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NEW CONGRUENCES AND DENSITY RESULTS FOR t-REGULAR PARTITIONS WITH DISTINCT EVEN PARTS

AJIT SINGH

ABSTRACT. Let $t \geq 2$ be a fixed positive integer. Let $ped_t(n)$ denote the number of t-regular partitions of n wherein the even parts are distinct and the odd parts are unrestricted. In this article, we establish infinite families of congruences for $ped_t(n)$ modulo certain positive integers M, for specific values of t. We next study the distribution of $ped_t(n)$ for $t = 3,5,7,9$. We prove that the series $\sum_{n=0}^{\infty} ped_t(2n+1)q^n$ is lacunary modulo arbitrary powers of 2 for $t = 3,5,9$. We also prove that the series $\sum_{n=0}^{\infty} ped_7(2n+1)q^n$ is lacunary modulo 2. We use arithmetic properties of modular forms and Hecke eigenforms to prove our results.

1. Introduction and statement of results

A partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n. Let $t \geq 2$ be a fixed positive integer. A t-regular partition of a positive integer n is a partition of n such that none of its parts is divisible by t. Let $b_t(n)$ be the number of t-regular partitions of n. In a recent paper [5], Hemanthkumar, Bharadwaj, and Naika studied the partition function $ped_t(n)$ which counts the number of t-regular partitions of n wherein the even parts are distinct and the odd parts are unrestricted. For example, $b_3(7) = 9$ with the relevant partitions being $7, 5+2, 5+1+1, 4+2+1, 4+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$; and $ped_3(7) = 7$ with the relevant partitions being $7, 5+2, 5+1+1, 4+2+1, 4+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$. The generating function of $ped_t(n)$ is given by [5]

$$\sum_{n=0}^{\infty} ped_t(n)q^n = \frac{f_4 f_t}{f_1 f_{4t}}, \tag{1.1}$$

where $f_k := (q^k;q^k)_\infty = \prod_{j=1}^{\infty}(1-q^{jk})$ and k is a positive integer.

In a recent paper [3], Drema and Saikia studied arithmetic properties of the partition function $ped_t(n)$ for certain values of t. Using q-series manipulations they proved several infinite families of congruences modulo small powers of 2 and 3. The objective of this paper is to study divisibility properties of $ped_t(n)$ for $t = 3,5,7,9$. To be specific, we use the theory of Hecke eigenforms to establish the following infinite families of congruences for $ped_t(n)$ modulo 2, 12, 8, and 18, respectively.

Theorem 1.1. *Let k,n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 3$ is prime such that $p_i \equiv 3 \pmod 4$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$ped_5\left(2p_1^2 \cdots p_{k+1}^2 n + \frac{p_1^2 \cdots p_k^2 p_{k+1}(4j+p_{k+1})+1}{2}\right) \equiv 0 \pmod 2.$$

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Let $p \geq 3$ be a prime such that $p \equiv 3 \pmod 4$. By taking $p_1 = p_2 = \cdots = p_{k+1} = p$ in Theorem 1.1, we obtain the following infinite family of congruences for $ped_5(n)$: For $k \geq 0$ and $n \geq 0$,

$$ped_5\left(2p^{2(k+1)}n + 2p^{2k+1}j + \frac{p^{2(k+1)}+1}{2}\right) \equiv 0 \pmod 2,$$

where $j \not\equiv 0 \pmod p$.

Theorem 1.2. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if p_i is prime such that $p_i \equiv 3 \pmod 4$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$ped_9\left(8p_1^2 \cdots p_{k+1}^2 n + 2p_1^2 \cdots p_k^2 p_{k+1}(4j + p_{k+1}) + 1\right) \equiv 0 \pmod{12}.$$

Let $p \geq 3$ be a prime such that $p \equiv 3 \pmod 4$. By taking $p_1 = p_2 = \cdots = p_{k+1} = p$ in Theorem 1.2, we obtain the following infinite family of congruences for $ped_9(n)$: For $k, n \geq 0$,

$$ped_9\left(8p^{2(k+1)}n + 8p^{2k+1}j + 2p^{2(k+1)} + 1\right) \equiv 0 \pmod{12},$$

where $j \not\equiv 0 \pmod p$.

Theorem 1.3. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 3$ is prime such that $p_i \not\equiv 1 \pmod 6$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$ped_9\left(6p_1^2 \cdots p_{k+1}^2 n + p_1^2 \cdots p_k^2 p_{k+1}(6j + p_{k+1}) + 1\right) \equiv 0 \pmod 8.$$

Let p be a prime such that $p \not\equiv 1 \pmod 6$. By taking $p_1 = p_2 = \cdots = p_{k+1} = p$ in Theorem 1.3, we obtain the following infinite family of congruences for $ped_9(n)$: For $k, n \geq 0$,

$$ped_9\left(6p^{2(k+1)}n + 6p^{2k+1}j + p^{2(k+1)} + 1\right) \equiv 0 \pmod 8,$$

where $j \not\equiv 0 \pmod p$.

Theorem 1.4. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 5$ is prime such that $p_i \not\equiv 1 \pmod 3$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$ped_9\left(12p_1^2 \cdots p_{k+1}^2 n + 4p_1^2 \cdots p_k^2 p_{k+1}(3j + p_{k+1}) + 1\right) \equiv 0 \pmod{18}.$$

Let $p \geq 5$ be a prime such that $p \not\equiv 1 \pmod 3$. By taking $p_1 = p_2 = \cdots = p_{k+1} = p$ in Theorem 1.4, we obtain the following infinite family of congruences for $ped_9(n)$: For $k, n \geq 0$,

$$ped_9\left(12p^{2(k+1)}n + 12p^{2k+1}j + 4p^{2(k+1)} + 1\right) \equiv 0 \pmod{18},$$

where $j \not\equiv 0 \pmod p$.

In addition to the study of Ramanujan-type congruences, it is an interesting problem to study the distribution of the partition function modulo positive integers M. To be precise, given an integral power series $F(q) := \sum_{n=0}^{\infty} a(n)q^n$ and $0 \leq r < M$, we define

$$\delta_r(F, M; X) := \frac{\#\{n \leq X : a(n) \equiv r \pmod M\}}{X}.$$

An integral power series F is called *lacunary modulo M* if

$$\lim_{X \to \infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M. For any fixed positive integer k, Gordon and Ono [4] proved that the partition function $b_t(n)$ is divisible by 2^k for almost all n. Similar studies are done for some other partition functions, for example see [11, 13, 14, 15]. In a recent paper [2], Cotron et al. proved lacunarity of certain eta-quotients modulo arbitrary powers of primes. We phrase their theorem as follows:

Theorem 1.5. [2, Theorem 1.1] *Let* $G(z) = \frac{\prod_{i=1}^{u} f_{\alpha_i}^{r_i}}{\prod_{i=1}^{t} f_{\beta_i}^{s_i}}$, *and* p *is a prime such that* p^a *divides* $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_u)$ *and*

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \beta_i s_i}{\sum_{i=1}^{u} \frac{r_i}{\alpha_i}}},$$

then $G(z)$ *is lacunary modulo* p^j *for any positive integer* j.

In this article, we study the arithmetic densities of $ped_t(2n+1)$ modulo arbitrary powers of 2 when $t = 3, 5, 9$. We also prove that $ped_7(2n+1)$ is almost always even. Also, the generating functions of these do not satisfy the conditions in the result of Cotron et al. In the following theorems, we prove that the partition functions $ped_3(2n+1)$, $ped_5(2n+1)$, and $ped_9(2n+1)$ are almost always divisible by arbitrary powers of 2 and $ped_7(2n+1)$ is lacunary modulo 2. To be specific, we prove the following results.

Theorem 1.6. *Let* k *be a positive integer and* $t \in \{3, 5, 9\}$. *Then the series* $\sum_{n=0}^{\infty} ped_t(2n+1)q^n$ *is lacunary modulo* 2^k, *namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 \leq n \leq X : ped_t(2n+1) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Theorem 1.7. *The series* $\sum_{n=0}^{\infty} ped_7(2n+1)q^n$ *is lacunary modulo* 2, *namely,*

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : ped_7(2n+1) \equiv 0 \pmod{2}\}}{X} = 1.$$

We prove Theorem 1.7 using the approach of Landau [8]. However, we couldn't find a similar proof for Theorem 1.6. We use a density result of Serre [12] to prove Theorem 1.6.

2. Preliminaries

We recall some definitions and basic facts on modular forms. For more details, see for example [10, 7]. We first define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where N is a positive integer. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the level of Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$. We identify ∞ with $\dfrac{1}{0}$ and define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \dfrac{r}{s} = \dfrac{ar+bs}{cr+ds}$, where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2} (cz+d)^{-\ell} f(\gamma z).$$

Definition 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

(1) We have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_\ell \gamma)(z)$ has a Fourier expansion of the form

$$(f|_\ell \gamma)(z) = \sum_{n \geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_\ell(\Gamma)$.

Definition 2.2. [10, Definition 1.15] If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

In this paper, the relevant modular forms are those that arise from eta-quotients. Recall that the Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer. We now recall two theorems from [10, p. 18] which will be used to prove our results.

Theorem 2.3. [10, Theorem 1.64] *If $f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N}\delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N}\frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^\ell s}{d}\right)$, where $s := \prod_{\delta|N}\delta^{r_\delta}$.

Suppose that f is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight ℓ is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N),\chi)$. The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

Theorem 2.4. [10, Theorem 1.65] *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. If f is an eta-quotient satisfying the conditions of Theorem 2.3 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}.$$

We now recall a density result of Serre [12] about the divisibility of Fourier coefficients of modular forms.

Theorem 2.5 (Serre)**.** *Let $f(z)$ be a modular form of positive integer weight k on some congruence subgroup of $SL_2(\mathbb{Z})$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

where $a(n)$ are algebraic integers in some number field. If m is a positive integer, then there exists a constant $c>0$ such that there are $O\left(\frac{X}{(\log X)^c}\right)$ integers $n \leq X$ such that $a(n)$ is not divisible by m.

We finally recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N),\chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty}\left(\sum_{d|\gcd(n,m)} \chi(d)d^{\ell-1}a\left(\frac{nm}{d^2}\right)\right)q^n.$$

In particular, if $m=p$ is prime, we have

$$f(z)|T_p := \sum_{n=0}^{\infty}\left(a(pn)+\chi(p)p^{\ell-1}a\left(\frac{n}{p}\right)\right)q^n. \tag{2.1}$$

We note that $a(n)=0$ unless n is a nonnegative integer.

Definition 2.6. A modular form $f(z)=\sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N),\chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exists a complex number $\lambda(m)$ for which

$$f(z)|T_m = \lambda(m)f(z). \tag{2.2}$$

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Setting $t=5$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} ped_5(n)q^n = \frac{f_4f_5}{f_1f_{20}}. \tag{3.1}$$

We now recall the following identity from [6]:

$$\frac{f_5}{f_1} = \frac{f_8f_{20}^2}{f_2^2f_{40}} + q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}. \tag{3.2}$$

Employing (3.2) in (3.1), we obtain

$$\sum_{n=0}^{\infty} ped_5(n)q^n = \frac{f_4f_8f_{20}}{f_2^2f_{40}} + q\frac{f_4^4f_{10}f_{40}}{f_2^3f_8f_{20}^2}. \tag{3.3}$$

Extracting the terms involving odd powers of q on both sides of (3.3), we get

$$\sum_{n=0}^{\infty} ped_5(2n+1)q^n = \frac{f_2^4f_5f_{20}}{f_1^3f_4f_{10}^2}. \tag{3.4}$$

This gives

$$\sum_{n=0}^{\infty} ped_5(2n+1)q^{4n+1} \equiv \eta(4z)\eta(20z) \pmod 2.$$

Let $\eta(4z)\eta(20z) := \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n)=0$ if $n \not\equiv 1 \pmod 4$ and for all $n \geq 0$,

$$ped_5(2n+1) \equiv a(4n+1) \pmod 2. \tag{3.5}$$

By Theorem 2.3, we have $\eta(4z)\eta(20z) \in S_1(\Gamma_0(80), \chi_0)$, where χ_0 is a Nebentypus character and is given by $\chi_0(\bullet) = (\frac{-5}{\bullet})$. Since $\eta(4z)\eta(20z)$ is a Hecke eigenform (see, for example [9]), (2.1) and (2.2) yield

$$\eta(4z)\eta(20z)|T_p = \sum_{n=1}^{\infty}\left(a(pn) + \chi_0(p)a\left(\frac{n}{p}\right)\right)q^n = \lambda(p)\sum_{n=1}^{\infty}a(n)q^n,$$

which implies

$$a(pn) + \chi_0(p)a\left(\frac{n}{p}\right) = \lambda(p)a(n). \tag{3.6}$$

Putting $n = 1$ and noting that $a(1) = 1$, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod 4$, we have $\lambda(p) = 0$. From (3.6), we obtain

$$a(pn) + \chi_0(p)a\left(\frac{n}{p}\right) = 0. \tag{3.7}$$

From (3.7), we derive that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2n + pr) = 0 \tag{3.8}$$

and

$$a(p^2n) = -\chi_0(p)a(n) \equiv a(n) \pmod 2. \tag{3.9}$$

Substituting n by $4n - pr + 1$ in (3.8) and together with (3.5), we find that

$$ped_5\left(2p^2n + \frac{p^2-1}{2} + pr\frac{1-p^2}{2} + 1\right) \equiv 0 \pmod 2. \tag{3.10}$$

Substituting n by $4n + 1$ in (3.9) and using (3.5), we obtain

$$ped_5\left(2p^2n + \frac{p^2-1}{2} + 1\right) \equiv ped_5(2n+1) \pmod 2. \tag{3.11}$$

Since $p \geq 3$ is prime, so $2 \mid (1-p^2)$ and $\gcd\left(\frac{1-p^2}{2}, p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{2}r$. Thus (3.10) can be rewritten as

$$ped_5\left(p^2n + \frac{p^2-1}{2} + pj + 1\right) \equiv 0 \pmod 2, \tag{3.12}$$

where $p \nmid j$.

Now, $p_i \geq 3$ are primes such that $p_i \not\equiv 1 \pmod 4$. Since

$$p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{2} = p_1^2\left(p_2^2 \dots p_k^2 n + \frac{p_2^2 \dots p_k^2 - 1}{2}\right) + \frac{p_1^2 - 1}{2},$$

using (3.11) repeatedly, we obtain that

$$ped_5\left(2p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{2} + 1\right) \equiv ped_5(2n+1) \pmod 2. \tag{3.13}$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$. Then (3.12) and (3.13) yield

$$ped_5\left(2p_1^2\dots p_{k+1}^2 n + \frac{p_1^2\dots p_k^2 p_{k+1}(4j+p_{k+1})+1}{2}\right) \equiv 0 \pmod{2}.$$

This completes the proof of the theorem. □

Proof of Theorem 1.2. Putting $t=9$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} ped_9(n)q^n = \frac{f_4 f_9}{f_1 f_{36}}. \tag{3.14}$$

We now recall the following identity from [16]:

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{3.15}$$

Employing (3.15) in (3.14), we obtain

$$\sum_{n=0}^{\infty} ped_9(n)q^n = \frac{f_4 f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}^2} + q\frac{f_4^3 f_6}{f_2^3 f_{12}}. \tag{3.16}$$

Extracting the terms involving odd powers of q on both sides of (3.16), we get

$$\sum_{n=0}^{\infty} ped_9(2n+1)q^n = \frac{f_2^3 f_3}{f_1^3 f_6}. \tag{3.17}$$

Again we recall the following identity from [5]:

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{3.18}$$

Employing (3.18) in (3.17), we obtain

$$\sum_{n=0}^{\infty} ped_9(2n+1)q^n = \frac{f_4^6 f_6^2}{f_2^7 f_{12}^2} + 3q\frac{f_4^2 f_{12}^2}{f_2^4}. \tag{3.19}$$

Extracting the terms involving odd powers of q on both sides of (3.19), we get

$$\sum_{n=0}^{\infty} ped_9(4n+3)q^n = 3\frac{f_2^2 f_6^2}{f_1^4} \equiv 3f_6^2 \pmod{12}. \tag{3.20}$$

Again extracting the terms involving even powers of q on both sides of (3.20), we obtain

$$\sum_{n=0}^{\infty} ped_9(8n+3)q^n \equiv 3f_3^2 \pmod{12}.$$

This gives

$$\sum_{n=0}^{\infty} ped_9(8n+3)q^{4n+1} \equiv 3\eta^2(12z) \pmod{12}.$$

Let $\eta^2(12z) := \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n)=0$ if $n \not\equiv 1 \pmod{4}$ and for all $n \geq 0$,

$$ped_9(8n+3) \equiv 3a(4n+1) \pmod{12}. \tag{3.21}$$

By Theorem 2.3, we have $\eta^2(12z) \in S_1(\Gamma_0(144), \chi_2)$, where χ_2 is a Nebentypus character given by $\chi_2(\bullet) = (\frac{-1}{\bullet})$. Since $\eta^2(12z)$ is a Hecke eigenform (see, for example [9]), (2.1) and (2.2) yield

$$\eta^2(12z)|T_p = \sum_{n=1}^{\infty} \left(a(pn) + \chi_2(p) a\left(\frac{n}{p}\right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} a(n) q^n,$$

which implies

$$a(pn) + \chi_2(p) a\left(\frac{n}{p}\right) = \lambda(p) a(n). \tag{3.22}$$

Putting $n = 1$ and noting that $a(1) = 1$, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod 4$, we have $\lambda(p) = 0$. From (3.22), we obtain

$$a(pn) + \chi_2(p) a\left(\frac{n}{p}\right) = 0. \tag{3.23}$$

From (3.23), we derive that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2 n + pr) = 0 \tag{3.24}$$

and noting that $\chi_2(p) = -1$ for the primes $p \equiv 3 \pmod 4$, we have

$$a(p^2 n) = -\chi_2(p) a(n) \equiv a(n) \pmod 4. \tag{3.25}$$

Substituting n by $4n - pr + 1$ in (3.24) and together with (3.21), we find that

$$ped_9\left(8p^2 n + 2(p^2 - 1) + 2pr(1 - p^2) + 3\right) \equiv 0 \pmod{12}. \tag{3.26}$$

Substituting n by $4n + 1$ in (3.25) and using (3.21), we obtain

$$ped_9\left(8p^2 n + 2(p^2 - 1) + 3\right) \equiv ped_9(8n + 3) \pmod{12}. \tag{3.27}$$

Since $p \geq 3$ is prime, so $\gcd\left((1 - p^2), p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $(1 - p^2)r$. Thus (3.26) can be rewritten as

$$ped_9\left(8p^2 n + 2(p^2 - 1) + 2pj + 3\right) \equiv 0 \pmod{12}, \tag{3.28}$$

where $p \nmid j$.

Now, $p_i \geq 3$ are primes such that $p_i \not\equiv 1 \pmod 4$. Since

$$p_1^2 \ldots p_k^2 n + p_1^2 \ldots p_k^2 - 1 = p_1^2\left(p_2^2 \ldots p_k^2 n + p_2^2 \ldots p_k^2 - 1\right) + p_1^2 - 1,$$

using (3.27) repeatedly, we obtain that

$$ped_9\left(8p_1^2 \ldots p_k^2 n + 2(p_1^2 \ldots p_k^2 - 1) + 3\right) \equiv ped_9(8n + 3) \pmod{12}. \tag{3.29}$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$. Then (3.28) and (3.29) yield

$$ped_9\left(8p_1^2 \ldots p_{k+1}^2 n + 2p_1^2 \ldots p_k^2 p_{k+1}(4j + p_{k+1}) + 1\right) \equiv 0 \pmod{12}.$$

This completes the proof of the theorem. □

4. Proof of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. First we recall the following identity from [3, (8.4)]:

$$\sum_{n=0}^{\infty} ped_9(6n+2)q^n \equiv 2f_2^2 \pmod 8.$$

This gives

$$\sum_{n=0}^{\infty} ped_9(6n+2)q^{6n+1} \equiv 2\eta^2(12z) \pmod 8.$$

Let $\eta^2(12z) := \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n)=0$ if $n \not\equiv 1 \pmod 6$ and for all $n \geq 0$,

$$ped_9(6n+2) \equiv 2a(6n+1) \pmod 8. \tag{4.1}$$

By Theorem 2.3, we have $\eta^2(12z) \in S_1(\Gamma_0(144),\chi_2)$, where χ_2 is a Nebentypus character and is given by $\chi_2(\bullet) = (\frac{-1}{\bullet})$. Since $\eta^2(12z)$ is a Hecke eigenform (see, for example [9]), (2.1) and (2.2) yield

$$\eta^2(12z)|T_p = \sum_{n=1}^{\infty}\left(a(pn)+\chi_2(p)a\left(\frac{n}{p}\right)\right)q^n = \lambda(p)\sum_{n=1}^{\infty} a(n)q^n,$$

which implies

$$a(pn)+\chi_2(p)a\left(\frac{n}{p}\right) = \lambda(p)a(n). \tag{4.2}$$

Putting $n=1$ and noting that $a(1)=1$, we readily obtain $a(p)=\lambda(p)$. Since $a(p)=0$ for all $p \not\equiv 1 \pmod 6$, we have $\lambda(p)=0$. From (4.2), we obtain

$$a(pn)+\chi_2(p)a\left(\frac{n}{p}\right) = 0. \tag{4.3}$$

From (4.3), we derive that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2n+pr) = 0 \tag{4.4}$$

and

$$a(p^2n) \equiv -\chi_2(p)a(n) \pmod 4. \tag{4.5}$$

Let $A(n) := a(6n+1)$. Let p be a prime such that $p \equiv 5 \pmod 6$. Now, replacing n by $6n-pr+1$ in (4.4), we find that

$$A\left(p^2n+\frac{p^2-1}{6}+pr\frac{1-p^2}{6}\right) = 0. \tag{4.6}$$

Substituting n by $6n+1$ in (4.5), we obtain

$$A\left(p^2n+\frac{p^2-1}{6}\right) \equiv -\chi_2(p)A(n) \pmod 4. \tag{4.7}$$

Since $p\geq 5$ is prime, so $6 \mid (1-p^2)$ and $\gcd\left(\frac{1-p^2}{6},p\right)=1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{6}r$. Thus (4.6) can be rewritten as

$$A\left(p^2n+\frac{p^2-1}{6}+pj\right)\equiv 0 \pmod 4, \tag{4.8}$$

where $p\nmid j$.

Now, $p_i\geq 5$ are primes such that $p_i\not\equiv 1 \pmod 6$. Since

$$p_1^2\dots p_k^2n+\frac{p_1^2\dots p_k^2-1}{6}=p_1^2\left(p_2^2\dots p_k^2n+\frac{p_2^2\dots p_k^2-1}{6}\right)+\frac{p_1^2-1}{6},$$

using (4.7) repeatedly, we obtain that

$$A\left(p_1^2\dots p_k^2n+\frac{p_1^2\dots p_k^2-1}{6}\right)\equiv(-\chi_2(p))^kA(n) \pmod 4. \tag{4.9}$$

Let $j\not\equiv 0 \pmod{p_{k+1}}$. Then (4.8) and (4.9) yield

$$A\left(p_1^2\dots p_{k+1}^2n+\frac{p_1^2\dots p_k^2p_{k+1}^2-1}{6}+p_1^2\dots p_k^2p_{k+1}j\right)\equiv 0 \pmod 4.$$

We complete the proof by using the fact that $ped_9(6n+2)\equiv 2A(n) \pmod 8$. □

Proof of Theorem 1.4. First we recall the following identity from [3, (10.3)]:

$$\sum_{n=0}^{\infty}ped_9(12n+5)q^n\equiv 6f_1^2f_3^2 \pmod{18}.$$

This gives

$$\sum_{n=0}^{\infty}ped_9(12n+5)q^{3n+1}\equiv 6\eta^2(3z)\eta^2(9z) \pmod{18}.$$

Let $\eta^2(3z)\eta^2(9z):=\sum_{n=1}^{\infty}c(n)q^n$. Then $c(n)=0$ if $n\not\equiv 1 \pmod 3$ and for all $n\geq 0$,

$$ped_9(12n+5)\equiv 6c(3n+1) \pmod{18}. \tag{4.10}$$

By Theorem 2.3, we have $\eta^2(3z)\eta^2(9z)\in S_2(\Gamma_0(27))$. Since $\eta^2(3z)\eta^2(9z)$ is a Hecke eigenform (see, for example [9]), (2.1) and (2.2) yield

$$\eta^2(3z)\eta^2(9z)|T_p=\sum_{n=1}^{\infty}\left(c(pn)+pc\left(\frac{n}{p}\right)\right)q^n=\lambda(p)\sum_{n=1}^{\infty}c(n)q^n,$$

which implies

$$c(pn)+pc\left(\frac{n}{p}\right)=\lambda(p)c(n). \tag{4.11}$$

Putting $n=1$ and noting that $c(1)=1$, we readily obtain $c(p)=\lambda(p)$. Since $c(p)=0$ for all $p\not\equiv 1 \pmod 3$, we have $\lambda(p)=0$. From (4.11), we obtain

$$c(pn)+pc\left(\frac{n}{p}\right)=0. \tag{4.12}$$

From (4.12), we derive that for all $n \geq 0$ and $p \nmid r$,

$$c(p^2n + pr) = 0 \tag{4.13}$$

and

$$c(p^2n) = -pc(n) \equiv c(n) \pmod 3. \tag{4.14}$$

Let $B(n) := c(3n+1)$. Let p be a prime such that $p \equiv 2 \pmod 3$. Now, replacing n by $3n - pr + 1$ in (4.13), we find that

$$B\left(p^2n + \frac{p^2-1}{3} + pr\frac{1-p^2}{3}\right) = 0. \tag{4.15}$$

Substituting n by $3n+1$ in (4.14), we obtain

$$B\left(p^2n + \frac{p^2-1}{3}\right) \equiv B(n) \pmod 3. \tag{4.16}$$

Since $p \geq 5$ is prime, so $3 \mid (1-p^2)$ and $\gcd\left(\frac{1-p^2}{3}, p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{3}r$. Thus (4.15) can be rewritten as

$$B\left(p^2n + \frac{p^2-1}{3} + pj\right) \equiv 0 \pmod 3, \tag{4.17}$$

where $p \nmid j$.

Now, $p_i \geq 5$ are primes such that $p_i \not\equiv 1 \pmod 3$. Since

$$p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{3} = p_1^2\left(p_2^2 \dots p_k^2 n + \frac{p_2^2 \dots p_k^2 - 1}{3}\right) + \frac{p_1^2 - 1}{3},$$

using (4.16) repeatedly, we obtain that

$$B\left(p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{3}\right) \equiv B(n) \pmod 3. \tag{4.18}$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$. Then (4.17) and (4.18) yield

$$B\left(p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}^2 - 1}{3} + p_1^2 \dots p_k^2 p_{k+1} j\right) \equiv 0 \pmod 3.$$

We complete the proof by using the fact that $ped_9(12n+5) \equiv 6B(n) \pmod{18}$. □

5. Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6. Putting $t = 3$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} ped_3(n)q^n = \frac{f_4 f_3}{f_1 f_{12}}. \tag{5.1}$$

We now recall the following identity from [3, (2.22)]:

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{5.2}$$

Employing (5.2) in (5.1) and extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} ped_3(2n+1)q^n = \frac{f_2 f_3 f_4^2 f_{24}}{f_1^2 f_6 f_8 f_{12}}. \tag{5.3}$$

Let

$$A(z) := \prod_{n=1}^{\infty} \frac{(1-q^{96n})^2}{(1-q^{192n})} = \frac{\eta^2(96z)}{\eta(192z)}.$$

Then using the binomial theorem we have

$$A^{2^k}(z) = \frac{\eta^{2^{k+1}}(96z)}{\eta^{2^k}(192z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_k(z)$ by

$$\begin{aligned} B_k(z) &:= \left(\frac{\eta(16z)\eta(24z)\eta^2(32z)\eta(192z)}{\eta^2(8z)\eta(48z)\eta(64z)\eta(96z)} \right) A^{2^k}(z) \\ &= \frac{\eta(16z)\eta(24z)\eta^2(32z)\eta^{2^{k+1}-1}(96z)}{\eta^2(8z)\eta(48z)\eta(64z)\eta^{2^k-1}(192z)}. \end{aligned}$$

Modulo 2^{k+1}, we have

$$B_k(z) \equiv \frac{\eta(16z)\eta(24z)\eta^2(32z)\eta(192z)}{\eta^2(8z)\eta(48z)\eta(64z)\eta(96z)} = q^3 \left(\frac{f_{16} f_{24} f_{32}^2 f_{192}}{f_8^2 f_{48} f_{64} f_{96}} \right). \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$B_k(z) \equiv \sum_{n=0}^{\infty} ped_3(2n+1)q^{8n+3} \pmod{2^{k+1}}. \tag{5.5}$$

Now, $B_k(z)$ is an eta-quotient with $N = 192$. We next prove that $B_k(z)$ is a modular form for all $k \geq 5$. We know that the cusps of $\Gamma_0(192)$ are represented by fractions $\frac{c}{d}$, where $d \mid 192$ and $\gcd(c,d) = 1$. By Theorem 2.4, we find that $B_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned} &\left(2^{k+1}-1\right)\frac{\gcd(d,96)^2}{96} + \frac{\gcd(d,16)^2}{16} + \frac{\gcd(d,24)^2}{24} + 2\frac{\gcd(d,32)^2}{32} - 3\frac{\gcd(d,8)^2}{8} \\ &- \frac{\gcd(d,48)^2}{48} - \frac{\gcd(d,64)^2}{64} - (2^k-1)\frac{\gcd(d,192)^2}{192} \geq 0. \end{aligned}$$

Equivalently, if and only if

$$L := (2^{k+2}-2)G_1 + 12G_2 + 8G_3 + 12G_4 - 48G_5 - 4F_6 - 3G_7 - 2^k + 1 \geq 0,$$

where $G_1 = \frac{\gcd(d,96)^2}{\gcd(d,192)^2}$, $G_2 = \frac{\gcd(d,16)^2}{\gcd(d,192)^2}$, $G_3 = \frac{\gcd(d,24)^2}{\gcd(d,192)^2}$, $G_4 = \frac{\gcd(d,32)^2}{\gcd(d,192)^2}$, $G_5 = \frac{\gcd(d,8)^2}{\gcd(d,192)^2}$, $G_6 = \frac{\gcd(d,48)^2}{\gcd(d,192)^2}$, and $G_7 = \frac{\gcd(d,64)^2}{\gcd(d,192)^2}$ respectively.

We now consider the following two cases according to the divisors of 192 and find the values of G_i for $i = 1, 2, \ldots, 7$. Let d be a divisor of $N = 192$.

Case (i). For $d|192$ and $d \neq 192$, we find that $G_1 = 1$, $1/144 \leq G_2 \leq 1, 1/64 \leq G_3 \leq 1$, $1/36 \leq G_4 \leq 1$, $1/576 \leq G_5 \leq 1$, $1/16 \leq G_6 \leq 1$, and $1/9 \leq G_7 \leq 1$. Hence,

$$L \geq 2^{k+2} - 2 + 1/12 + 1/8 + 1/3 - 48 - 4 - 3 - 2^k + 1 = 3 \cdot 2^k + 13/24 - 56.$$

Since $k \geq 5$, we have $L \geq 0$.
Case (ii). For $d = 192$, we find that $G_1 = 1/4$, $G_2 = 1/144$, $G_3 = 1/64$, $G_4 = 1/36$, $G_5 = 1/576$, $G_6 = 1/16$, and $G_7 = 1/9$. Hence, $L = 3/8$.
Hence, $B_k(z)$ is holomorphic at every cusp $\frac{c}{d}$ for all $k \geq 5$. Using Theorem 2.3, we find that the weight of $B_k(z)$ is equal to 2^{k-1}. Also, the associated character for $B_k(z)$ is given by $\chi_1(\bullet) = \left(\frac{4 \cdot 3^{3 \cdot 2^k + 2}}{\bullet}\right)$. This proves that $B_k(z) \in M_{2^{k-1}}(\Gamma_0(192), \chi_1)$ for all $k \geq 5$. Also, the Fourier coefficients of $B_k(z)$ are all integers. Hence by Theorem 2.5, the Fourier coefficients of $B_k(z)$ are almost always divisible by $m = 2^k$, for any positive integer k. Due to (5.5), the same holds for $ped_3(2n+1)$ and the theorem is established for $t = 3$.

We now prove Theorem 1.6 for the case $t = 5$. By (3.4), we have

$$\sum_{n=0}^{\infty} ped_5(2n+1)q^n = \frac{f_2^4 f_5 f_{20}}{f_1^3 f_4 f_{10}^2}. \tag{5.6}$$

Let

$$E(z) := \prod_{n=1}^{\infty} \frac{(1-q^{40n})^2}{(1-q^{80n})} = \frac{\eta^2(40z)}{\eta(80z)}.$$

Then using binomial theorem we have

$$E^{2^k}(z) = \frac{\eta^{2^{k+1}}(40z)}{\eta^{2^k}(80z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $F_k(z)$ by

$$F_k(z) := \left(\frac{\eta^4(8z)\eta(20z)\eta(80z)}{\eta^3(4z)\eta(16z)\eta^2(40z)}\right) E^{2^k}(z) = \frac{\eta^4(8z)\eta(20z)\eta^{2^{k+1}-2}(40z)}{\eta^3(4z)\eta(16z)\eta^{2^k-1}(80z)}. \tag{5.7}$$

Modulo 2^{k+1}, we have

$$F_k(z) \equiv \frac{\eta^4(8z)\eta(20z)\eta(80z)}{\eta^3(4z)\eta(16z)\eta^2(40z)} = q\frac{f_8^4 f_{20} f_{80}}{f_4^3 f_{16} f_{40}^2}. \tag{5.8}$$

Combining (5.6) and (5.8), we obtain

$$F_k(z) \equiv \sum_{n=0}^{\infty} ped_5(2n+1)q^{4n+1} \pmod{2^{k+1}}. \tag{5.9}$$

Now, $F_k(z)$ is an eta-quotient with $N = 80$. We next prove that $F_k(z)$ is a modular form for all $k \geq 5$. We know that the cusps of $\Gamma_0(80)$ are represented by fractions $\frac{c}{d}$, where $d \mid 80$ and $\gcd(c,d) = 1$. By Theorem 2.4, we find that $F_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$R := \left(2^{k+1} - 2\right)\frac{\gcd(d,40)^2}{40} + 4\frac{\gcd(d,8)^2}{8} + \frac{\gcd(d,20)^2}{20} - 3\frac{\gcd(d,4)^2}{4} - \frac{\gcd(d,16)^2}{16}$$

$$-(2^k-1)\frac{\gcd(d,80)^2}{80} \geq 0.$$

As shown in the case of $t=3$, we verify that $R \geq 0$ for all $d|80$ and for all $k \geq 5$. Hence, $F_k(z) \in M_{2^{k-1}}(\Gamma_0(80))$ for all $k \geq 5$. Now, using Serre's Theorem 2.5 as shown in the proof for $t=3$, we arrive at the desired result due to (5.9).

We next prove Theorem 1.6 for $t=9$. By (3.17), we have

$$\sum_{n=0}^{\infty} ped_9(2n+1)q^n = \frac{f_2^3 f_3}{f_1^3 f_6}. \tag{5.10}$$

As in the proof for $t=3$, let

$$G(z) := \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{6n})} = \frac{\eta^2(3z)}{\eta(6z)}.$$

Then using binomial theorem we have

$$G^{2^k}(z) = \frac{\eta^{2^{k+1}}(3z)}{\eta^{2^k}(6z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $H_k(z)$ by

$$H_k(z) := \left(\frac{\eta^3(2z)\eta(3z)}{\eta^3(z)\eta(6z)}\right) G^{2^k}(z) = \frac{\eta^3(2z)\eta^{2^{k+1}+1}(3z)}{\eta^3(z)\eta^{2^k+1}(6z)}. \tag{5.11}$$

Modulo 2^{k+1}, we have

$$H_k(z) \equiv \frac{\eta^3(2z)\eta(3z)}{\eta^3(z)\eta(6z)} = \frac{f_2^3 f_3}{f_1^3 f_6}. \tag{5.12}$$

Combining (5.10) and (5.12), we obtain

$$H_k(z) \equiv \sum_{n=0}^{\infty} ped_9(2n+1)q^{n+1} \pmod{2^{k+1}}. \tag{5.13}$$

Now, $H_k(z)$ is an eta-quotient with $N=18$. We next prove that $H_k(z)$ is a modular form for all $k \geq 3$. We know that the cusps of $\Gamma_0(18)$ are represented by fractions $\frac{c}{d}$, where $d \mid 18$ and $\gcd(c,d)=1$. By Theorem 2.4, we find that $H_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$S := \left(2^{k+1}+1\right)\frac{\gcd(d,3)^2}{3} + 3\frac{\gcd(d,2)^2}{2} - 3\frac{\gcd(d,1)^2}{1} - (2^k+1)\frac{\gcd(d,6)^2}{6} \geq 0.$$

As shown in the case of $t=3$, we verify that $S \geq 0$ for all $d|18$ and for all $k \geq 3$. Hence, $H_k(z) \in M_{2^{k-1}}(\Gamma_0(18))$ for all $k \geq 3$. Now, using Serre's Theorem 2.5 as shown in the proof for $t=3$, we arrive at the desired result due to (5.13). This completes the proof of the theorem. □

We now prove Theorem 1.7. We recall the following classical result due to Landau [8].

Lemma 5.1. *Let $r(n)$ and $s(n)$ be quadratic polynomials. Then*

$$\left(\sum_{n\in\mathbb{Z}} q^{r(n)}\right)\left(\sum_{n\in\mathbb{Z}} q^{s(n)}\right)$$

is lacunary modulo 2.

Proof of Theorem 1.7. We first recall the following identity [3, (7.2)]:

$$\sum_{n=0}^{\infty} ped_7(2n+1)q^n \equiv f_1 f_2 \pmod{2}. \tag{5.14}$$

We now recall Euler's pentagonal number theorem [1, Corollary 1.3.5]. For $|q|<1$,

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod{2}. \tag{5.15}$$

Now, magnifying (5.15) by $q \to q^2$, we have

$$f_2 \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)} \pmod{2}. \tag{5.16}$$

Finally combining (5.14), (5.15), and (5.16), and then applying Lemma 5.1 we complete the proof of the theorem. □

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI, ASSAM, INDIA, PIN- 781039
E-mail address: ajit18@iitg.ac.in