, it is not unital as $F(1) = u \neq 1$. The unit in B is not the unit in B_0 while the unit in A is the same as the unit in A_0. There is also a different behavior w.r.t. the counit. For the counit ε on B_0 we do have $\varepsilon \circ F = \varepsilon$ because $\varepsilon(u) = 1$. For the conditional expectation E on A_0 we do not have $\varepsilon \circ E = \varepsilon$ as we observed in the remark after Proposition 4.2. These two phenomena are of course related.

Still the two results are of the same type because $\Delta(b) = (F \otimes \iota)\Delta_0(b) = (\iota \otimes F)\Delta_0(b)$ for $b \in B$, just like we have for E on A_0.

When the group H is a normal subgroup, the element u commutes with all elements of B_0 and then, $\Delta(b) = \Delta_0(b)(1 \otimes u)$. It follows again that Δ is a homomorphism on B. Remark however that here Δ is not equal to the restriction of Δ_0 to B as was the case for A_0.

Also the converse is true. Indeed, if Δ is a homomorphism, then B is a Hopf algebra. We will show below that B is the dual of A. Then A is also a Hopf algebra and the coproduct on A is also a homomorphism. We have seen that then H is a normal subgroup of G, see Proposition 4.6.

The pairing of A with B

Consider the natural pairing of the function algebra A_0 with the group algebra B_0 of G given by $\langle f, \lambda_p \rangle = f(p)$ for all p. We use the conditional expectations $E : A_0 \mapsto A$ and $F : B_0 \mapsto B$.

4.10. Proposition For all $f \in A_0$ and $b \in B_0$ we have $\langle E(f), b \rangle = \langle f, F(b) \rangle$. The restriction of the pairing to $A \times B$ is a pairing of *-algebras. The coproducts induced by the pairing are the given coproducts on the algebras A and B.

Proof: Take $f \in C(G)$ and $p \in G$. Then

$$\langle E(f), \lambda_p \rangle = E(f)(p) = \frac{1}{n^2} \sum_{h,k \in H} f(hpk) = \frac{1}{n^2} \sum_{h,k \in H} \langle f, \lambda_{hpk} \rangle$$

and we see that $\langle E(f), \lambda_p \rangle = \langle f, u\lambda_p u \rangle = \langle f, F(\lambda_p) \rangle$.

The other statements in the proposition are easy consequences of this equality.. $\square$

The construction we have used here, namely to get quantum hypergroups from quantum groups, is found also in [4]. What we have done here is just a special case of the general procedure used in [4]. We have another application in Section 6, see Remark 6.17.

5. Two dimensional examples

We will now consider a special case. We take for G the group S_3 of permutations of the set $\{1, 2, 3\}$. We denote by σ_1 and σ_2 the permutations $(1, 2)(3)$ and $(1)(2, 3)$ respectively. They generate the group. They satisfy

$$\sigma_1^2 = \sigma_2^2 = e \qquad \text{and} \qquad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

We denote $\sigma_1\sigma_2\sigma_1$ by σ_3. It is the permutation $(13)(2)$ and also satisfies $\sigma_3^2 = e$. Further let $p = \sigma_1\sigma_2$, then

$$p^2 = \sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_2\sigma_2 = \sigma_2\sigma_1.$$

We see that $p^3 = e$. This gives the elements of the group: $\{e, p, p^2, \sigma_1, \sigma_2, \sigma_3\}$.

We now consider the subgroup H with elements e and σ_1.

5.1. Proposition The left cosets are $H = \{e, \sigma_1\}$, $\{p, \sigma_3\}$ and $\{p^2, \sigma_2\}$. The right cosets are $H = \{e, \sigma_1\}$, $\{p, \sigma_2\}$ and $\{p^2, \sigma_3\}$. The double cosets are $H = \{e, \sigma_1\}$ and V where $V = \{p, p^2, \sigma_2, \sigma_3\}$.

This is obtained easily from the properties of the generators σ_1 and σ_2.

The *-algebra A is spanned by the two characteristic functions $u = \chi_H$ and $v = \chi_V$. They satisfy $u^2 = u$, $v^2 = v$ and $uv = vu = 0$. For the involution we have $u^* = u$ and $v^* = v$. The sum $u + v$ is the identity of A.

For the coproduct Δ on A we get the following from the general theory, see Theorem 4.4.

5.2. Proposition We have $\Delta(1) = 1 \otimes 1$ and

$$\Delta(u) = u \otimes u + \frac{1}{2} v \otimes v$$
$$\Delta(v) = u \otimes v + v \otimes u + \frac{1}{2} v \otimes v.$$

For the counit, we get $\varepsilon(u) = 1$ and $\varepsilon(v) = 0$. For the antipode we have $S(u) = u$ and $S(v) = v$.

Let us finally look at the integrals. Because the coproduct is coabelian, the left integral is also a right integral. We obtain it from the general formula, again see Theorem 4.4.

5.3. Proposition The left integral φ is given by $\varphi(u) = 2$ and $\varphi(v) = 4$.

We verify the necessary formulas. We have

$$(\iota \otimes \varphi)\Delta(u) = \varphi(u)u + \frac{1}{2}\varphi(v)v = 2(u+v) = \varphi(u)1$$
$$(\iota \otimes \varphi)\Delta(v) = \varphi(u)v + \varphi(v)u + \frac{1}{2}\varphi(v)v = 4(u+v) = \varphi(v)1.$$

This proves that φ is left invariant.

In this case, it follows that it is a left integral. We claim that

$$(\iota \otimes \varphi)(\Delta(x)(1 \otimes y)) = (\iota \otimes \varphi)((1 \otimes x)\Delta(y)) \tag{5.1}$$

for all x and y in A. Because S is the identity map it follows that φ is a left integral.

When $x = 1$ or $y = 1$, Equation (5.1) follows from the left invariance of φ. On the other hand, when $x = u$ and $y = u$, it follows because the algebra is abelian. Then (5.1) is true for all x, y as the algebra is spanned by 1 and u.

In fact, as we see, a left invariant functional on such a two-dimensional abelian algebra is always a left integral for the trivial antipode.

All of this suggest a set of examples shown in the following propositions.

5.4. Proposition Let A be the algebra spanned by the identity 1 and an idempotent element v. Take any complex number α not equal to -1. Define Δ on A by $\Delta(1) = 1 \otimes 1$ and

$$\Delta(v) = v \otimes 1 + 1 \otimes v + \alpha\, v \otimes v.$$

Then (A, Δ) is a quantum hypergroup. If α is real and $v^* = v$, then it is a *-quantum hypergroup.

Proof: i) It is easy to verify that Δ is coassociative. If we let $\varepsilon(1) = 1$ and $\varepsilon(v) = 0$, we see that ε is a counit for Δ. It is a homomorphism.

ii) Define φ by $\varphi(v) = 1$ and $\varphi(1) = -\alpha$. It is left invariant.

iii) If $u = 1 - v$ we have $\varphi(u) = \varphi(1) - \varphi(v) = -(1 + \alpha)$. Because we assume that $\alpha \neq -1$, we see that φ is faithful.

iv) That φ is a left integral for the antipode defined as the identity follows as before. Remark that the identity map is an anti-isomorphism because the algebra is abelian.

v) The last statement is obvious. Remark that $a \mapsto S(a)^*$ is the involution itself and it is a homomorphism because the algebra is abelian. $\square$

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

For $\Delta(u)$ and $\Delta(v)$ we find that

$$\Delta(u) = u \otimes u - (\alpha+1)\, v \otimes v \qquad \text{and} \qquad \Delta(v) = u \otimes v + v \otimes u + (\alpha+2) v \otimes v.$$

We see that for $\alpha = -1$ we have $\Delta(u) = u \otimes u$. Then for a left integral we get $\varphi(u) = 0$ and so φ can not be faithful.

For $\alpha = -2$ we get $\Delta(u) = u \otimes u + v \otimes v$ and $\Delta(v) = u \otimes v + v \otimes u$. Only in this case Δ is a homomorphism. This gives the Hopf algebra of the group with two elements.

For $\alpha = -\frac{3}{2}$ we get the first example in Proposition 5.2.

For the dual of this more general example we have to consider two cases.

5.5. Proposition Let A be the algebra generated by 1 and an element v satisfying $v^2 = v$ and let B be the algebra generated by 1 and an element w satisfying $w^2 = w$. Assume that $\alpha \neq 0$. Define a pairing of A with B by

$$\begin{array}{ccc} \langle 1, 1\rangle = 1 & \text{and} & \langle v, 1\rangle = 0 \\ \langle 1, w\rangle = 0 & \text{and} & \langle v, w\rangle = \alpha^{-1}. \end{array}$$

Then the coproduct Δ on A is unital and $\Delta(v) = v \otimes 1 + 1 \otimes v + \alpha\, v \otimes v$

Proof: Because $(1, v')$, with $v' = \alpha v$, and $(1, w)$ are dual bases of A and B respectively, the coproduct can easily be derived from the product. Indeed, because in B we have

$$\begin{array}{ccc} 1 \cdot 1 = 1 & \text{and} & 1 \cdot w = w \\ w \cdot 1 = w & \text{and} & w \cdot w = w, \end{array}$$

we must have

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 \\ \Delta(v') &= 1 \otimes v' + v' \otimes 1 + v' \otimes v'. \end{aligned}$$

If we replace v' by αv in the last equality, we get the result. $\square$

By symmetry, the coproduct on B is also unital and $\Delta(w) = 1 \otimes w + w \otimes 1 + \alpha w \otimes w$.

We get the following duality result.

5.6. Proposition Assume that $\alpha \neq -1$ and $\alpha \neq 0$, then the above pairing is a pairing of quantum hypergroups.

Indeed, the pairing of A with B induces the coproduct as in Proposition 5.2. Hence A is a quantum hypergroup. We need $\alpha \neq -1$ for this to hold. And we need $\alpha \neq 0$ for Proposition 5.5 to hold.

However, we also have the following result.

5.7. Proposition Let A be generated by 1 and v satisfying $v^2 = v$. Now let D be generated by 1 and an element y satisfying $y^2 = 0$. Again let $\alpha \neq 0$. Define a pairing by

$$\begin{array}{ccc} \langle 1, 1\rangle = 1 & \text{and} & \langle v, 1\rangle = 0 \\ \langle 1, y\rangle = 0 & \text{and} & \langle v, y\rangle = \alpha^{-1}. \end{array}$$

The induced coproduct Δ on A is unital and satisfies

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

while the coproduct on D is unital and

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \alpha y \otimes y.$$

Again this follows easily from the formulas for the product in A and in D because we have dual bases.

Now we get again a pairing of quantum hypergroups.

5.8. Proposition The algebra D with the coproduct as in the above proposition is again a quantum hypergroup. As in the previous proposition, we have a pairing of quantum hypergroups.

The left integral on A is given by $\varphi(v) = 1$ and $\varphi(1) = 0$. The left integral on D is as before, we have $\varphi(y) = 1$ and $\varphi(1) = -\alpha$.

Remark that we can assume that $\alpha = 1$ by replacing y by αy.

This suggest still another possibility.

5.9. Proposition Let C be the unital algebra generated by an element x satisfying $x^2 = 0$ and D the unital algebra generated by y satisfying $y^2 = 0$. Again let $\alpha \neq 0$. Define a pairing by

$$\begin{aligned} \langle 1, 1\rangle &= 1 \qquad \text{and} \qquad \langle x, 1\rangle = 0 \\ \langle 1, y\rangle &= 0 \qquad \text{and} \qquad \langle y, y\rangle = \alpha^{-1}. \end{aligned}$$

Then the coproducts on C and on D are given by the same formula:

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x \\ \Delta(y) &= y \otimes 1 + 1 \otimes y. \end{aligned}$$

Again we can assume that $\alpha = 1$. And we get a pairing of quantum hypergroups.

5.10. Remark In all these cases we have a pairing of *-quantum hypergroups. The generators v, w, x, y are assumed to be self-adjoint. Only for the first case however (Proposition 5.5), we have a good involutive structure in the sense that we have operator algebras. Indeed, if $\varphi(1) = 0$, then φ can not be positive. In an operator algebra, we also can not have non-zero self-adjoint elements y satisfying $y^2 = 0$.

In the extended version of this paper [7] we have included some more examples of this type to illustrate the difference between invariant functionals and integrals. The first example is obtained from the two set groupoid. In that case, integrals exist but no invariant functionals. The other examples have invariant functionals, but no integrals. By a remark made earlier (before Proposition 5.4), we need 3- or 4-dimensional algebras to achieve this.

6. Two subgroups

In this section, we consider some examples of finite quantum hypergroups of a different nature. They are inspired by the theory of bicrossproducts. We will formulate the results, but only for a few cases we will include a proof. Most of the arguments are obvious and easy. We leave them as an exercise to the reader. Moreover, details can be found in the expanded version of this paper [7] and for the more general cases in [10].

Finite quantum hypergroups from a group with two finite subgroups

The starting point is any group G, not necessarily finite, with two finite subgroups H and K. The only requirement is that $H \cap K = \{e\}$ where e is the identity of G.

6.1. Notation Denote by Ω the set of pairs (h,k) in $H \times K$ satisfying $hk \in KH$. For $(h,k) \in \Omega$ we define elements $h \triangleright k \in K$ and $h \triangleleft k \in H$ by $hk = (h \triangleright k)(h \triangleleft k)$. These elements are well-defined because $hk \in KH$ and $H \cap K = \{e\}$.

One easily shows that $(h,k) \mapsto h \triangleright k$ is a partially defined left action of H on K and that $(h,k) \mapsto h \triangleleft k$ is a partial right action of K on H. The actions are unital. On the other hand, for all $h, h' \in H$ and $k, k' \in K$ we have $h \triangleright e = e$, $e \triangleleft k = e$ and

$$h \triangleright kk' = (h \triangleright k)((h \triangleleft k) \triangleright k') \qquad \text{and} \qquad h'h \triangleleft k = (h' \triangleleft (h \triangleright k))(h \triangleleft k)$$

whenever these expressions are defined.

All these formulas follow easily from the definitions.

We introduce two finite-dimensional unital *-algebras A and B. The underlying space for both is the set $C(\Omega)$ of complex functions on Ω, but the product and the involutions are different.

6.2. Proposition Let $f, f_1, f_2 \in C(\Omega)$. If we define $f_1 f_2$ and f^* in $C(\Omega)$ by

$$(f_1 f_2)(h,k) = \sum_v f_1(h,v) f_2(h \triangleleft v, v^{-1}k) \tag{6.1}$$
$$f^*(h,k) = \overline{f(h \triangleleft k, k^{-1})},$$

then $C(\Omega)$ is a unital *-algebra. The sum is taken over the elements v in K satisfying $(h,v) \in \Omega$. The unit is the function f given by $f(h,k) = 0$ except for $k = e$ and $f(h,e) = 1$ for all h.

Having a sum like in Equation (6.1), to be precise, we always should mention explicitly the set over which the sum is taken. Most of the time however, this should be clear. If that is not the case, we will indicate it.

6.3. Proposition Let $g, g_1, g_2 \in C(\Omega)$. If we define $g_1 g_2$ and g^* in $C(\Omega)$ by

$$(g_1 g_2)(h,k) = \sum_u g_1(hu^{-1}, u \triangleright k) g_2(u,k)$$
$$g^*(h,k) = \overline{g(h^{-1}, h \triangleright k)},$$

then $C(\Omega)$ is a unital *-algebra. We take the sum over elements $u \in H$ satisfying $(u,k) \in \Omega$. The unit is the function g given by $g(h,k) = 0$ except for $h = e$ and $g(e,k) = 1$ for all k.

These two algebras are groupoid algebras, see Remark 6.14 below.

We will use $C(\widehat{\Omega})$ when we consider $C(\Omega)$ with this second product and involution. We keep the notation $C(\Omega)$ for the first algebra. We will systematically use the *algebra* $C(\Omega)$ in the first case and the *algebra* $C(\widehat{\Omega})$ in the second case. If we only consider the vector space, we simply mention it as the space $C(\Omega)$. Besides, we will systematically use f, f_1, f_2 for elements in the algebra $C(\Omega)$ and g, g_1, g_2 for elements in the algebra $C(\widehat{\Omega})$.

6.4. Theorem Define a pairing of the algebras $C(\Omega)$ and $C(\widehat{\Omega})$ by

$$\langle f, g\rangle = \sum_{(h,k)\in\Omega} f(h,k)g(h,k).$$

Then we have a dual pair of finite *-quantum hypergroups.

We split the proof in a couple of partial results.

First we have the necessary property of the counits, induced by the pairing.

6.5. Proposition The counits are given by

$$\varepsilon(f) = \sum_{k\in K} f(e,k) \quad \text{for } f \in C(\Omega),$$
$$\varepsilon(g) = \sum_{h\in H} g(h,e) \quad \text{for } g \in C(\widehat{\Omega}).$$

The counits are *-homomorphisms.

For the coproducts, we get the following formulas.

6.6. Proposition The coproducts are given by

$$\Delta(f)(u,v;h,k) = f(uh,k)\delta_{h\triangleright k}(v) \quad \text{for } f \in C(\Omega),$$
$$\Delta(g)(u,v;h,k) = g(u,vk)\delta_{u\triangleleft v}(h) \quad \text{for } g \in C(\widehat{\Omega}).$$

where (u,v) and (h,k) are Ω. They are *-maps and they are unital.

We use δ for the Kronecker delta function.

In general, the coproducts are not homomorphisms. There are simple examples to illustrate this. In fact, we have the following result, see Proposition 3.12 in [10].

6.7. Proposition The coproducts are homomorphisms if and only if $HK = KH$.

For a proof see also [7].

6.8. Remark i) If $HK = KH$ then HK is a subgroup and we can as well assume that it is all of G. In that case, we have a matched pair of subgroups and we find us in the theory of bicrossproducts.

ii) We get an example at the other extreme if G is the free group generated by H and K. Then Ω only contains the pairs (h,e) and (e,k). We will consider this special case further in this section.

We can verify that the following linear functionals are invariant.

6.9. Proposition i) Define φ on $C(\Omega)$ by

$$\varphi(f) = \sum_{h \in H} f(h, e).$$

Then φ is left and right invariant on $C(\Omega)$.

ii) Define φ on $C(\widehat{\Omega})$ by

$$\varphi(g) = \sum_{k \in K} g(e, k).$$

Then again φ is left and right invariant on $C(\widehat{\Omega})$.

Finally we obtain the antipodes.

6.10. Proposition i) Define S on the algebra $C(\Omega)$ by

$$S(f)(h,k) = f((h \triangleleft k)^{-1}, (h \triangleright k)^{-1})$$

for $(h,k) \in \Omega$. Then S is an antipode for the linear functional φ.

ii) We have the same formula for the antipode on the dual algebra.

Observe that $hk = (h \triangleright k)(h \triangleleft k)$ so that $(hk)^{-1} = (h \triangleleft k)^{-1}(h \triangleright k)^{-1}$. If we define $\widetilde{f}$ on G by $\widetilde{f}(hk) = f(h,k)$ when $(h,k) \in \Omega$ and $\widetilde{f}(p) = 0$ in other points, we see that

$$\widetilde{S(f)}(p) = \widetilde{f}(p^{-1})$$

for all $p \in G$.

The formulas for the maps $f \to S(f)^*$ on $C(\Omega)$ and $g \mapsto S(g)^*$ on $C(\widehat{\Omega})$ are easier to handle.

6.11. Proposition We have

$$S(f)^*(h,k) = \overline{f(h^{-1}, h \triangleright k)} \qquad \text{for } f \in C(\Omega)$$
$$S(g)^*(h,k) = \overline{g(h \triangleleft k, k^{-1})} \qquad \text{for } g \in C(\widehat{\Omega}).$$

One can verify these formulas easily.

We can use this to show that S on $C(\Omega)$ is an antipode for φ on $C(\Omega)$:

6.12. Proposition Assume that φ is the integral on the algebra $C(\Omega)$ as defined in item i) of Proposition 6.9. Let $f_1, f_2 \in C(\Omega)$ and define

$$f_3 = (\iota \otimes \varphi)(\Delta(f_1)(1 \otimes f_2^*)) \qquad \text{and} \qquad f_4 = (\iota \otimes \varphi)(\Delta(f_2)(1 \otimes f_1^*)).$$

Then $S(f_3)^* = f_4$.

For a proof, see [7] and [10].

One still has to verify that the antipodes are anti-isomorphisms. Again it is easier to show that the maps $f \mapsto S(f)^*$ and $g \mapsto S(g)^*$ are (conjuqate linear) isomorphisms. The arguments are of the same type as for showing that we have involutive algebras in Propositions 6.2 and 6.3.

This will complete the proof of the fact that $C(\Omega)$ and $C(\widehat{\Omega})$ are finite *-quantum hypergroups.

That they are dual to each other is proven in the following proposition.

6.13. Proposition Let $f, f' \in C(\Omega)$ and $g, g' \in C(\widehat{\Omega})$. Then

$$\langle \Delta(f), g \otimes g' \rangle = \langle f, gg' \rangle \qquad \text{and} \qquad \langle f \otimes f', \Delta(g) \rangle = \langle ff', g \rangle.$$

Also observe that, for $f, f' \in C(\Omega)$,

$$\varphi(ff') = \sum_h (ff')(h, e) = \sum_{h,v} f(h, v) f'(h \triangleleft v, v^{-1}).$$

The second sum is over elements $v \in K$ satisfying $(h, v) \in \Omega$. So the map $f' \mapsto \kappa(f')$ where

$$\kappa(f')(h, v) = f'(h \triangleleft v, v^{-1})$$

is the bijective map of the space $C_c(\Omega)$ that realizes the dual of $C(\Omega)$ with $C(\widehat{\Omega})$.

6.14. Remark i) The set Ω has two groupoid structures. One is given by $(h, k)(h', k') = (h, kk')$ if $h \triangleleft k = h'$. The other one is $(h, k)(h', k') = (hh', k')$ if $k = h' \triangleright k'$. The inverse of (h, k) is $(h \triangleleft k, k^{-1})$ in the first case and $(h^{-1}, h \triangleright k)$ in the second case.

ii) The algebra A is the groupoid algebra of Ω with the first groupoid structure and B is the groupoid algebra of Ω with the second one.

iii) The map $(h, k) \mapsto (h \triangleleft k, k^{-1})$ is an anti-isomorphism of the first groupoid and an isomorphism of the second one. This explains why $f \mapsto f^*$ is an anti-isomorphism and $f \mapsto S(f)^*$ an isomorphism of A and similarly for B.

iv) Finally remark again that

$$(hk)^{-1} = ((h \triangleright k)(h \triangleleft k))^{-1} = (h \triangleleft k)^{-1}(h \triangleright k)^{-1}.$$

If we put $h' = (h \triangleleft k)^{-1}$ and $k' = (h \triangleright k)^{-1}$ we see that $h' \triangleright k' = k^{-1}$ and $h' \triangleleft k' = h^{-1}$. In other words, if $(h', k') = ((h \triangleleft k)^{-1}, (h \triangleright k)^{-1})$ we have $h'k' = (hk)^{-1}$. As mentioned already, this clarifies the formula for the antipodes.

A special case

We consider now the case of two finite groups H and K satisfying $HK \cap KH = H \cup K$, i.e. when Ω is as small as possible. Then $hk \in KH$ can only occur when either $h = e$ or $k = e$. Hence the set Ω consist of elements (h, e) and (e, k) where $h \in H$ and $k \in K$. Remark that (e, e) is common to both. This makes the system not completely trivial as we will see.

We have the following characterizations of the algebras A and B in this case.

6.15. Proposition Let A_0 be the tensor product $F(H) \otimes \mathbb{C}K$ of the function algebra $F(H)$ of the set H with the group algebra $\mathbb{C}K$ of the group K. Then A is the subalgebra of functions f on $H \times K$ with support in Ω. The map $E : A_0 \to A$ given by restricting a function to Ω is a self-adjoint conditional expectation satisfying $E(1) = 1$.

Proof: i) We identify elements of A_0 with complex functions on $H \times K$. The product ff' of functions $f, f' \in A$ is given by the formula

$$(ff')(h, k) = \sum_v f(h, v) f'(h \triangleleft v, v^{-1}k).$$

Because f has support in Ω, then either $h = e$ or $v = e$. When $h = e$ we have $h \triangleleft v = e$ and so $h \triangleleft v = h$. This also holds for $v = e$. It follows that

$$(ff')(h,k) = \sum_v f(h,v)f'(h,v^{-1}k)$$

and we see that A is a subalgebra of A_0.

ii) Now assume that f has support in Ω and that f' is any functional in A_0. Consider the product ff' in A_0. If we assume that $h \neq e$ then $(ff')(h,k) = f(h,e)f'(h,k)$ and then $ff'(h,e) \neq 0$ implies $f'(h,e) \neq 0$. This means that $E(ff') = fE(f')$. On the other hand, assume that f' has support in Ω. Again when $h \neq e$ we have $(ff')(h,k) = f(h,k)f'(h,e)$ and $E(ff') = E(f)f'$ for the same reason.

iii) The identity in A_0 is given by the function f that takes the value 1 in (h,e) and 0 in other points of $H \times K$. This function has support in Ω and therefore $E(1) = 1$.

iv) Finally, if $(h,k) \in \Omega$, then $(h \triangleleft k, k^{-1}) \in \Omega$. But for $(h,k) \in \Omega$ we have either $h = e$ or $k = e$ so that $(h \triangleleft k, k^{-1}) = (h, k^{-1})$ in both cases. As the adjoint in A_0 is given by $f^*(h,k) = \overline{f(h,k^{-1})}$, we see that A is a *-subalgebra of A_0 and that $E(f^*) = E(f)^*$. $\square$

We have a similar result for the algebra B_0, defined as $\mathbb{C}H \otimes F(K)$.

We now consider the natural pairing of the function algebras with the group algebras and the associated tensor product pairing of A_0 with B_0, This is a pairing of Hopf *-algebras. As the pairing is given by $\langle f, g\rangle = \sum_{h,k} f(h,k)g(h,k)$ we have $\langle E(f), g\rangle = \langle f, E(g)\rangle$ for all $f \in A_0$ and $g \in B_0$.
It is an immediate consequence that $(E \otimes E)\Delta_0(f) = (E \otimes \iota)\Delta_0(f) = (\iota \otimes E)\Delta_0(f)$ when $f \in A$. Indeed, given any $g, g' \in B_0$ we find

$$\langle (E \otimes \iota)\Delta(f), g \otimes g'\rangle = \langle f, E(g)g'\rangle = \langle f, E(E(g)g')\rangle = \langle f, E(g)E(g')\rangle$$

and we see that $(E \otimes \iota)\Delta_0(f) = (E \otimes E)\Delta_0(f)$. Similarly on the other side.
One can now easily get the main result for this example:

6.16. Theorem The pairing of A with B makes A and B into a dual pair of *-quantum hypergroups.

The arguments needed to prove this result are very similar to the ones used to prove Theorem 4.4 and 4.8.

6.17. Remark i) Indeed, when we compare this with the Theorems 4.4 and 4.8 we see that in all these cases we have a conditional expectation E from a bigger algebra A_0 to the subalgebra A. The algebra A_0 is a Hopf algebra. The coproduct Δ on A is obtained from the coproduct Δ_0 on A_0 by the formula $\Delta(a) = (E \otimes E)\Delta_0(a)$ when $a \in A$. We use that $E(1)$ is the identity in A when 1 is the identity in A_0. We also use that the antipode S of A_0 leaves A globally invariant. Then the crucial property is that

$$(E \otimes E)\Delta_0(a) = (E \otimes \iota)\Delta_0(a) = (\iota \otimes E)\Delta_0(a)$$

for $a \in A$. The integrals on A are the restrictions to A of the integrals on A_0.

ii) For the dual B we also have such a conditional expectation F on a larger algebra B_0. When (A_0, B_0) is a dual pair of Hopf algebras, we need that

$$\langle E(a), b\rangle = \langle a, F(b)\rangle$$

for $a \in A_0$ and $b \in B_0$. Then the restriction of the pairing to $A \times B$ results in a pairing of quantum hypergroups.

The algebra A_0 of the previous example is generated by elements $\delta_h \otimes \lambda_k$ where δ_h is the Dirac function.

We now look at the simplest case where both H and K are groups with two elements $\{e, h\}$ and $\{e, k\}$ respectively. Then the algebra A is abelian and is spanned by the elements

$$p = \delta_h \otimes \lambda_e, \qquad u = \delta_e \otimes \lambda_e, \qquad v = \delta_e \otimes \lambda_k.$$

We have the following relations

$$\begin{array}{ll} p^2 = p, & pu = pv = 0, \\ u^2 = u, & uv = v, \\ v^2 = u. & \end{array}$$

The dual algebra B i is spanned by the elements

$$v' = \lambda_h \otimes \delta_e, \qquad u' = \lambda_e \otimes \delta_e, \qquad p' = \lambda_e \otimes \delta_k$$

and we have the same relations as above.

The pairing is given by

$$\begin{array}{lll} \langle p, v'\rangle = 1, & \langle u, v'\rangle = 0, & \langle v, v'\rangle = 0, \\ \langle p, u'\rangle = 0, & \langle u, u'\rangle = 1, & \langle v, u'\rangle = 0, \\ \langle p, p'\rangle = 0, & \langle u, p'\rangle = 0, & \langle v, p'\rangle = 1. \end{array}$$

Since we have dual bases, we can easily find the expressions for the coproduct on A from the product of the generators in B. One can also use the general formulas for the coproduct on the function algebras and the group algebras.

Observe that the algebras A and B are both $\mathbb{C}^3$. In A we can look at

$$\left\{p, \frac{1}{2}(u+v), \frac{1}{2}(u-v)\right\}$$

and we get 3 orthogonal projections. The pairing however is not the obvious pairing of $\mathbb{C}^3$ with itself as can be seen from the pairing of the generators above. If we denote these elements in A by e_1, e_2, e_3 and if we use f_1, f_2, f_3 for the elements

$$\left\{p', \frac{1}{2}(u'+v'), \frac{1}{2}(u'-v')\right\}$$

we find for the pairing

$$\begin{array}{lll} \langle e_1, f_1\rangle = 0, & \langle e_2, f_1\rangle = \frac{1}{2}, & \langle e_3, f_1\rangle = -\frac{1}{2}, \\ \langle e_1, f_2\rangle = \frac{1}{2}, & \langle e_2, f_2\rangle = \frac{1}{4}, & \langle e_3, f_2\rangle = \frac{1}{4}, \\ \langle e_1, f_3\rangle = -\frac{1}{2}, & \langle e_2, f_3\rangle = \frac{1}{4}, & \langle e_3, f_3\rangle = \frac{1}{4}. \end{array}$$

7. Conclusions and further research

In [10] we obtain a pair of *-algebraic quantum hypergroups from a pair (H, K) of subgroups of a group G satisfying $H \cap K = \{e\}$. In [11] we make an attempt to generalize this result for a pair of closed subgroups of a locally compact group. We no longer get algebraic quantum hypergroups but topological quantum hypergroups. Even in the case where the locally compact group G has a compact open subgroup, contrary to what was expected, we also still do not get a pair of algebraic quantum hypergroups, see [10].

This has inspired us to initiate the study of what would be considered as topological quantum hypergroups in [9]. In that paper, we emphasize on the development of the notion and suggest further study. We do not need this for the examples we find in the [10, 11].

While writing this paper on topological quantum hypergroups we found it instructive to collect the purely algebraic features behind this construction by treating finite quantum hypergroups. Finite quantum hypergroups are special cases of algebraic quantum hypergroups as studied in [3, 4]. In this paper, we have a treatment independent of the more general case of algebraic quantum hypergroups, especially written for the purpose of generalizing it to topological quantum hypergroups in [9].

In Section 2 we have defined left and right integrals and obtained a duality for pairs of finite-dimensional algebras with an antipode and such integrals. Finite quantum hypergroups require one more condition, namely that the coproducts are unital. Duality for finite quantum hypergroups is obtained in Section 3. In Section 5 we constructed examples of pairs where we have invariant functionals but no integrals. It suggests that the existence of integrals is much stronger than the existence of invariant functionals.

The duality of the more general objects, algebras with an antipode and integral that are not quantum hypergroups, deserves further study. In particular, there is a need for a suitable name for these objects. In [8] we treat such pairs of possibly infinite-dimensional non-degenerate algebras.

Finally, we spent a great deal of this paper to illustrate the theory with examples. It would be interesting to look for more examples of finite quantum hypergroups and by doing so understand the algebraic requirements.

References

[1] E. Abe: *Hopf algebras.* Cambridge Tracts in Mathematics, Vol. 74, Cambridge University Press, Cambridge, New-York (1977).

[2] A. Chapovsky and L.I. Vainerman, Compact quantum hypergroups, J. Operator Theory **41** (1999), 261-289.

[3] L. Delvaux, A. Van Daele, *Algebraic quantum hypergroups*, Adv. Math. **226** (2011), 1134–1167.

[4] L. Delvaux, A. Van Daele, *Algebraic quantum hypergroups II. Construction and examples*, Int. J. Math. **22** (2011), 407–434.

[5] M. Enock and J.-M. Schwartz, *Kac algebras and duality of locally compact groups*, Springer-Verlag, Berlin (1992).

[6] M.B. Landstad & A. Van Daele, *Compact and discrete subgroups of algebraic quantum groups*, arXiv: math.OA/0702458.

[7] M. B. Landstad and A. Van Daele, *Finite quantum hypergroups.* arXiv: 2209.13282v1 [math.QA].

[8] M. B. Landstad and A. Van Daele, *Duality of algebras with an antipode and integrals.* In Preparation.

12 Oct 2023 06:24:46 PDT
230425-VanDaele Version 3 - Submitted to Rocky Mountain J. Math.

[9] M. B. Landstad and A. Van Daele, *Dual pairs of quantum hypergroups*. In Preparation.

[10] M. B. Landstad and A. Van Daele, *Algebraic quantum hypergroups arising from a pair of subgroups*, In preparation.

[11] M. B. Landstad and A. Van Daele, *Topological quantum hypergroups arising from two closed subgroups*. In Preparation.

[12] R.G. Larson & M.E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*. Amer. J. Math. 91 ((1969), 75–93.

[13] T. Nakayama, *On Frobeniusean Algebras. 1*, Ann. of Math., 40, (1939), 611-633.

[14] G. K. Pedersen, *C*-Algebras and their Automorphism Groups*, London Mathematical Society. Monographs No. 14, Academic Press, (1979).

[15] D.E. Radford, *Hopf algebras*. Series on Knots and Everything, Vol. 49, World Scientific, Singapore (2012).

[16] T. Timmermann, A. Van Daele and S.H. Wang, *Pairing and duality of algebraic quantum groupoids*, Int. J. Math. Vol. 33, No. 07, 2250055 (2022).

[17] A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc. **342** (1994), 917–932.

[18] A. Van Daele, *An algebraic framework for group duality*, Adv. in Math. 140 (1998), 323-366.

[19] A. Van Daele *The Fourier transform in quantum group theory*, New techniques in Hopf algebras and graded ring theory (Brussels 2006), Conference Proceedings, Eds S. Caenepeel & F. Van Oystaeyen (2007), 187-196. See also arXiv:0609502.v3 [math.RA]

[20] A. Van Daele, *From Hopf algebras to topological quantum groups. A short history, various aspcects and some problems*, in "Quantum Dynamics", Banah Center Publications, Vol. 120, Institute of Mathematics, Polish Academy of Sciences, Warszawa (2020), 179-220.

[21] A. Van Daele & S. Wang: *The Larson-Sweedler theorem for multiplier Hopf algebras*. Journal of Algebra 296 (2006), 75–95.

[22] A. Van Daele and S. Wang *Weak multiplier Hopf algebras III. Integrals and duality*, See arXiv:1701.04951v4 [math.RA]

[23] M. Sweedler, *Hopf algebras*, Benjamin, New-York (1969).