ON THE CONSTRUCTION OF A CLASS OF DISCRETE HILBERT-TYPE INEQUALITIES IN THE WHOLE PLANE

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ABSTRACT. In this work, we first construct a new discrete homogeneous kernel function which includes some classical kernels, and the range of parameters in the newly constructed kernel is limited to two special sets. Then, a new Hilbert-type inequality which is defined in the whole plane and involves the new kernel is established, and it is proved that the constant factor of the newly obtained inequality is the best possible. Additionally, assigning special values to the parameters, and employing the partial fraction expansion of trigonometric functions and some other techniques of real analysis, a variety of new and special discrete Hilbert-type inequalities in the whole plane are established at the end of the paper.

1. Introduction. In this work, it is supposed that $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$,

$$\Omega_1 := \left\{x : x = \frac{2j}{2m+1}, j, m \in \mathbb{Z}^+ \cup \{0\}\right\},$$

$$\Omega_2 := \left\{x : x = \frac{2j+1}{2m+1}, j, m \in \mathbb{Z}^+ \cup \{0\}\right\}.$$

Assume that $\boldsymbol{a} = \{a_m\}_{m=1}^{\infty} \in l_2$ and $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty} \in l_2$ are two sequences of real numbers. Then [1]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2, \tag{1}$$

where the constant factor π is the best possible.

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Inequality (1) was first put forward by German mathematician Hilbert in his lecture on integral equations, and therefore it is commonly named as Hilbert double series inequality. By introducing a pair of conjugate parameters (p, q), Hardy and Riesz [2] established the extended form of (1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}}\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{2}$$

where the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ is also the best possible.

For more than 100 years since Hilbert inequality was proposed, it has been a hot topic for researchers, especially in the past 30 years. In 1991, Chinese mathematician Hsu [3] proposed the weight coefficient method, and established a strengthened form of (1), that is,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \|\boldsymbol{a}\|_{\mu,2}\|\boldsymbol{b}\|_{\nu,2}, \tag{3}$$

where $\mu_m = \pi - \frac{\gamma_0}{\sqrt{m}}$, $\nu_n = \pi - \frac{\gamma_0}{\sqrt{n}}$ $(\gamma_0 = 1.1213^+)$.

After 1998, by optimizing the weight coefficient method and introducing special functions such as beta function, researchers established various extensions of (1) and (2) with parameters, such as [4]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\gamma}} < B\left(\gamma_1,\gamma_2\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4}$$

where $0 < \gamma_1, \gamma_2 \leq 2$, $\gamma_1 + \gamma_2 = \gamma$, $\mu_m = m^{p(1-\gamma_1)-1}$, $\nu_n = n^{q(1-\gamma_2)-1}$, and $B(u, v)$ is the beta function [5]. With regard to other extensions of (2), we refer to [6–12]. Besides the inequalities mentioned above, there are some other classical discrete inequalities similar to (2), such as [1, 7, 10]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{5}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log\frac{m}{n}}{m-n}a_m b_n < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q. \tag{6}$$

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Such inequalities like (4), (5) and (6) are commonly referred to as Hilbert-type inequality. In the past 20 years, by introducing parameters and special functions, and employing the techniques of real analysis, especially Euler-Maclaurin summation formula, researchers established a great many extensions, analogues, strengthened forms and reverses of these classical Hilbert-type inequalities (see [7,10,13–15]). In addition, with the construction of some new kernels and the consideration of integral and half-discrete forms, various new Hilbert-type inequalities [16–22] have also been established in the past twenty year. Such a large number of inequalities have already grown into a vast inequality system and are critical to the development of modern analysis.

It should be pointed out that discrete Hilbert-type inequalities are commonly constructed in the first quadrant. There are few results based on the whole plane in the literature, because when a kernel function is extended to the whole plane, there is no guarantee that it is nonnegative, monotonic, and integrable. In this work, we first construct a new discrete kernel function where the range of parameters is limited to two special sets, and then we deserve a class of Hilbert-type inequalities based on the whole plane. Furthermore, by employing the partial fraction expansions of trigonometric functions, some new Hilbert-type inequalities are established at the end of the paper.

2. Some Lemmas.

Lemma 2.1. *Let* $\theta \in \{1, -1\}$, $\alpha \in (0, 1)$ *and* $\lambda \in \mathbb{R}^{+} \cup \{0\}$. *Suppose that* α, β, γ *satisfy one of the following conditions:*
(i) $0 < \beta < \gamma$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = 1$;
(ii) $0 \leq \beta < \min\{1 - \alpha, \gamma\}$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = -1$;
(iii) $0 < \beta < \min\{1 - \alpha, \gamma\}$ *when* $\beta, \gamma \in \Omega_2$.
Let

$$k(t) := \frac{1 - \theta t^{\beta}}{(1 - \theta t^{\gamma}) \max\{1, |t|^{\lambda}\}}, \tag{7}$$

where $t \in \mathbb{R} \setminus \{1, -1\}$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = 1$, $t \in \mathbb{R}$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = -1$, $t \in \mathbb{R} \setminus \{1\}$ *when* $\beta, \gamma \in \Omega_2$ *and* $\theta = 1$, *and* $t \in \mathbb{R} \setminus \{-1\}$ *when* $\beta, \gamma \in \Omega_2$ *and* $\theta = -1$. *Let*

$$L(t) := k(t)|t|^{\alpha - 1}. \tag{8}$$

Then $L(t)$ decreases with t for $t \in \mathbb{R}^+$, and increases with t for $t \in \mathbb{R}^-$.

Proof. To begin with, we consider the case where $\beta, \gamma \in \Omega_1$ and $\theta = -1$, then $0 < \beta < \gamma$. Let

$$h(t) := \frac{1 - t^\beta}{1 - t^\gamma},$$

then we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{\beta-1}}{(1-t^\gamma)^2}\left[\gamma t^{\gamma-\beta} + (\beta - \gamma)t^\gamma - \beta\right] := \frac{h_1(t)t^{\beta-1}}{(1-t^\gamma)^2},$$

and it can be easily obtained that

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} = \gamma(\gamma - \beta)t^{\gamma-\beta-1}(1 - t^\beta). \tag{9}$$

It follows from $0 < \beta < \gamma$ that $\frac{\mathrm{d}h_1}{\mathrm{d}t} > 0$ when $t \in (0, 1)$, and $\frac{\mathrm{d}h_1}{\mathrm{d}t} < 0$ when $t \in (1, \infty)$. Thus, we have $h_1(t) \leq h_1(1) = 0$, and it implies that $\frac{\mathrm{d}h}{\mathrm{d}t} < 0$ $(t \neq 1)$. Set $h(1) := \frac{\beta}{\gamma}$, then $h(t)$ is continuous on $\mathbb{R}^+$, and decreases with t $(t \in \mathbb{R}^+)$ obviously. In addition, it can be easily proved that $\frac{t^{\alpha-1}}{\max\{1,t^\lambda\}}$ decreases with t $(t \in \mathbb{R}^+)$ owing to $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}^+ \cup \{0\}$. Therefore, $L(t)$ decreases with t for $t \in \mathbb{R}^+$. Furthermore, observing that $L(t)$ is an even function owing to $\beta, \gamma \in \Omega_1$, it can be obtained that $L(t)$ increases with t for $t \in \mathbb{R}^-$.

In what follows, we will consider the case where $\beta, \gamma \in \Omega_1$ and $\theta = 1$. Let

$$g(t) := \frac{t^{\alpha-1} + t^{\alpha+\beta-1}}{1 + t^\gamma} \ (t \in \mathbb{R}^+),$$

then we obtain

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{-t^{\alpha-2}}{(1+t^\gamma)^2}&\big[(1-\alpha) + (1-\alpha-\beta)t^\beta \\ &+ (1-\alpha+\gamma)t^\gamma + (1-\alpha-\beta+\gamma)t^{\beta+\gamma}\big].\end{aligned}$$

In view of that $\alpha \in (0, 1)$ and $\alpha+\beta < 1$, we have $\frac{\mathrm{d}g}{\mathrm{d}t} < 0$. It implies that $g(t)$ decreases with t $(t \in \mathbb{R}^+)$, and therefore $L(t) = \frac{g(t)}{\max\{1,t^\lambda\}}$ decreases with t $(t \in \mathbb{R}^+)$. Additionally, it can also be proved that $L(t)$ increases with t $(t \in \mathbb{R}^-)$ according to the symmetry of even function.

Lastly, we consider the case of $\beta,\gamma\in\Omega_2$. According to the discussions in the above two cases, it can be easily proved that $L(t)$ decreases with t for $t\in\mathbb{R}^+$, and increases with t for $t\in\mathbb{R}^-$, whether $\theta=1$ or $\theta=-1$. Lemma 2.1 is proved. $\square$

Lemma 2.2. *Let $\theta\in\{1,-1\}$, $\alpha\in(0,1)$ and $\lambda\in\mathbb{R}^+\cup\{0\}$. Suppose that $\alpha,\beta,\gamma,\lambda$ satisfy one of the following conditions:*
(i) $0<\beta<\gamma+\lambda-\alpha$ *when* $\beta,\gamma\in\Omega_1$ *and* $\theta=1$*;*
(ii) $0\le\beta<\gamma+\lambda-\alpha$ *when* $\beta,\gamma\in\Omega_1$ *and* $\theta=-1$*;*
(iii) $0<\beta<\gamma+\lambda-\alpha$ *when* $\beta,\gamma\in\Omega_2$.
Let $k(t)$ be defined by (7)*, and*

(10)
$$F(\alpha,\beta,\gamma,\lambda,\theta)=\begin{cases}2\sum_{s=0}^{\infty}\left(\frac{\theta^s}{\gamma s+\alpha}-\frac{\theta^{s+1}}{\gamma s+\gamma+\lambda-\alpha}\right)+\\ 2\sum_{s=0}^{\infty}\left(\frac{\theta^s}{\gamma s+\gamma+\lambda-\alpha-\beta}-\frac{\theta^{s+1}}{\gamma s+\alpha+\beta}\right), & \beta,\gamma\in\Omega_1,\\ \\ 2\sum_{s=0}^{\infty}\left(\frac{1}{2\gamma s+\alpha}-\frac{1}{2\gamma s+2\gamma+\lambda-\alpha}\right)+\\ 2\sum_{s=0}^{\infty}\left(\frac{1}{2\gamma s+\gamma+\lambda-\alpha-\beta}-\frac{1}{2\gamma s+\alpha+\beta+\gamma}\right), & \beta,\gamma\in\Omega_2.\end{cases}$$

Then

$$\int_{t\in\mathbb{R}}k(t)|t|^{\alpha-1}\mathrm{d}t=F(\alpha,\beta,\gamma,\lambda,\theta). \tag{11}$$

Proof. we first prove (11) under the condition where $\beta,\gamma\in\Omega_1$ and $\theta=-1$. In view of that $k(t)|t|^{\alpha-1}$ is an even function, we have

(12)
$$\begin{aligned}\int_{t\in\mathbb{R}}k(t)|t|^{\alpha-1}\mathrm{d}t&=2\int_0^1\frac{t^{\alpha-1}+t^{\alpha+\beta-1}}{1+t^\gamma}\mathrm{d}t+2\int_1^\infty\frac{t^{\alpha-\lambda-1}+t^{\alpha+\beta-\lambda-1}}{1+t^\gamma}\mathrm{d}t\\&=2\int_0^1\frac{t^{\alpha-1}+t^{\alpha+\beta-1}}{1+t^\gamma}\mathrm{d}t+2\int_0^1\frac{t^{\gamma+\lambda-\alpha-\beta-1}+t^{\gamma+\lambda-\alpha-1}}{1+t^\gamma}\mathrm{d}t\\&=2\int_0^1\frac{t^{\alpha-1}+t^{\gamma+\lambda-\alpha-1}+t^{\gamma+\lambda-\alpha-\beta-1}+t^{\alpha+\beta-1}}{1+t^\gamma}\mathrm{d}t\\&:=2(I_1+I_2+I_3+I_4).\end{aligned}$$

Expanding $\frac{1}{1+t^\gamma}$ $(t\in(0,1))$ into Maclaurin series, and using Lebesgue term-by-term integration theorem, we have

(13)
$$I_1=\int_0^1\sum_{s=0}^{\infty}(-1)^s t^{\gamma s+\alpha-1}\mathrm{d}t=\sum_{s=0}^{\infty}\int_0^1(-1)^s t^{\gamma s+\alpha-1}\mathrm{d}t=\sum_{s=0}^{\infty}\frac{(-1)^s}{\gamma s+\alpha}.$$

Similarly, since $\alpha+\beta<\gamma+\lambda$, we have

$$I_2=\sum_{s=0}^{\infty}\int_0^1(-1)^s t^{\gamma s+\gamma+\lambda-\alpha-1}\mathrm{d}t=\sum_{s=0}^{\infty}\frac{(-1)^s}{\gamma s+\gamma+\lambda-\alpha},\tag{14}$$

$$I_3=\sum_{s=0}^{\infty}\int_0^1(-1)^s t^{\gamma s+\gamma+\lambda-\alpha-\beta-1}\mathrm{d}t=\sum_{s=0}^{\infty}\frac{(-1)^s}{\gamma s+\gamma+\lambda-\alpha-\beta},\tag{15}$$

$$I_4=\sum_{s=0}^{\infty}\int_0^1(-1)^s t^{\gamma s+\alpha+\beta-1}\mathrm{d}t=\sum_{s=0}^{\infty}\frac{(-1)^s}{\gamma s+\alpha+\beta}.\tag{16}$$

Plugging (13), (14), (15) and (16) back into (12), we arrive at (11) under the condition where $\beta,\gamma\in\Omega_1$ and $\theta=-1$.

Secondly, we consider the case where $\beta,\gamma\in\Omega_1$ and $\theta=1$. Then

(17)
$$\begin{aligned}\int_{t\in\mathbb{R}}k(t)|t|^{\alpha-1}\mathrm{d}t&=2\int_0^1\frac{t^{\alpha-1}-t^{\alpha+\beta-1}}{1-t^{\gamma}}\mathrm{d}t+2\int_1^{\infty}\frac{t^{\alpha-\lambda-1}-t^{\alpha+\beta-\lambda-1}}{1-t^{\gamma}}\mathrm{d}t\\&=2\int_0^1\frac{t^{\alpha-1}-t^{\alpha+\beta-1}}{1-t^{\gamma}}\mathrm{d}t+2\int_0^1\frac{t^{\gamma+\lambda-\alpha-\beta-1}-t^{\gamma+\lambda-\alpha-1}}{1-t^{\gamma}}\mathrm{d}t\\&=2\int_0^1\frac{t^{\alpha-1}-t^{\gamma+\lambda-\alpha-1}}{1-t^{\gamma}}\mathrm{d}t+2\int_0^1\frac{t^{\gamma+\lambda-\alpha-\beta-1}-t^{\alpha+\beta-1}}{1-t^{\gamma}}\mathrm{d}t\\&:=2(J_1+J_2).\end{aligned}$$

Expand $\frac{1}{1-t^{\gamma}}$ $(t\in(0,1))$ into Maclaurin series, and use Lebesgue term-by-term integration theorem, then

(18)
$$J_1=\sum_{s=0}^{\infty}\int_0^1\left(t^{\gamma s+\alpha-1}-t^{\gamma s+\gamma+\lambda-\alpha-1}\right)\mathrm{d}t=\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+\alpha}-\frac{1}{\gamma s+\gamma+\lambda-\alpha}\right),$$

$$
\begin{aligned}
(19) \qquad J_2 &= \sum_{s=0}^{\infty} \int_0^1 \left(t^{\gamma s+\gamma+\lambda-\alpha-\beta-1} - t^{\gamma s+\alpha+\beta-1}\right) \mathrm{d}t \\
&= \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s+\gamma+\lambda-\alpha-\beta} - \frac{1}{\gamma s+\alpha+\beta} \right).
\end{aligned}
$$

Applying (18) and (19) to (17), we obtain (11) under the condition where $\beta, \gamma \in \Omega_1$ and $\theta = 1$.

At last, we consider the case where $\beta, \gamma \in \Omega_2$. Then

$$
\begin{aligned}
(20) \qquad & \int_{t\in\mathbb{R}} k(t)|t|^{\alpha-1}\mathrm{d}t \\
&= \int_{[-1,1]} \frac{1-\theta t^{\beta}}{1-\theta t^{\gamma}} |t|^{\alpha-1}\mathrm{d}t + \int_{\mathbb{R}\setminus[-,1]} \frac{1-\theta t^{\beta}}{1-\theta t^{\gamma}} |t|^{\alpha-\lambda-1}\mathrm{d}t \\
&= \int_0^1 \left(\frac{1-\theta t^{\beta}}{1-\theta t^{\gamma}} + \frac{1+\theta t^{\beta}}{1+\theta t^{\gamma}} \right) t^{\alpha-1}\mathrm{d}z \\
&+ \int_1^{\infty} \left(\frac{1-\theta t^{\beta}}{1-\theta t^{\gamma}} + \frac{1+\theta t^{\beta}}{1+\theta t^{\gamma}} \right) t^{\alpha-\lambda-1}\mathrm{d}t \\
&= 2\int_0^1 \frac{t^{\alpha-1} - t^{\alpha+\beta+\gamma-1}}{1-t^{2\gamma}}\mathrm{d}t + 2\int_1^{\infty} \frac{t^{\alpha-\lambda-1} - t^{\alpha+\beta+\gamma-\lambda-1}}{1-t^{2\gamma}}\mathrm{d}t \\
&= 2\int_0^1 \frac{t^{\alpha-1} - t^{\alpha+\beta+\gamma-1}}{1-t^{2\gamma}}\mathrm{d}t + 2\int_0^1 \frac{t^{\gamma+\lambda-\alpha-\beta-1} - t^{2\gamma+\lambda-\alpha-1}}{1-t^{2\gamma}}\mathrm{d}t \\
&= 2\int_0^1 \frac{t^{\alpha-1} - t^{2\gamma+\lambda-\alpha-1}}{1-t^{2\gamma}}\mathrm{d}t + 2\int_0^1 \frac{t^{\gamma+\lambda-\alpha-\beta-1} - t^{\alpha+\beta+\gamma-1}}{1-t^{2\gamma}}\mathrm{d}t \\
&:= 2(L_1 + L_2).
\end{aligned}
$$

By using the Maclaurin expansion of $\frac{1}{1-t^{2\gamma}}$ and Lebesgue term-by-term integration theorem, we have

$$
(21) \qquad L_1 = \sum_{s=0}^{\infty} \left(\frac{1}{2\gamma s+\alpha} - \frac{1}{2\gamma s+2\gamma+\lambda-\alpha} \right)
$$

$$
(22) \qquad L_2 = \sum_{s=0}^{\infty} \left(\frac{1}{2\gamma s+\gamma+\lambda-\alpha-\beta} - \frac{1}{2\gamma s+\alpha+\beta+\gamma} \right).
$$

Plugging (21) and (22) back into (20), we arrive at (11). Lemma 2.2 is proved. □

Lemma 2.3. *Let $\theta \in \{1,-1\}$, $\alpha, \tau \in (0,1)$ and $\lambda \in \mathbb{R}^+ \cup \{0\}$. Suppose that $\alpha, \tau, \beta, \gamma, \lambda$ satisfy $\alpha + \tau + \beta = \gamma + \lambda$ and one of the following conditions:*
(i) $0 < \beta < \gamma$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = 1$*;*
(ii) $0 \leq \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = -1$*;*
(iii) $0 < \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *when* $\beta, \gamma \in \Omega_2$.
Let $k(z)$ be defined by (7)*, and*

$$\hat{\boldsymbol{a}} := \{\hat{a}_m\}_{m\in\mathbb{Z}^0} := \left\{|m|^{\alpha-1-\frac{2}{pz}}\right\}_{m\in\mathbb{Z}^0},$$
$$\hat{\boldsymbol{b}} := \left\{\hat{b}_n\right\}_{n\in\mathbb{Z}^0} := \left\{|n|^{\tau-1-\frac{2}{qz}}\right\}_{n\in\mathbb{Z}^0},$$

where z is a positive integer which is large enough. Let

$$K(m,n) := \frac{m^\beta - \theta n^\beta}{(m^\gamma - \theta n^\gamma)\max\{|m|^\lambda, |n|^\lambda\}}, \tag{23}$$

where $m \neq \pm n$ when $\beta, \gamma \in \Omega_1$ and $\theta = 1$, $m \neq n$ when $\beta, \gamma \in \Omega_2$ and $\theta = 1$, and $m \neq -n$ when $\beta, \gamma \in \Omega_2$ and $\theta = -1$. Then

$$\begin{aligned}\hat{J} :&= \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} K(m,n)\,\hat{a}_m\hat{b}_n \\ &> z\left[\int_{[-1,1]} k(t)|t|^{\alpha-1+\frac{2}{qz}}\mathrm{d}t + \int_{\mathbb{R}\setminus[-1,1]} k(t)|t|^{\alpha-1-\frac{2}{pz}}\mathrm{d}t\right].\end{aligned} \tag{24}$$

Proof. Write

$$\begin{aligned}\hat{J} &= \sum_{n\in\mathbb{Z}^-}\sum_{m\in\mathbb{Z}^-} K(m,n)\hat{a}_m\hat{b}_n + \sum_{n\in\mathbb{Z}^-}\sum_{m\in\mathbb{Z}^+} K(m,n)\hat{a}_m\hat{b}_n \\ &\quad + \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^-} K(m,n)\hat{a}_m\hat{b}_n + \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+} K(m,n)\hat{a}_m\hat{b}_n \\ &:= \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4.\end{aligned}$$

We first consider the case where $m, n \in \mathbb{Z}^-$. Observing that $\alpha+\tau+\beta = \gamma+\lambda$, and using Lemma 2.1, it can be proved that

$$K(m,n)\hat{a}_m\hat{b}_n = |n|^{-2-\frac{2}{qz}} L\left(\frac{m}{n}\right)|m|^{-\frac{2}{pz}}$$

increases with m ($m \in \mathbb{Z}^-$) for a fixed n ($n \in \mathbb{Z}^-$). Similarly, it can also be proved that $K(m,n)\hat{a}_m\hat{b}_n$ increases with n ($n \in \mathbb{Z}^-$) for a fixed

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m $(m \in \mathbb{Z}^-)$. It follows therefore that

$$\sigma_1 > \int_{-\infty}^{-1} |y|^{\tau-1-\frac{2}{qz}} \int_{-\infty}^{-1} K(x,y) |x|^{\alpha-1-\frac{2}{pz}} \mathrm{d}x\mathrm{d}y := T_1.$$

Furthermore, we can obtain the following relations by the similar way:

$$\sigma_2 > \int_{-\infty}^{-1} |y|^{\tau-1-\frac{2}{qz}} \int_{1}^{\infty} K(x,y) |x|^{\alpha-1-\frac{2}{pz}} \mathrm{d}x\mathrm{d}y := T_2.$$

$$\sigma_3 > \int_{1}^{\infty} |y|^{\tau-1-\frac{2}{qz}} \int_{-\infty}^{-1} K(x,y) |x|^{\alpha-1-\frac{2}{pz}} \mathrm{d}x\mathrm{d}y := T_3.$$

$$\sigma_4 > \int_{1}^{\infty} |y|^{\tau-1-\frac{2}{qz}} \int_{1}^{\infty} K(x,y) |x|^{\alpha-1-\frac{2}{pz}} \mathrm{d}x\mathrm{d}y := T_4.$$

Setting $x = yt$, and using Fubini's theorem, we have

$$\begin{aligned}
(25) \qquad T_1 &= \int_{-\infty}^{-1} |y|^{-1-\frac{2}{z}} \int_{-\frac{1}{y}}^{\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t\mathrm{d}x \\
&= \int_{-\infty}^{-1} |y|^{-1-\frac{2}{z}} \int_{1}^{\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t\mathrm{d}x \\
&+ \int_{-\infty}^{-1} |y|^{-1-\frac{2}{z}} \int_{-\frac{1}{y}}^{1} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t\mathrm{d}x \\
&= \frac{z}{2} \int_{1}^{\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t \\
&+ \int_{0}^{1} k(t)|t|^{\alpha-1-\frac{2}{pz}} \int_{-\infty}^{-\frac{1}{u}} |y|^{-1-\frac{2}{z}} \mathrm{d}y\mathrm{d}t \\
&= \frac{z}{2} \left[\int_{0}^{1} k(t)|t|^{\alpha-1+\frac{2}{qz}} \mathrm{d}t + \int_{1}^{\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t \right]
\end{aligned}$$

Moreover, it can also be obtained that

$$(26) \qquad T_2 = T_3 = \frac{z}{2} \left[\int_{-1}^{0} k(t)|t|^{\alpha-1+\frac{2}{qz}} \mathrm{d}t + \int_{-\infty}^{-1} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t \right],$$

$$(27) \qquad T_4 = T_1 = \frac{z}{2} \left[\int_{0}^{1} k(t)|t|^{\alpha-1+\frac{2}{qz}} \mathrm{d}t + \int_{1}^{\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}} \mathrm{d}t \right].$$

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It follows from (26) and (27) that

$$\begin{aligned}\hat{J} &> T_1 + T_2 + T_3 + T_4 \\ &= z\left[\int_{[-1,1]} k(t)|t|^{\alpha-1+\frac{2}{qz}}\mathrm{d}t + \int_{\mathbb{R}\setminus[-1,1]} k(t)|t|^{\alpha-1-\frac{2}{pz}}\mathrm{d}t\right].\end{aligned}$$

Lemma 2.3 is proved. □

Lemma 2.4. *Let $a, b > 0$ and $a + b = \kappa$. Let $\Phi(t) = \cot t$ and $\Psi(t) = \csc t$. Then*

$$\sum_{s=0}^{\infty}\left(\frac{1}{\kappa s + a} - \frac{1}{\kappa s + b}\right) = \frac{\pi}{\kappa}\Phi\left(\frac{a\pi}{\kappa}\right), \tag{28}$$

$$\sum_{s=0}^{\infty}\left(\frac{(-1)^s}{\kappa s + a} + \frac{(-1)^s}{\kappa s + b}\right) = \frac{\pi}{\kappa}\Psi\left(\frac{a\pi}{\kappa}\right). \tag{29}$$

Proof. Expand $\Phi(t) = \cot t$ $(0 < t < \pi)$ into a partial fraction as follows:

$$\Phi(t) = \frac{1}{t} + \sum_{s=1}^{\infty}\left(\frac{1}{t + s\pi} + \frac{1}{t - s\pi}\right).$$

Let $t = \frac{a\pi}{\kappa}$, then we have

$$\begin{aligned}\Phi\left(\frac{a\pi}{\kappa}\right) &= \frac{\kappa}{\pi}\left[\frac{1}{a} + \sum_{s=1}^{\infty}\left(\frac{1}{\kappa s + a} + \frac{1}{a - \kappa s}\right)\right] \\ &= \frac{\kappa}{\pi}\lim_{n\to\infty}\left(\sum_{s=0}^{n}\frac{1}{\kappa s + a} + \sum_{s=1}^{n}\frac{1}{a - \kappa s}\right) \\ &= \frac{\kappa}{\pi}\lim_{n\to\infty}\left(\sum_{s=0}^{n}\frac{1}{\kappa s + a} - \sum_{s=0}^{n-1}\frac{1}{\kappa s + b}\right) \\ &= \frac{\kappa}{\pi}\lim_{n\to\infty}\left[\sum_{s=0}^{n}\left(\frac{1}{\kappa s + a} - \frac{1}{\kappa s + b}\right) + \frac{1}{\kappa n + b}\right] \\ &= \frac{\kappa}{\pi}\sum_{s=0}^{\infty}\left(\frac{1}{\kappa s + a} - \frac{1}{\kappa s + b}\right).\end{aligned} \tag{30}$$

Inequality (28) follows form (30) naturally. Furthermore, observing that

$$2\Psi(t) = \Phi\left(\frac{t}{2}\right) + \Phi\left(\frac{\pi}{2} - \frac{t}{2}\right) \ (0 < t < \pi),$$

and using (28), we have

$$\begin{aligned}
\Psi\left(\frac{a\pi}{\kappa}\right) &= \frac{1}{2}\Phi\left(\frac{a\pi}{2\kappa}\right) + \frac{1}{2}\Phi\left(\frac{(\kappa-a)\pi}{2\kappa}\right) \\
&= \frac{\kappa}{\pi}\sum_{s=0}^{\infty}\left(\frac{1}{2\kappa s + a} - \frac{1}{2\kappa s + 2\kappa - a}\right) \\
&+ \frac{\kappa}{\pi}\sum_{s=0}^{\infty}\left(\frac{1}{2\kappa s + \kappa - a} - \frac{1}{2\kappa s + \kappa + a}\right) \\
&= \frac{\kappa}{\pi}\sum_{s=0}^{\infty}\left[\left(\frac{1}{2\kappa s + a} - \frac{1}{2\kappa s + \kappa + a}\right)\right. \\
&+ \left.\left(\frac{1}{2\kappa s + b} - \frac{1}{2\kappa s + \kappa + b}\right)\right] \\
&= \frac{\kappa}{\pi}\sum_{s=0}^{\infty}(-1)^s\left(\frac{1}{\kappa s + a} + \frac{1}{\kappa s + b}\right).
\end{aligned}$$

It follows therefore that (29) holds true, and Lemma 2.4 is proved. □

3. Main Results.

Theorem 3.1. *Let $\theta \in \{1, -1\}$, $\alpha, \tau \in (0, 1)$ and $\lambda \in \mathbb{R}^+ \cup \{0\}$. Suppose that $\alpha, \tau, \beta, \gamma, \lambda$ satisfy $\alpha + \tau + \beta = \gamma + \lambda$ and one of the following conditions:*
(i) $0 < \beta < \gamma$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = 1$;
(ii) $0 \le \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *when* $\beta, \gamma \in \Omega_1$ *and* $\theta = -1$;
(iii) $0 < \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *when* $\beta, \gamma \in \Omega_2$.
Let $a_m, b_n > 0$ with $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ and $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, where $\mu_m = |m|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. Let $F(\alpha, \beta, \gamma, \lambda, \theta)$ and $K(m, n)$ be defined by (10) and (23), respectively. Then

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} K(m,n)a_m b_n < F(\alpha,\beta,\gamma,\lambda,\theta)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{31}$$

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where the constant factor $F(\alpha,\beta,\gamma,\lambda,\theta)$ in (31) is the best possible.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
(32)\qquad & \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} K(m,n)a_m b_n \\
&= \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \left\{[K(m,n)]^{\frac{1}{p}}\,|n|^{\frac{\tau-1}{p}}\,|m|^{\frac{1-\alpha}{q}} a_m\right\} \\
&\times \left\{[K(m,m)]^{\frac{1}{q}}\,|m|^{\frac{\alpha-1}{q}}\,|n|^{\frac{1-\tau}{p}} b_n\right\} \\
&\le \left[\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} K(m,n)|n|^{\tau-1}|m|^{\frac{p(1-\alpha)}{q}} a_m^p\right]^{\frac{1}{p}} \\
&\times \left[\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} K(m,n)|m|^{\alpha-1}|n|^{\frac{q(1-\tau)}{p}} b_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{m\in\mathbb{Z}^0} \phi(m)|m|^{\frac{p(1-\alpha)}{q}} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n\in\mathbb{Z}^0} \psi(n)|n|^{\frac{q(1-\tau)}{p}} b_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$\phi(m) = \sum_{n\in\mathbb{Z}^0} K(m,n)|n|^{\tau-1},\ \psi(n) = \sum_{m\in\mathbb{Z}^0} K(m,n)|m|^{\alpha-1}.$$

It is easy to find that

$$K(m,n)|m|^{\alpha-1} = |n|^{\beta-\gamma-\lambda+\alpha-1} L\left(\frac{m}{n}\right),$$

where $L(t)$ is defined by (8). By using Lemma 2.1, it follows that $L\left(\frac{m}{n}\right)$ decreases with $m(m\in\mathbb{Z}^+)$ and increases with $m(m\in\mathbb{Z}^-)$ for a fixed n, whether $n>0$ or $n<0$. Therefore, setting $\frac{z}{n}=t$, we have

$$(33)\quad \psi(n) < \int_{z\in\mathbb{R}} K(z,n)|z|^{\alpha-1}\mathrm{d}z = |n|^{\alpha+\beta-\gamma-\lambda}\int_{z\in\mathbb{R}} k(t)|t|^{\alpha-1}\mathrm{d}t.$$

Observing that $\alpha+\beta+\tau=\gamma+\lambda$, we have $\beta<\gamma+\lambda-\alpha$. Therefore, applying (11) to (33), we have

$$(34)\qquad \psi(n) < |n|^{-\tau} F(\alpha,\beta,\gamma,\lambda,\theta).$$

Similarly, we have

$$\phi(m) < |m|^{-\alpha} F(\tau, \beta, \gamma, \lambda, \theta). \tag{35}$$

Since $\tau = \gamma + \lambda - \alpha - \beta$, it can be proved that

$$F(\alpha, \beta, \gamma, \lambda, \theta) = F(\tau, \beta, \gamma, \lambda, \theta).$$

Thus, plugging (34) and (35) back into (32), we arrive at (31).

What follows is the proof of the optimality of the constant factor in (31). In fact, assume that there exists a real number C such that

$$0 < C \le F(\alpha, \beta, \gamma, \lambda, \theta), \tag{36}$$

and

$$\sum_{n\in\mathbb{Z}^0} \sum_{m\in\mathbb{Z}^0} K(m,n) a_m b_n < C \|\boldsymbol{a}\|_{p,\mu} \|\boldsymbol{b}\|_{q,\nu}. \tag{37}$$

Replace a_m and b_n in (37) by $\hat{a}_m$ and $\hat{b}_n$ which are defined in Lemma 2.3, respectively, then

$$\begin{aligned} \hat{J} &< C\|\hat{\boldsymbol{a}}\|_{p,\mu}\|\hat{\boldsymbol{b}}\|_{q,\nu} = C\left(\sum_{m\in\mathbb{Z}^0} |m|^{-1-\frac{2}{z}}\right)^{\frac{1}{p}} \left(\sum_{n\in\mathbb{Z}^0} |n|^{-1-\frac{2}{z}}\right)^{\frac{1}{q}} \\ &= C \sum_{m\in\mathbb{Z}^0} |m|^{-1-\frac{2}{z}} = 2C\left(1 + \sum_{m=2}^{\infty} m^{-1-\frac{2}{z}}\right) \\ &< 2C\left(1 + \int_1^{\infty} t^{-1-\frac{2}{z}} \mathrm{d}t\right) = 2C + Cz. \end{aligned} \tag{38}$$

Combine (24) and (38), and use Fatou's lemma and (11), then we have

$$\begin{aligned} F(\alpha, \beta, \gamma, \lambda, \theta) &= \int_{t\in\mathbb{R}} k(t)|t|^{\alpha-1}\mathrm{d}t \\ &= \int_{[-1,1]} \varliminf_{z\to\infty} k(t)|t|^{\alpha-1+\frac{2}{qz}}\mathrm{d}t + \int_{\mathbb{R}\setminus[-1,1]} \varliminf_{z\to\infty} k(t)|t|^{\alpha-1-\frac{2}{pz}}\mathrm{d}t \\ &\leqslant \varliminf_{z\to\infty}\left[\int_{[-1,1]} k(t)|t|^{\alpha-1+\frac{2}{qz}}\mathrm{d}t + \int_{\mathbb{R}\setminus[-1,1]} k(t)|t|^{\alpha-1-\frac{2}{pz}}\mathrm{d}t\right] \\ &\leqslant \varliminf_{z\to\infty}\left(\frac{2C}{z} + C\right) = C. \end{aligned} \tag{39}$$

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It follows from (36) and (39) that

$$C = F(\alpha, \beta, \gamma, \lambda, \theta).$$

Hence, the constant factor $F(\alpha, \beta, \gamma, \lambda, \theta)$ in (31) is the best possible, and therefore Theorem 3.1 is proved. □

Remark 3.2. *Theorem* 3.1 *implies a Hardy-type inequality as follows:*

(40)
$$I := \sum_{n\in\mathbb{Z}^0} |n|^{p\tau-1} \left[\sum_{m\in\mathbb{Z}^0} K\left(m,n\right) a_m \right]^p < \left[F(\alpha, \beta, \gamma, \lambda, \theta)\right]^p \|\boldsymbol{a}\|_{p,\mu}^p,$$

where the constant factor $[F(\alpha, \beta, \gamma, \lambda, \theta)]^p$ *in* (40) *is the best possible. In fact, set* $\boldsymbol{y} := \{y_n\}_{n\in\mathbb{Z}^0}$*,*

$$y_n := |n|^{p\tau-1} \left[\sum_{m\in\mathbb{Z}^0} K\left(m,n\right) a_m \right]^{p-1},$$

and use (31)*, we have*

(41)
$$\begin{aligned} I &= \sum_{n\in\mathbb{Z}^0} \sum_{m\in\mathbb{Z}^0} K\left(m,n\right) a_m y_n \\ &< F(\alpha, \beta, \gamma, \lambda, \theta) \|\boldsymbol{a}\|_{p,\mu} \|\boldsymbol{y}\|_{q,\nu} \\ &= F(\alpha, \beta, \gamma, \lambda, \theta) \|\boldsymbol{a}\|_{p,\mu} I^{1/q}. \end{aligned}$$

Inequality (40) *follows from* (41) *naturally.*

4. Some Corollaries. Let $\beta, \gamma \in \Omega_1$, $\theta = 1$ and $\lambda = 0$ in Theorem 3.1, and use Lemma (2.4), then we obtain the following corollary.

Corollary 4.1. *Let* $\alpha, \tau \in (0, 1)$*,* $\beta, \gamma \in \Omega_1$*. Suppose that* $\alpha, \tau, \beta, \gamma$ *satisfy* $0 < \beta < \gamma$ *and* $\alpha + \tau + \beta = \gamma$*. Let* $\Phi(t) = \cot t$*,* $a_m, b_n > 0$ *with* $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$*, where* $\mu_m =$

$|m|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. *Then*

(42)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{m^\beta - n^\beta}{m^\gamma - n^\gamma}a_m b_n < \frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{\tau\pi}{\gamma}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $\frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{\tau\pi}{\gamma}\right)\right]$ *in* (42) *is the best possible.*

Let $\gamma = 2^s\beta$ $(s \in \mathbb{Z}^+)$ in (42), we have $\alpha + \tau = (2^s - 1)\beta$, where $\alpha, \tau \in (0, 1)$ and $\beta \in \Omega_1$, and (42) reduces to

(43)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\prod_{j=0}^{s-1}(m^{2^j\beta}+n^{2^j\beta})} < \frac{\pi}{2^{s-1}\beta}\left[\Phi\left(\frac{\alpha\pi}{2^s\beta}\right)+\Phi\left(\frac{\tau\pi}{2^s\beta}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$.

Let $s = 1$ in (43), then we have $\alpha + \tau = \beta$, where $\alpha, \tau \in (0, 1)$ and $\beta \in \Omega_1$. Since

$$\Phi\left(\frac{\alpha\pi}{2\beta}\right)+\Phi\left(\frac{\tau\pi}{2\beta}\right) = 2\Psi\left(\frac{\alpha\pi}{\beta}\right),$$

where $\Psi(t) = \csc t$, inequality (43) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{m^\beta + n^\beta} < \frac{2\pi}{\beta}\Psi\left(\frac{\alpha\pi}{\beta}\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{44}$$

Let $s = 2$, $\alpha = \beta$ and $\tau = 2\beta$ in (43), then $0 < \beta < \frac{1}{2}$ $(\beta \in \Omega_1)$, and

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{(m^\beta + n^\beta)(m^{2\beta}+n^{2\beta})} < \frac{\pi}{2\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{45}$$

where $\mu_m = |m|^{p(1-\beta)-1}$ and $\nu_n = |n|^{q(1-2\beta)-1}$.

Let $\gamma = (2s + 1)\beta$ $(s \in \mathbb{Z}^+)$ in (42), then we have $\alpha + \tau = 2s\beta$, where

$\alpha, \tau \in (0,1)$ and $\beta \in \Omega_1$, and (42) reduces to

(46)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\sum_{j=0}^{2s} m^{j\beta}n^{(2s-j)\beta}} < \frac{2\pi}{(2s+1)\beta}\left[\Phi\left(\frac{\alpha\pi}{(2s+1)\beta}\right)+\Phi\left(\frac{\tau\pi}{(2s+1)\beta}\right)\right] \times \|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\tau)-1}$.

Let $s = 1$ and $\alpha = \tau = \beta$ in (46), then we have $0 < \beta < 1$ ($\beta \in \Omega_1$), and

(47)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{m^{2\beta}+m^\beta n^\beta + n^{2\beta}} < \frac{4\sqrt{3}\pi}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\beta)-1}$ and $\nu_n = |n|^{q(1-\beta)-1}$.

Let $\beta, \gamma \in \Omega_1$, $\theta = 1$ and $\lambda = \gamma$ in Theorem 3.1, then $\alpha + \tau + \beta = 2\gamma$.

If $0 < \alpha < \alpha + \beta < \gamma$, then it follows from Lemma (2.4) that

$$\begin{aligned}
F(\alpha,\beta,\gamma,\lambda,\theta) &= 2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+\alpha}-\frac{1}{\gamma s+2\gamma-\alpha}\right)\\
&+2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+2\gamma-\alpha-\beta}-\frac{1}{\gamma s+\alpha+\beta}\right)\\
&=2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+\alpha}-\frac{1}{\gamma s+\gamma-\alpha}\right)\\
&+2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+\gamma-\alpha-\beta}-\frac{1}{\gamma s+\alpha+\beta}\right)\\
&+\frac{2}{\gamma-\alpha}-\frac{2}{\gamma-\alpha-\beta}\\
&=\frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{(\gamma-\alpha-\beta)\pi}{\gamma}\right)+P_0\right]\\
&=\frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{\tau\pi}{\gamma}\right)+P_0\right],
\end{aligned}$$

where
$$P_0 = \frac{\beta\gamma}{\pi(\gamma-\alpha)(\gamma-\tau)}.$$

If $0 < \alpha < \alpha + \beta = \gamma$, then $\tau = \gamma$, and

$$\begin{aligned}
F(\alpha, \beta, \gamma, \lambda, \theta) &= 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + \alpha} - \frac{1}{\gamma s + 2\gamma - \alpha} \right) \\
&= 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + \alpha} - \frac{1}{\gamma s + \gamma - \alpha} \right) + \frac{2}{\gamma - \alpha} \\
&= \frac{2}{\beta} - \frac{2\pi}{\gamma} \Phi \left(\frac{\beta \pi}{\gamma} \right).
\end{aligned}$$

If $0 < \alpha < \gamma < \alpha + \beta < 2\gamma$, then we have

$$\begin{aligned}
F(\alpha, \beta, \gamma, \lambda, \theta) &= 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + \alpha} - \frac{1}{\gamma s + \gamma - \alpha} \right) \\
&+ 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + 2\gamma - \alpha - \beta} - \frac{1}{\gamma s + \alpha + \beta - \gamma} \right) \\
&+ \frac{2}{\gamma - \alpha} - \frac{2}{\gamma - \alpha - \beta} \\
&= \frac{2\pi}{\gamma} \left[\Phi \left(\frac{\alpha \pi}{\gamma} \right) + \Phi \left(\frac{(2\gamma - \alpha - \beta)\pi}{\gamma} \right) + P_0 \right] \\
&= \frac{2\pi}{\gamma} \left[\Phi \left(\frac{\alpha \pi}{\gamma} \right) + \Phi \left(\frac{\tau \pi}{\gamma} \right) + P_0 \right].
\end{aligned}$$

If $0 < \alpha = \gamma < \alpha + \beta < 2\gamma$, then we have

$$\begin{aligned}
F(\alpha, \beta, \gamma, \lambda, \theta) &= 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + \gamma - \beta} - \frac{1}{\gamma s + \gamma + \beta} \right) \\
&= 2 \sum_{s=0}^{\infty} \left(\frac{1}{\gamma s + \gamma - \beta} - \frac{1}{\gamma s + \beta} \right) + \frac{2}{\beta} \\
&= \frac{2}{\beta} - \frac{2\pi}{\gamma} \Phi \left(\frac{\beta \pi}{\gamma} \right).
\end{aligned}$$

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If $0<\gamma<\alpha<\alpha+\beta<2\gamma$, then we have

$$\begin{aligned}
F(\alpha,\beta,\gamma,\lambda,\theta) &= 2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+\alpha-\gamma}-\frac{1}{\gamma s+2\gamma-\alpha}\right)\\
&+2\sum_{s=0}^{\infty}\left(\frac{1}{\gamma s+2\gamma-\alpha-\beta}-\frac{1}{\gamma s+\alpha+\beta-\gamma}\right)\\
&+\frac{2}{\gamma-\alpha}-\frac{2}{\gamma-\alpha-\beta}\\
&=\frac{2\pi}{\gamma}\left[\Phi\left(\frac{(\alpha-\gamma)\pi}{\gamma}\right)+\Phi\left(\frac{(2\gamma-\alpha-\beta)\pi}{\gamma}\right)+P_0\right]\\
&=\frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{\tau\pi}{\gamma}\right)+P_0\right].
\end{aligned}$$

Let

$$F_0(\alpha,\beta,\gamma,\tau)=\begin{cases}\frac{2}{\beta}-\frac{2\pi}{\gamma}\Phi\left(\frac{\beta\pi}{\gamma}\right), & \gamma=\alpha \text{ or } \gamma=\tau,\\[2ex] \frac{2\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{\gamma}\right)+\Phi\left(\frac{\tau\pi}{\gamma}\right)+P_0\right], & \gamma\neq\alpha \text{ and } \gamma\neq\tau.\end{cases}$$

Then the following corollary holds ture.

Corollary 4.2. *Let $\alpha,\tau\in(0,1)$, $\beta,\gamma\in\Omega_1$. Suppose that α,τ,β,γ satisfy $0<\beta<\gamma$ and $\alpha+\tau+\beta=2\gamma$. Let $a_m,b_n>0$ with $\boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where $\mu_m=|m|^{p(1-\alpha)-1}$, $\nu_n=|n|^{q(1-\tau)-1}$. Then*

(48)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{m^\beta-n^\beta}{(m^\gamma-n^\gamma)\max\{m^\gamma,n^\gamma\}}a_mb_n<F_0(\alpha,\beta,\gamma,\tau)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $F_0(\alpha,\beta,\gamma,\lambda)$ in (48) is the best possible.

Let $\gamma=\alpha=2^s\beta$, $\tau=(2^s-1)\beta$ $(s\in\mathbb{Z}^+)$ in (48), then we have

$0<\beta<\frac{1}{2^s}$ $(\beta\in\Omega_1)$, and (48) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{2^s\beta},n^{2^s\beta}\}\prod_{j=0}^{s-1}(m^{2^j\beta}+n^{2^j\beta})} < \left[\frac{2}{\beta}-\frac{\pi}{2^{s-1}\beta}\Phi\left(\frac{\pi}{2^s}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{49}$$

where $\mu_m=|m|^{p(1-2^s\beta)-1}$ and $\nu_n=|n|^{q[1-(2^s-1)\beta]-1}$.

Let $s=1$ in (49), then $0<\beta<\frac{1}{2}$ $(\beta\in\Omega_1)$, and (49) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{2\beta},n^{2\beta}\}(m^\beta+n^\beta)} < \frac{2}{\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{50}$$

where $\mu_m=|m|^{p(1-2\beta)-1}$ and $\nu_n=|n|^{q(1-\beta)-1}$.

Let $\gamma=\tau=2^s\beta,\ \alpha=(2^s-1)\beta\ \ (s\in\mathbb{Z}^+)$ in (48), then we can get (49) with $\mu_m=|m|^{p[1-(2^s-1)\beta]-1}$ and $\nu_n=|n|^{q(1-2^s\beta)-1}$.

Let $\gamma=2^s\beta,\ \alpha=(2^s+1)\beta,\ \tau=(2^s-2)\beta\ \ (s\in\mathbb{Z}^+\setminus\{1\})$ in (48), then we have $0<\beta<\frac{1}{2^s+1}$ $(\beta\in\Omega_1)$, and (48) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{2^s\beta},n^{2^s\beta}\}\prod_{j=0}^{s-1}(m^{2^j\beta}+n^{2^j\beta})} < \frac{\pi}{2^{s-1}\beta}\left[\Phi\left(\frac{\pi}{2^s}\right)-\Phi\left(\frac{\pi}{2^{s-1}}\right)-\frac{2^{s-1}}{\pi}\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{51}$$

where $\mu_m=|m|^{p[1-(2^s+1)\beta]-1}$ and $\nu_n=|n|^{q[1-(2^s-2)\beta]-1}$.

Let $s=2$ in (51), then $0<\beta<\frac{1}{5}$ $(\beta\in\Omega_1)$, and (51) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{4\beta},n^{4\beta}\}(m^\beta+n^\beta)(m^{2\beta}+n^{2\beta})} < \frac{\pi-2}{2\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p(1-5\beta)-1}$ and $\nu_n=|n|^{q(1-2\beta)-1}$.

Let $\gamma=\alpha=(2s+1)\beta,\ \tau=2s\beta\ (s\in\mathbb{Z}^+)$ in (48), then we have

$0<\beta<\frac{1}{2s+1}$ $(\beta\in\Omega_1)$, and (48) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_mb_n}{\max\left\{m^{(2s+1)\beta},n^{(2s+1)\beta}\right\}\sum_{j=0}^{2s}m^{j\beta}n^{(2s-j)\beta}} < \left[\frac{2}{\beta}-\frac{2\pi}{(2s+1)\beta}\Phi\left(\frac{\pi}{2s+1}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{52}$$

where $\mu_m=|m|^{p[1-(2s+1)\beta]-1}$ and $\nu_n=|n|^{q(1-2s\beta)-1}$.

Let $s=1$ in (52), then $0<\beta<\frac{1}{3}$ $(\beta\in\Omega_1)$, and (52) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_mb_n}{\max\left\{m^{3\beta},n^{3\beta}\right\}(m^{2\beta}+m^\beta n^\beta+n^{2\beta})}<\frac{18-2\sqrt{3}\pi}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p(1-3\beta)-1}$ and $\nu_n=|n|^{q(1-2\beta)-1}$.

Let $\gamma=(2s+1)\beta,\ \alpha=s\beta,\ \tau=(3s+1)\beta\ \ (s\in\mathbb{Z}^+)$ in (48), then we have $0<\beta<\frac{1}{3s+1}$ $(\beta\in\Omega_1)$, and (48) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_mb_n}{\max\left\{m^{(2s+1)\beta},n^{(2s+1)\beta}\right\}\sum_{j=0}^{2s}m^{j\beta}n^{(2s-j)\beta}} < \left[\frac{4\pi}{(2s+1)\beta}\Phi\left(\frac{s\pi}{2s+1}\right)-\frac{2}{s(s+1)\beta}\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{53}$$

where $\mu_m=|m|^{p(1-s\beta)-1}$ and $\nu_n=|n|^{q[1-(3s+1)\beta]-1}$.

Let $s=1$ in (53), then $0<\beta<\frac{1}{4}$ $(\beta\in\Omega_1)$, and (53) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_mb_n}{\max\left\{m^{3\beta},n^{3\beta}\right\}(m^{2\beta}+m^\beta n^\beta+n^{2\beta})}<\frac{4\sqrt{3}\pi-9}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p(1-\beta)-1}$ and $\nu_n=|n|^{q(1-4\beta)-1}$.

Letting $\beta,\gamma\in\Omega_1$, $\theta=-1$ and $\lambda=0$ in Theorem 3.1, and using Lemma 2.4, we obtain the following corollary.

Corollary 4.3. *Let $\alpha,\tau\in(0,1)$, $\beta,\gamma\in\Omega_1$. Suppose that α,τ,β,γ satisfy $0\le\beta<\min\{1-\alpha,1-\tau,\gamma\}$ and $\alpha+\tau+\beta=\gamma$. Let $\Psi(t)=\csc t$, $a_m,b_n>0$ with $\boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where*

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$\mu_m = |m|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\tau)-1}$. Then

(54)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{m^\beta+n^\beta}{m^\gamma+n^\gamma}a_m b_n < \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $\frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)\right]$ in (54) is the best possible.

Let $\beta = 0$ in (54), then (54) is transformed into (44).

Let $\gamma = (2s+1)\beta$ $(s \in \mathbb{Z}^+)$ in (54), then we have $\alpha + \tau = 2s\beta$, and (54) reduces to

$$(55)\quad \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\sum_{j=0}^{2s}(-1)^j m^{j\beta}n^{(2s-j)\beta}} < \frac{2\pi}{(2s+1)\beta} \times\left[\Psi\left(\frac{\alpha\pi}{(2s+1)\beta}\right)+\Psi\left(\frac{\tau\pi}{(2s+1)\beta}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\tau)-1}$.

Let $s = 1$, $\alpha = \tau = \beta$ in (55), then $0 < \beta < \frac{1}{2}$ $(\beta \in \Omega_1)$, and

$$(56)\qquad \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{m^{2\beta}-m^\beta n^\beta+n^{2\beta}} < \frac{8\sqrt{3}\pi}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\beta)-1}$ and $\nu_n = |n|^{q(1-\beta)-1}$.

Let $\beta, \gamma \in \Omega_1$, $\theta = -1$ and $\lambda = 2\gamma$ in Theorem 3.1, then $\alpha+\tau+\beta = 3\gamma$.

If $0 < \alpha < \alpha + \beta < \gamma$, then it follows from Lemma 2.4 that

$$\begin{aligned}
F(\alpha,\beta,\gamma,\lambda,\theta) &= 2\sum_{s=0}^{\infty}\left(\frac{(-1)^s}{\gamma s+\alpha}+\frac{(-1)^s}{\gamma s+3\gamma-\alpha}\right)\\
&\quad+2\sum_{s=0}^{\infty}\left(\frac{(-1)^s}{\gamma s+3\gamma-\alpha-\beta}+\frac{(-1)^s}{\gamma s+\alpha+\beta}\right)\\
&=2\sum_{s=0}^{\infty}\left(\frac{(-1)^s}{\gamma s+\alpha}+\frac{(-1)^s}{\gamma s+\gamma-\alpha}\right)\\
&\quad+2\sum_{s=0}^{\infty}\left(\frac{(-1)^s}{\gamma s+\gamma-\alpha-\beta}+\frac{(-1)^s}{\gamma s+\alpha+\beta}\right)\\
&=\frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)-Q_0\right].
\end{aligned}$$

where $Q_0 = Q_1 + Q_2$, and

$$Q_1 := \frac{\gamma^2}{\pi(\gamma-\alpha)(2\gamma-\alpha)},\ Q_2 := \frac{\gamma^2}{\pi(\gamma-\tau)(2\gamma-\tau)}.$$

If $0 < \alpha < \alpha + \beta = \gamma$, then $\tau = 2\gamma$, and

$$F(\alpha,\beta,\gamma,\lambda,\theta) = \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\frac{1}{\pi}-Q_1\right].$$

If $0 < \alpha < \gamma < \alpha + \beta < 2\gamma$, then

$$F(\alpha,\beta,\gamma,\lambda,\theta) = \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)-Q_0\right].$$

If $0 < \alpha = \gamma < \alpha + \beta < 2\gamma$, then

$$F(\alpha,\beta,\gamma,\lambda,\theta) = \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\tau\pi}{\gamma}\right)+\frac{1}{\pi}-Q_2\right].$$

If $0 < \gamma < \alpha < \alpha + \beta = 2\gamma$, then $\tau = \gamma$, and

$$F(\alpha,\beta,\gamma,\lambda,\theta) = \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\frac{1}{\pi}-Q_1\right].$$

If $0 < \gamma < \alpha < 2\gamma < \alpha + \beta$, then

$$F(\alpha,\beta,\gamma,\lambda,\theta) = \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)-Q_0\right].$$

If $0<\gamma<\alpha=2\gamma<\alpha+\beta<3\gamma$, then

$$F(\alpha,\beta,\gamma,\lambda,\theta)=\frac{2\pi}{\gamma}\left[\Psi\left(\frac{\tau\pi}{\gamma}\right)+\frac{1}{\pi}-Q_2\right].$$

Let

$$F_1(\alpha,\beta,\gamma,\tau)=\begin{cases}\frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\frac{1}{\pi}-Q_1\right], & \tau=\gamma \text{ or } \tau=2\gamma;\\ \\ \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\alpha\pi}{\gamma}\right)+\Psi\left(\frac{\tau\pi}{\gamma}\right)-Q_0\right], & \alpha\neq\gamma,2\gamma \text{ and } \tau\neq\gamma,2\gamma;\\ \\ \frac{2\pi}{\gamma}\left[\Psi\left(\frac{\tau\pi}{\gamma}\right)+\frac{1}{\pi}-Q_2\right], & \alpha=\gamma \text{ or } \alpha=2\gamma.\end{cases}$$

Then we obtain the following corollary.

Corollary 4.4. *Let $\alpha,\tau\in(0,1)$, $\beta,\gamma\in\Omega_1$. Suppose that α,τ,β,γ satisfy $0<\beta<\min\{1-\alpha,1-\tau,\gamma\}$ and $\alpha+\tau+\beta=3\gamma$. Let $a_m,b_n>0$ with $\boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where $\mu_m=|m|^{p(1-\alpha)-1}$ and $\nu_n=|n|^{q(1-\tau)-1}$. Then*

(57)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{m^\beta+n^\beta}{(m^\gamma+n^\gamma)\max\{m^{2\gamma},n^{2\gamma}\}}a_mb_n<F_1(\alpha,\beta,\gamma,\tau)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $F_1(\alpha,\beta,\gamma,\lambda)$ in (57) is the best possible.

Let $\gamma=\alpha=(2s+1)\beta,\ \tau=(4s+1)\beta\ (s\in\mathbb{Z}^+)$ in (57), then we have $0<\beta<\frac{1}{4s+2}$ $(\beta\in\Omega_1)$, and (57) reduces to

(58)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_mb_n}{\max\left\{m^{(4s+2)\beta},n^{(4s+2)\beta}\right\}\sum_{j=0}^{2s}(-1)^jm^{j\beta}n^{(2s-j)\beta}}$$
$$<\left[\frac{2}{(2s+1)\beta}+\frac{2s+1}{s\beta}-\frac{2\pi}{(2s+1)\beta}\Psi\left(\frac{\pi}{2s+1}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p[1-(2s+1)\beta]-1}$ and $\nu_n=|n|^{q[1-(4s+1)\beta]-1}$.

Let $s=1$ in (58), then $0<\beta<\frac{1}{6}$ ($\beta\in\Omega_1$), and (58) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{6\beta},n^{6\beta}\}(m^{2\beta}-m^\beta n^\beta+n^{2\beta})}<\frac{33-4\sqrt{3}\pi}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p(1-3\beta)-1}$ and $\nu_n=|n|^{q(1-5\beta)-1}$.

Let $\gamma=\tau=(2s+1)\beta,\ \alpha=(4s+1)\beta$ ($s\in\mathbb{Z}^+$) in (57), then (58) is also valid with $\mu_m=|m|^{p[1-(4s+1)\beta]-1}$ and $\nu_n=|n|^{q[1-(2s+1)\beta]-1}$.

Let $\gamma=(2s+1)\beta,\ \alpha=(4s+2)\beta,\ \tau=2s\beta$ ($s\in\mathbb{Z}^+$) in (57), then we have $0<\beta<\frac{1}{4s+3}$ ($\beta\in\Omega_1$), and (57) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\left\{m^{(4s+2)\beta},n^{(4s+2)\beta}\right\}\sum_{j=0}^{2s}(-1)^j m^{j\beta}n^{(2s-j)\beta}}$$
$$<\left[\frac{2\pi}{(2s+1)\beta}\Psi\left(\frac{2s\pi}{2s+1}\right)+\frac{2}{(2s+1)\beta}-\frac{2s+1}{(s+1)\beta}\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{59}$$

where $\mu_m=|m|^{p[1-(4s+2)\beta]-1}$ and $\nu_n=|n|^{q(1-2s\beta)-1}$.

Let $s=1$ in (59), then $0<\beta<\frac{1}{7}$ ($\beta\in\Omega_1$), and (59) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\{m^{6\beta},n^{6\beta}\}(m^{2\beta}-m^\beta n^\beta+n^{2\beta})}<\frac{8\sqrt{3}\pi-15}{18\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m=|m|^{p(1-6\beta)-1}$ and $\nu_n=|n|^{q(1-2\beta)-1}$.

Let $\gamma=(2s+1)\beta,\ \alpha=\tau=(3s+1)\beta$ ($s\in\mathbb{Z}^+$) in (57), then we have $0<\beta<\frac{1}{3s+2}$ ($\beta\in\Omega_1$), and (57) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\frac{a_m b_n}{\max\left\{m^{(4s+2)\beta},n^{(4s+2)\beta}\right\}\sum_{j=0}^{2s}(-1)^j m^{j\beta}n^{(2s-j)\beta}}$$
$$<\left[\frac{8s+4}{s(s+1)\beta}-\frac{4\pi}{(2s+1)\beta}\Psi\left(\frac{s\pi}{2s+1}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{60}$$

where $\mu_m=|m|^{p[1-(3s+1)\beta]-1}$ and $\nu_n=|n|^{q[1-(3s+1)\beta]-1}$.

Letting $\beta,\gamma\in\Omega_2$ and $\lambda=0$ in Theorem 3.1, and using Lemma 2.4, then Theorem 3.1 is transformed into the following corollary.

Corollary 4.5. *Let* $\alpha, \tau \in (0,1)$, $\beta, \gamma \in \Omega_2$. *Suppose that* $\alpha, \tau, \beta, \gamma$ *satisfy* $0 < \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *and* $\alpha+\tau+\beta = \gamma$. *Let* $\Phi(t) = \cot t$, $a_m, b_n > 0$ *with* $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, *where* $\mu_m = |m|^{p(1-\alpha)-1}$ *and* $\nu_n = |n|^{q(1-\tau)-1}$. *Then*

(61)
$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{m^\beta - \theta n^\beta}{m^\gamma - \theta n^\gamma} a_m b_n < \frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) + \Phi\left(\frac{\tau\pi}{2\gamma}\right)\right] \|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $\frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) + \Phi\left(\frac{\tau\pi}{2\gamma}\right)\right]$ *in* (61) *is the best possible.*

Let $\gamma = (2s+1)\beta$ $(s \in \mathbb{Z}^+)$ in (61), then we have $\alpha + \tau = 2s\beta$, and (61) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{a_m b_n}{\sum_{j=0}^{2s} \theta^j m^{j\beta} n^{(2s-j)\beta}} < \frac{\pi}{(2s+1)\beta}\left[\Phi\left(\frac{\alpha\pi}{(2s+1)\beta}\right) + \Phi\left(\frac{\tau\pi}{(2s+1)\beta}\right)\right]$$
$$\times \|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{62}$$

where $\mu_m = |m|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\tau)-1}$.

Let $s = 1$, $\alpha = \tau = \beta$ in (62), then $0 < \beta < \frac{1}{2}$ $(\beta \in \Omega_2)$, and (62) is transformed into

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{a_m b_n}{(m^{2\beta} + \theta m^\beta n^\beta + n^{2\beta})} < \frac{2\sqrt{3}\pi}{9\beta}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p(1-\beta)-1}$ and $\nu_n = |n|^{q(1-\beta)-1}$.

Let $\beta, \gamma \in \Omega_2$ and $\lambda = 2\gamma$ in Theorem 3.1, and

$$F_2(\alpha,\beta,\gamma,\tau) = \begin{cases} \frac{\pi}{\gamma}\Phi\left(\frac{\tau\pi}{2\gamma}\right) + \frac{2}{2\gamma-\tau}, & \alpha = 2\gamma, \\ \frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) + \Phi\left(\frac{\tau\pi}{2\gamma}\right) + W_0\right], & \alpha \neq 2\gamma \text{ and } \tau \neq 2\gamma, \\ \frac{\pi}{\gamma}\Phi\left(\frac{\alpha\pi}{2\gamma}\right) + \frac{2}{2\gamma-\alpha}, & \tau = 2\gamma, \end{cases}$$

where
$$W_0 := \frac{2\gamma(\beta+\gamma)}{\pi(2\gamma-\alpha)(2\gamma-\tau)}.$$

Then the following corollary holds true.

Corollary 4.6. *Let* $\alpha, \tau \in (0,1)$, $\beta, \gamma \in \Omega_2$. *Suppose that* $\alpha, \tau, \beta, \gamma$ *satisfy* $0 < \beta < \min\{1-\alpha, 1-\tau, \gamma\}$ *and* $\alpha + \tau + \beta = 3\gamma$. *Let* $a_m, b_n > 0$ *with* $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, *where* $\mu_m = |m|^{p(1-\alpha)-1}$ *and* $\nu_n = |n|^{q(1-\tau)-1}$. *Then*

(63)

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{m^\beta - \theta n^\beta}{(m^\gamma - \theta n^\gamma)\max\{m^{2\gamma}, n^{2\gamma}\}} a_m b_n < F_2(\alpha,\beta,\gamma,\tau)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where the constant factor $F_2(\alpha,\beta,\gamma,\lambda)$ *in* (63) *is the best possible .*

Let $\gamma = (2s+1)\beta,\ \alpha = (4s+2)\beta,\ \tau = 2s\beta\ (s \in \mathbb{Z}^+)$ in (63), then we have $0 < \beta < \frac{1}{4s+3}$ $(\beta \in \Omega_2)$, and (63) reduces to

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{a_m b_n}{\max\left\{m^{(4s+2)\beta}, n^{(4s+2)\beta}\right\}\sum_{j=0}^{2s}\theta^j m^{j\beta} n^{(2s-j)\beta}} < \left[\frac{\pi}{(2s+1)\beta}\Phi\left(\frac{s\pi}{2s+1}\right) + \frac{1}{(s+1)\beta}\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{64}$$

where $\mu_m = |m|^{p[1-(4s+2)\beta]-1}$ and $\nu_n = |n|^{q(1-2s\beta)-1}$.

Let $\gamma = (2s+1)\beta,\ \alpha = \tau = (3s+1)\beta\ (s \in \mathbb{Z}^+)$ in (63), then we have $0 < \beta < \frac{1}{3s+2}$ $(\beta \in \Omega_2)$, and (63) reduces to

(65)

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} \frac{a_m b_n}{\max\left\{m^{(4s+2)\beta}, n^{(4s+2)\beta}\right\}\sum_{j=0}^{2s}\theta^j m^{j\beta} n^{(2s-j)\beta}} < \left[\frac{4}{(s+1)\beta} - \frac{2\pi}{(2s+1)\beta}\Phi\left(\frac{(s+1)\pi}{4s+2}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu},$$

where $\mu_m = |m|^{p[1-(3s+1)\beta]-1}$ and $\nu_n = |n|^{q[1-(3s+1)\beta]-1}$.

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