WEAKLY CONFLUENT CLASSES OF DENDRITES

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Abstract. Given continua X, Y and a class $\mathcal{F}$ of maps between continua, define $X \geq_{\mathcal{F}} Y$ if there exists an onto map $f : X \to Y$ belonging to $\mathcal{F}$. A map $f : X \to Y$ is *weakly confluent* if for each subcontinuum B of Y, there exists a subcontinuum A of X such that $f(A) = B$. In this paper we consider the class $\mathcal{W}$ of weakly confluent maps. Two continua X and Y are $\mathcal{W}$-*equivalent* provided that $X \leq_{\mathcal{W}} Y$ and $Y \leq_{\mathcal{W}} X$. We show that any Gehman Dendrite G_n is $\mathcal{W}$-equivalent to any universal dendrite D_m. We consider the class $[G_3]_{\mathcal{W}}$ of all dendrites that are $\mathcal{W}$-equivalent to G_3. We characterize the elements of $[G_3]_{\mathcal{W}}$ in two ways: (a) a dendrite D belongs to $[G_3]_{\mathcal{W}}$ if and only if D contains uncountably many end-points, and (b) a dendrite D belongs to $[G_3]_{\mathcal{W}}$ if and only if D is maximal with respect to the preorder $\leq_{\mathcal{W}}$

1. Introduction

A *continuum* is a compact connected metric space with more than one point. A *subcontinuum* of a continuum X is a nonempty closed connected subset of X, so one-point sets in X are subcontinua of X. A *map* is a continuous function.

Given an onto map $f : X \to Y$ between continua, we say that f is:
- *monotone* provided that for each subcontinuum B of Y, $f^{-1}(B)$ is a subcontinuum of X;
- *confluent* if for each subcontinuum B of Y and each component A of $f^{-1}(B)$, $f(A) = B$; and
- *weakly confluent* if for each subcontinuum B of Y, there is a subcontinuum A of X such that $f(A) = B$.

Note that

$$\text{monotone} \Rightarrow \text{confluent} \Rightarrow \text{weakly confluent}.$$

The class of monotone (respectively, confluent and weakly confluent) maps is denoted by $\mathcal{M}$ (respectively, $\mathcal{C}$ and $\mathcal{W}$). It is easy to show that classes $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{W}$ are closed under composition.

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Given continua X and Y, and a class of maps between continua $\mathcal{F}$, we define $X \geq_{\mathcal{F}} Y$ if there exists an onto map $f : X \to Y$ belonging to $\mathcal{F}$. Two continua X and Y are *$\mathcal{F}$-equivalent* (denoted by $X \simeq_{\mathcal{F}} Y$) provided that $X \leq_{\mathcal{F}} Y$ and $Y \leq_{\mathcal{F}} X$. Given a class of continua $\mathcal{E}$, a continuum $X \in \mathcal{E}$ is *$\mathcal{F}$-isolated* in the class $\mathcal{E}$ provided that the following implication holds: if $Y \in \mathcal{E}$ and $X \simeq_{\mathcal{F}} Y$, then X and Y are homeomorphic.

A *curve* is a 1-dimensional continuum. A *dendrite* is a locally connected continuum without simple closed curves. For a continuum X and a point $p \in X$ we use the *order of p in X* in the sense of Menger-Urysohn [4, Appendix A.2], which is denoted by $o(p, X)$. For dendrites D, $o(p, D)$ can be defined as the number of components of $D \setminus \{p\}$ (see [1, p. 2]). Then $o(p, D) \in \mathbb{N} \cup \{\omega\}$. Points of order one in X are *end-points*, and points of order greater than 2 are *ramification points*. The set of end-points of X is denoted by $E(X)$ and the set of ramification points of X is denoted by $R(X)$.

Given $n \in \mathbb{N}$ $(n \geq 3)$ and $m \in \mathbb{N} \cup \{\omega\}$ $(m \geq 3)$, two important dendrites we will use are the *Gehman dendrite* G_n and the *the universal dendrite* D_m. The Gehman dendrite G_n is characterized by having $E(G_n)$ homeomorphic to the Cantor set; all ramification points of G_n are of order n; and $E(G_n) = \mathrm{cl}_X(R(G_n)) \setminus R(G_n)$ (see [5, p. 21], and for a picture of G_3 see [10, p. 424]). The universal dendrite D_m is characterized by having the following properties: all ramification points are of order m and each arc in X contains ramification points [3, Theorem 3.1] (see [7, p. 61] for a picture of D_4).

In the realm of dendrites a very complete study of the preorder $\leq_{\mathcal{F}}$ was made by J. J. Charatonik, W. J. Charatonik and J. R. Prajs in [5]. Several families $\mathcal{F}$ were considered, but the most important results are related to monotone and open mappings.

For dendrites, the following facts are known.

(a) if X and Y are dendrites, then $X \simeq_{\mathcal{M}} Y$ if and only if $X \simeq_{\mathcal{C}} Y$ [5, Corollary 5.7],
(b) for every $n, m \in \mathbb{N} \cup \{\omega\}$ $(n, m \geq 3)$, $D_n \simeq_{\mathcal{M}} D_m$, $D_n \simeq_{\mathcal{C}} D_m$ and $D_n \simeq_{\mathcal{W}} D_m$ [5, Theorem 5.27],
(c) for each $n \geq 3$ and for each $m \in \mathbb{N} \cup \{\omega\}$ $(m \geq 3)$, G_n and D_m are not $\mathcal{M}$-equivalent (it follows from [9, Theorem 5.27]),
(d) trees are $\mathcal{W}$-isolated in the class of trees [8, Theorem 3.3],
(e) A finite graph X is not $\mathcal{W}$-isolated in the class of all continua if and only if X is either an arc, or a simple closed curve, or contains a cycle (a *cycle* is a simple closed curve with exactly one ramification point of X), or contains a ramification point contained in two distinct sticks (a *stick* is an edge joining a ramification point to an end-point) [8, Theorem 3.4],
(f) a dendrite X is $\mathcal{M}$-isolated in the class of all continua if and only if $R(X)$ is finite [9, Theorem 1.1],
(g) it follows from [2, Theorem 3.2] that: if two dendrites are monotone-equivalent, then they are quasi-homeomorphic (two dendrites X and Y are quasi-homeomorphic if for each $\varepsilon > 0$ there are ε-onto maps $f_\varepsilon : X \to Y$ and $g_\varepsilon : Y \to X$). However the converse is not true.

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The authors in [5, Theorem 5.27], gave a complete characterization of dendrites which are maximum elements with respect to the preorder $\leq_{\mathcal{M}}$ (equivalently, $\leq_{\mathcal{C}}$ [5, Corollary 5.7]), they showed that a dendrite D satisfies $X \leq_{\mathcal{M}} D$ for every dendrite X if and only if D contains the dendrite L_0 described in [5, 5.26].

The aim of this paper is to characterize the maximal dendrites with respect to the preorder $\leq_{\mathcal{W}}$. We prove that D is one of these dendrites if and only if $E(D)$ is uncountable. The proof of this result is based in the theorem that says that there exists a weakly confluent map f from the Gehman dendrite G_6 onto the universal dendrite D_4. Most of this paper is devoted to give a detailed construction of the map f.

2. Gehman and universal dendrites

Theorem 2.1. *For $n \geq 3$ and $m \in \{3, 4, \ldots\} \cup \{\omega\}$, the Gehman dendrite G_n and the universal dendrite D_m are weakly confluent equivalent.*

To prove this theorem it is enough to show that there exists a weakly confluent map $f : G_6 \to D_4$; the argument is as follows: By [1, Corollary 6.10], for all $n, m \geq 3$, G_n is a monotone image of G_m and, by [3, Corollary 6.4], for all $k, l \in \{3, 4, \ldots\} \cup \{\omega\}$, D_k is monotone equivalent to D_l. Let $n \geq 3$ and $m \in \{3, 4, \ldots\}$, since monotone maps are weakly confluent, there are weakly confluent maps $g_0 : D_m \to D_\omega$ and $g_1 : D_\omega \to G_n$ [3, Proposition 6.2]. Hence, $g = g_1 \circ g_0 : D_m \to G_n$ is a weakly confluent map. We can take monotone maps $f_1 : G_n \to G_6$ and $f_2 : D_4 \to D_m$. Thus, $f_3 = f_2 \circ f \circ f_1 : G_n \to D_m$ is weakly confluent. Therefore, G_n and D_m are weakly confluent equivalent.

This section is devoted to construct a weakly confluent map $f : G_6 \to D_4$.

For simplicity, the ramification and end points of a dendrite will also be called vertices. We will use the universal dendrite D_4. Recall that this dendrite is characterized by the following two properties [6, Theorem 6.2, p. 229]:

(a) each ramification point in D_4 has order 4, and
(b) each arc in D_4 contains points of order 4.

Since the proof that there exists a weakly confluent map from the Gehman dendrite G_6 onto D_4 requires some explicit formulas, we start by giving an appropriate description of D_4.

We will use the set of *dyadic numbers* $\mathcal{D}$ in the interval $[0, 1]$:

$$\mathcal{D} = \{\tfrac{k}{2^m} \in [0, 1] : m \in \mathbb{N} \text{ and } k \in \{0, 1, \ldots, 2^m\}\}.$$

Given $r \in \mathcal{D} \setminus \{0, 1\}$, the *degree* of r is the unique number $g(r) \in \mathbb{N}$ such that $r = \frac{k}{2^{g(r)}}$, where k is odd.

Lemma 2.2. *(a) Let $r, s \in \mathcal{D} \setminus \{0, 1\}$. Then $r - \frac{s}{2^{g(r)}} \in \mathcal{D} \setminus \{0, 1\}$ and $g(r - \frac{s}{2^{g(r)}}) = g(r) + g(s)$.*
(b) Let $[a, b]$ be a non-degenerate subinterval of $[0, 1]$. Then there exists a unique element $r \in [a, b] \cap (\mathcal{D} \setminus \{0, 1\})$ with minimal degree $g(r)$; if $g(r) > 1$, then $\frac{1}{2^{g(r)}} > max\{b - r, r - a\}$, and if $g(r) = 1$, then $r = \frac{1}{2}$.

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Proof. (a). Since $r \geq \frac{1}{2^{g(r)}}$, we have that $0 \leq r - \frac{1}{2^{g(r)}} < r - \frac{s}{2^{g(r)}} < r < 1$, so $r - \frac{s}{2^{g(r)}} \in \mathcal{D} \setminus \{0,1\}$. Let $m = g(r)$ and $n = g(s)$. Consider the dyadic representation of r and s: $r = \frac{r_1}{2^1} + \cdots + \frac{r_m}{2^m}$, $s = \frac{s_1}{2^1} + \cdots + \frac{s_n}{2^n}$, where each r_i and each s_i is in $\{0,1\}$ and $r_m = 1 = s_n$. Then $r - \frac{s}{2^{g(r)}} = \frac{r_1}{2^1} + \cdots + \frac{r_m}{2^m} - (\frac{s_1}{2^{m+1}} + \cdots + \frac{s_n}{2^{m+n}}) = \frac{2^{m+n-1}r_1 + \cdots + 2^n r_m - 2^{n-1}s_1 - \cdots - 2s_{n-1} - s_n}{2^{m+n}}$. This shows that $g(r - \frac{s}{2^{g(r)}}) = m + n = g(r) + g(s)$.

(b). Suppose to the contrary that $r_1 < r_2$ are elements with minimal degree in $[a,b]$ such that $g(r_1) = g(r_2) \in \mathbb{N}$. Then there exist odd numbers $k_1, k_2 \in \{1, \ldots, 2^{g(r_1)}\}$ such that $0 \leq r_1 = \frac{k_1}{2^{g(r_1)}} < \frac{k_1+1}{2^{g(r_1)}} < \frac{k_1+2}{2^{g(r_1)}} \leq \frac{k_2}{2^{g(r_2)}} = r_2 \leq 1$. Since $k_1 + 1$ is even, the number $r_0 = \frac{k_1+1}{2^{g(r_1)}}$ belongs to $[a,b] \cap (\mathcal{D} \setminus \{0,1\})$ and $g(r_0) < g(r_1)$, a contradiction. This proves the uniqueness of the element r of minimal degree. Suppose that $r = \frac{k}{2^{g(r)}}$, with k odd and $g(r) > 1$. If $r + \frac{1}{2^{g(r)}} = \frac{k+1}{2^{g(r)}} \leq b$, then $g(\frac{k+1}{2^{g(r)}}) < g(r)$, this contradicts the choice of r. Thus $b - r < \frac{1}{2^{g(r)}}$. Similarly, $r - a < \frac{1}{2^{g(r)}}$. □

2.1. Construction of D_4.

When we take points p and q in a dendrite, by pq we denote the unique arc joining them, if $p \neq q$, and $pq = \{p\}$, if $p = q$.

We consider the points $v = d = (0,0)$, $a = (0,1)$, $b = (0,-1)$, $c = (1,0)$ and $e = (-1,0)$ in the Euclidean plane $\mathbb{R}^2$. To construct D_4, we start with a cross and then we add smaller and smaller crosses in strategic points and strategic sizes. Points a, b, c, e will be useful for indicating if we will walk up, down, right or left.

Let $\mathcal{B}_L = \{d, a, b, c, e\}$ and $\mathcal{B}'_L = \{a, b, c, e\}$. Set $\beta = \frac{7}{8}$. We use the number β to short segments in order to avoid intersection of paths.

We define two types of elements in the set $\mathcal{B}'_L$, we say that a and b are of the *vertical type*; and c and e are of the *horizontal type*.

We consider the set D_4^* of points q in the plane $\mathbb{R}^2$ such that either $q = v$ or q is of the following form.

$$q = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}} + t\beta^m \frac{r_{m+1} z_{m+1}}{2^{g(1)+\cdots+g(r_m)}}, \tag{1}$$

where $m \geq 0$, $t \in (0,1]$, for each $i \in \{1, \ldots, m+1\}$, $r_i \in \mathcal{D} \setminus \{0,1\}$, $z_i \in \mathcal{B}'_L$, and, if $i > 1$, z_i is of distinct type than z_{i-1}, meaning $z_i \in \{a, b\}$ if and only if $z_{i+1} \in \{c, d\}$.

We will give a brief explanation of a point $q \in D_4^*$.

In the term $\frac{r_1 z_1}{2^0}$, z_1 indicates one of the four fundamental directions a, b, c or e and the dyadic number r_1 indicates how much we advance on the direction z_1. Similarly, in the term $\beta \frac{r_2 z_2}{2^{g(r_1)}}$, z_2 indicates the direction in which we move when we are standing on point $v + \frac{r_1 z_1}{2^0}$, we are asking that z_2 is of different type than z_1, so we change direction, and $\beta \frac{r_2}{2^{g(r_1)}}$ indicates how much we move in that direction. This movement is limited by the factor $\frac{1}{2^{g(r_1)}}$. For example if $r_1 = \frac{1}{2}$, since $r_2 \in (0,1)$, the length of this movement is less than $\frac{\beta}{2}$, if $r_1 \in \{\frac{1}{4}, \frac{3}{4}\}$, is less than $\frac{\beta}{4}$, if $r_1 \in \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$, is less than $\frac{\beta}{8}$, etcetera. The factor β allows us to avoid intersections of paths, so the arcs from the point v to any point in D_4^* is unique. We continue until we use the last term: $t\beta^m \frac{r_{m+1}z_{m+1}}{2^{g(1)+\cdots+g(r_m)}}$, here the number t indicates that we run on a complete segment.

On Figure 1, we illustrate the set covered by the elements in D_4^* with $m = 0$, and we also illustrate some elements with $m = 1$. In fact the complete elements for $m = 1$ include countably many segments perpendicular to the first cross.

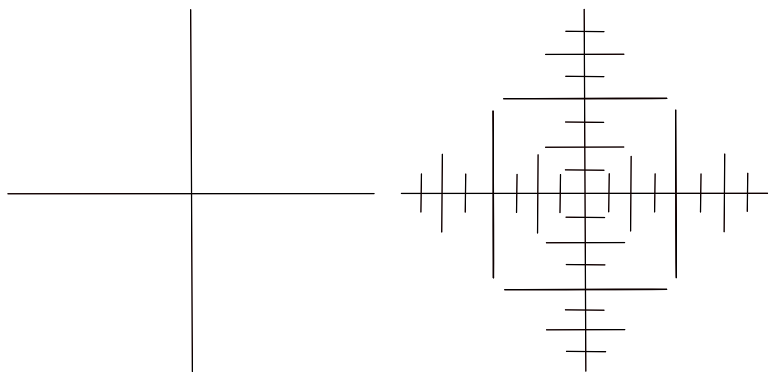

FIGURE 1. $m = 0$ and $m = 1$

In the case that q is written in the form (1), define the number $m(q) = m$ and the point

$$w(q) = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}}.$$

Notice that $m(q)$, $w(q)$ and z_{m+1} are uniquely determined by q. So we can write

$$q = w(q) + t\beta^{m(q)} \frac{r_{m(q)+1} z_{m(q)+1}}{2^{g(1)+\cdots+g(r_{m(q)})}}.$$

The expression in (1) is not unique since the number $tr_{m(q)+1}$ can be written in many ways. Observe that D_4^* includes exactly all points in D_4 of order 2 or 4. That is, $D_4 \setminus D_4^* = E(D_4)$ ($E(D_4)$ is the set of end-points of D_4). Then D_4^* is dense in D_4. The set of ramification points of D_4 is the set $R(D_4)$ of points $p \in D_4$ such that either $p = v$ or p is of the form

$$p = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}} + \beta^m \frac{r_{m+1} z_{m+1}}{2^{g(1)+\cdots+g(m)}} \tag{2}$$

where $m, r_1, \ldots, r_{m+1}$ and $z_1, \ldots, z_{m+1}$ satisfy the conditions described previously. Observe that the expression for points in $R(D_4)$ is unique.

Given $q \in D_4^*$, in the following definition we give name the segments we use to go from v to q.

Definition 2.3. *Given $q \in D_4^*$ (written as in (1)), define*

$$L_1(q) = \{v + s\tfrac{r_1 z_1}{2^0} : s \in (0,1]\},$$
$$L_2(q) = \{v + \tfrac{r_1 z_1}{2^0} + s\beta\tfrac{r_2 z_2}{2^{g(r_1)}} : s \in (0,1]\},$$
$$\vdots$$
$$L_m(q) = \{v + \tfrac{r_1 z_1}{2^0} + \cdots + \beta^{m-2}\tfrac{r_{m-1} z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + s\beta^{m-1}\tfrac{r_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}} : s \in (0,1]\} \text{ and}$$
$$L_{m+1}(q) = \{w(q) + st\beta^m\tfrac{r_{m+1} z_{m+1}}{2^{g(r_1)+\cdots+g(r_m)}} : s \in (0,1]\}.$$

Observe that each set $L_i(q)$ is homeomorphic to the interval $(0,1]$ and the unique arc in D_4 joining v and q (respectively, v and $w(q)$) is $vq = \{v\} \cup L_1(q) \cup \cdots \cup L_{m+1}(q)$ (respectively, $\{v\} \cup L_1(q) \cup \cdots \cup L_m(q)$). Observe that the rays $L_1(q), \cdots, L_{m+1}(q)$ are uniquely determined by q.

2.2. Description of the dendrite X.

Recall that the *Gehman dendrite* G_3 is characterized as the dendrite satisfying that its set of end-points is homeomorphic to the Cantor set, each ramification point is of order three and $E(G_3) = \mathrm{cl}_{G_3}(R(G_3)) \setminus R(G_3)$ [11, p. 100], see [12, p. 203], for a picture. Similarly, the Gehman dendrite of order 6, denoted by G_6, is characterized as the dendrite satisfying that its set of end-points is homeomorphic to the Cantor set, each ramification point is of order 6 and $E(G_6) = \mathrm{cl}_{G_6}(R(G_6)) \setminus R(G_6)$.

Instead of working directly with G_6, it is convenient for us to take G_6 but transforming (exactly) one point of order 6 into a point of order 5. This new space is named X.

Fix a ramification point v_{G_6} of G_6, let $C_1^*, \ldots, C_6^*$ be the components of $G_6 \setminus \{v_{G_6}\}$. Consider the continuum X obtained by shrinking the set $C_1^* \cup \{v_{G_6}\}$ into a point. Let $V \in X$ be the point corresponding to $C_1^* \cup \{v_{G_6}\}$. Then X is a dendrite such that its set of end-points is homeomorphic to the Cantor set, the point V has order 5, the rest of its ramification points are of order 6 and $E(X) = \mathrm{cl}_X(R(X)) \setminus R(X)$. Observe that X is a monotone (and then weakly confluent) image of G_6 ($X \leq_{\mathcal{W}} G_6$). We establish the following conventions on dendrite X.

As we did with D_4, we will describe X by starting at the vertex V, and then giving five possible directions (D, A, B, C and E) indicating the ways we can walk. So, the vertices of X will be described in the following way: V is the first vertex, VD, VA, VB, VC and VE are the five vertices adjacent to V in X. Besides V, the vertices adjacent to VA, are VAD, VAA, VAB, VAC and VAE, and we continue in this way.

Formally: fix five distinct labels D, A, B, C and E (all different from V). Let $\mathcal{B}_C = \{D, A, B, C, E\}$ and $\mathcal{B}'_C = \{A, B, C, E\}$. The ramification points of X are all the finite sequences of the form:

$$T = Z_0 Z_1 Z_2 \ldots Z_m,$$

where $m \geq 0$, $Z_0 = V$ and for each $i \in \{1, \ldots, m\}$, $Z_i \in \mathcal{B}_C$.

The maximal free arcs in X are the arcs of the form T_mT_{m+1}, where $T_m = Z_0Z_1Z_2 \dots Z_m$ and $T_{m+1} = Z_0Z_1Z_2 \dots Z_mZ_{m+1}$. Then the arc VT_m is the union of the arcs $Z_0(Z_0Z_1)$, $(Z_0Z_1)(Z_0Z_1Z_2)$,$\dots$,$(Z_0 \dots Z_{m-1})(Z_0 \dots Z_m)$. We fix a one-to-one onto map

$$\sigma(T_{m+1}) : [0,1] \to T_mT_{m+1}$$

such that $\sigma(T_{m+1})(0) = T_m$ and $\sigma(T_{m+1})(1) = T_{m+1}$. The set $\sigma(T_{m+1})([0,1])$ is the arc T_mT_{m+1} in X that joins T_m and T_{m+1}. Let

$$\eta(T_{m+1}) : T_mT_{m+1} \to [0,1]$$

be the inverse mapping of $\sigma(T_{m+1})$.

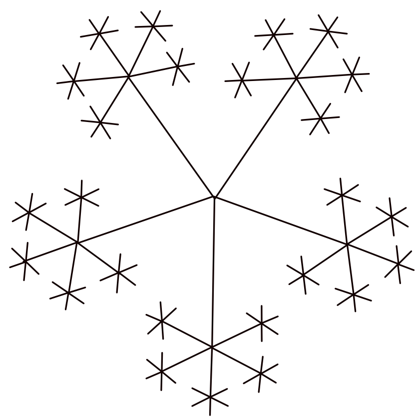

Figure 2. X_3

The end-points of X are the infinite sequences of the form:

$$R = Z_0Z_1Z_2 \dots$$

where $Z_0 = V$ and for each $i \in \mathbb{N}$, $Z_i \in \mathcal{B}_C$. The arc VR in X is given by:

$$VR = T_0T_1 \cup T_1T_2 \cup T_2T_3 \cup \cdots$$

where for each $m \geq 0$, $T_m = Z_0Z_1 \dots Z_m$. Then $T_0 = Z_0 = V$ and

$$X = \bigcup\{T_0R : R \text{ is an end-point of X}\}.$$

For each $m \geq 0$, let

$$X_m = \{T_0T_m \subseteq X : T_m = Z_0Z_1Z_2 \dots Z_m \text{ and, for each } i \in \{1, \dots, m\}, Z_i \in \mathcal{B}_C\}.$$

In Figure 2, we illustrate the set X_3.

For the definition of D_4, we used the set $\mathcal{B}_L = \{d, a, b, c, e\}$. Recall that the elements of the set $\mathcal{B}_C$ are denoted with the capital letters A, B, C, D, E we will use the following correspondence: $D \to d$, $A \to a$, $B \to b$, $C \to c$, $E \to e$. When we denote an element in $\mathcal{B}_C$ by Z_i, we consider the element $z_i \in \mathcal{B}_L$ defined with the previous correspondence for

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the element Z_i. Conversely, for each element $z \in \mathcal{B}_L$, we define the corresponding element $Z \in \mathcal{B}_C$.

We define two types of elements in the set $\mathcal{B}'_C$, we say that A and B are of the *vertical type*; and C and E are of the *horizontal type*.

2.3. **Definition of f.**

For a vertex $T_{m+1} = Z_0Z_1 \ldots Z_{m+1}$ of X, define a sequence $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$ as follows. Take $i \in \{1, 2, \ldots, m+1\}$.

(a) If $Z_i = D$, let $\lambda_i = 0$;
(b) if $Z_i \neq D$ and $\{Z_0, \ldots, Z_{i-1}\} = \{D\}$, let $\lambda_i = 1$;
(c) if $Z_i \neq D$ and $\{Z_0, \ldots, Z_{i-1}\} \neq \{D\}$, let $j_0 = \max\{j \in \{1, \ldots, i-1\} : Z_j \neq D\}$ and define $\lambda_i = \lambda_{j_0}$, in the case that Z_i is of the same type than Z_{j_0}; and $\lambda_i = \beta\lambda_{j_0}$ (recall that $\beta = \frac{7}{8}$), in the case that Z_i is of distinct type than Z_{j_0}. Then each λ_i belongs to the set $\{\beta^k : k \in \mathbb{N}\} \cup \{0, 1\}$

Define $f : X \to \mathbb{R}^2$ as follows. Set $f(V) = v$, and given a vertex $T_{m+1} = Z_0Z_1 \ldots Z_{m+1}$ of X and a point $p \in T_mT_{m+1}$, where $T_m = Z_0Z_1 \ldots Z_m$, define

$$f(p) = v + \frac{\lambda_1 z_1}{2^1} + \frac{\lambda_2 z_2}{2^2} + \cdots + \frac{\lambda_m z_m}{2^m} + \eta(T_{m+1})(p)\frac{\lambda_{m+1} z_{m+1}}{2^{m+1}} \tag{3}$$

where $\lambda_1, \ldots, \lambda_{m+1}$ are defined as previously, for the sequence T_{m+1}.

Given an end-point $p = Z_0Z_1Z_2 \ldots$ of X, define

$$f(p) = v + \frac{\lambda_1 z_1}{2^1} + \frac{\lambda_2 z_2}{2^2} + \frac{\lambda_3 z_3}{2^3} + \cdots ,$$

where for each $m \in \mathbb{N}$, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are defined as previously for the sequence $T_m = Z_0Z_1 \ldots Z_m$. Observe that each number λ_i is defined using only the elements $Z_1, \ldots, Z_i$, and it is independent of any number $k \geq i$.

Given $m \in \mathbb{N}$, observe that

$$\begin{aligned} f(X_m) &= \{f(T_0(Z_0Z_1 \ldots Z_m)) : Z_0Z_1Z_2 \ldots Z_m \text{ is a ramification point of } X\} \\ &= \{f(p) : p \in T_{n-1}T_n, 1 \leq n \leq m \text{ and } T_n \in R(X)\} \end{aligned}$$

is the minimum tree in $\mathbb{R}^2$ containing the points in the set

$$f(X_m) = \{f(Z_0Z_1 \ldots Z_m) : Z_0Z_1Z_2 \ldots Z_m \text{ is a ramification point of } X\}.$$

Since $\{Z_0Z_1 : Z_1 \in \mathcal{B}_C\} = \{VD, VA, VB, VC, VE\}$, we have that $f(X_1)$ is the minimum tree in the plane $\mathbb{R}^2$ containing the points v, $v + \frac{a}{2}$, $v + \frac{b}{2}$, $v + \frac{c}{2}$ and $v + \frac{e}{2}$.

Observe that $f(X_2)$ is the minimum tree in the plane containing the points:

v, $v + \frac{a}{2}$, $v + \frac{b}{2}$, $v + \frac{c}{2}$, $v + \frac{e}{2}$, (they come from VD, VA, VB, VC, VE, or VDD, VAD, VBD, VCD, VED);

$v + \frac{a}{4}$, $v + \frac{b}{4}$, $v + \frac{c}{4}$, $v + \frac{e}{4}$, (from VDA, VDB, VDC, VDE);

$v + \frac{3a}{4}$, $v + \frac{3b}{4}$, $v + \frac{3c}{4}$, $v + \frac{3e}{4}$, (from VAA, VBB, VCC, VEE);

$v + \frac{a}{4}$, $v + \frac{b}{4}$, $v + \frac{c}{4}$, $v + \frac{e}{4}$, (from VAB, VBA, VCE, VEC);

$v+\frac{a}{2}+\beta\frac{c}{4}$, $v+\frac{a}{2}+\beta\frac{e}{4}$, $v+\frac{b}{2}+\beta\frac{c}{4}$, $v+\frac{b}{2}+\beta\frac{e}{4}$, $v+\frac{c}{2}+\beta\frac{a}{4}$, $v+\frac{c}{2}+\beta\frac{b}{4}$, $v+\frac{e}{2}+\beta\frac{a}{4}$, $v+\frac{e}{2}+\beta\frac{b}{4}$ (from $VAC, VAE, VBC, VBE, VCA, VCB, VEA, VEB$).

In Figure 3 we picture the sets $f(X_1)$, $f(X_2)$ and $f(X_3)$.

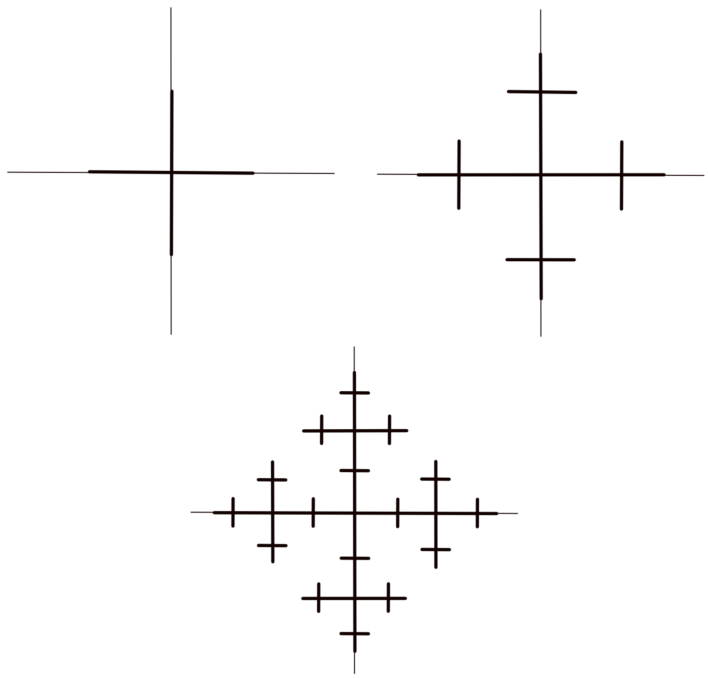

FIGURE 3. $f(X_1)$, $f(X_2)$ and $f(X_3)$.

Clearly f is continuous.

The following lemma is an easy consequence of the definitions.

Lemma 2.4. *Let $T_{m+1} = Z_0Z_1 \dots Z_{m+1}$ be a vertex of X and $T_m = Z_0Z_1 \dots Z_m$. Then:*

(a) $f(T_m) = v + \frac{\lambda_1 z_1}{2^1} + \frac{\lambda_2 z_2}{2^2} + \cdots + \frac{\lambda_m z_m}{2^m}$,
(b) if $Z_{m+1} = D$, then $f(T_mT_{m+1}) = \{f(T_m)\} = \{f(T_{m+1})\} = f(T_m)f(T_{m+1})$,
(c) if $Z_{m+1} \neq D$, then $f(T_mT_{m+1}) = f(T_m)f(T_{m+1})$. That is, $f(T_mT_{m+1}) = \{v + \frac{\lambda_1 z_1}{2^1} + \frac{\lambda_2 z_2}{2^2} + \cdots + \frac{\lambda_m z_m}{2^m} + t\frac{\lambda_{m+1} z_{m+1}}{2^{m+1}} \in D_4 : t \in [0,1]\}$.

Lemma 2.5. *Let $T = Z_0Z_1 \dots Z_m$ be a vertex of X and $Z \in \mathcal{B}'_C$. Suppose that $\{W_1, \dots, W_n\} \subset \{D, Z\}$. Define the sequence $S = Z_0Z_1 \dots Z_mW_1 \dots W_n$. For each $i \in \{1, \dots, n\}$, let $s_i = 0$, if $W_i = D$; and $s_i = 1$, if $W_i = Z$. Set $r = \frac{s_1}{2^1} + \cdots + \frac{s_n}{2^n} \in \mathcal{D}$. Then:*

(a) if $\{W_1, \dots, W_n\} = \{D\}$, then $f(TS) = \{f(T)\}$;
(b) if $Z \in \{W_1, \dots, W_n\}$, then $f(TS) = f(T)f(S)$; and
(c) if Z and Z_m are of different type and $Z_m \neq D$, then $f(S) = f(T) + \frac{\beta\lambda_m}{2^m} rz$, where λ_m is defined for the sequence T.

Proof. (a) follows from Lemma 2.4. To prove (b) and (c), suppose that $W_{i_1}, \dots, W_{i_k}$ are all the elements in $\{W_1, \dots, W_n\}$ which are equal to Z, where $k \in \mathbb{N}$ and $i_1 < \dots < i_k$. For each $l \in \{1, \dots, k\}$, let $S_l = Z_0 Z_1 \dots Z_m W_1 \dots W_{i_l}$.

Given $i \in \{1, \dots, n\}$, if $i \notin \{i_1, \dots, i_k\}$, then $w_i = d = (0,0)$ and $\lambda_{m+i} = 0$; if $i \in \{i_1, \dots, i_k\}$, then $w_i = z$ and $\lambda_{m+i} = \lambda_{m+i_1}$ (since there are not changes of types). Thus, by the definition of f, we obtain that

$$\begin{aligned} f(S_l) &= v + \frac{\lambda_1 z_1}{2^1} + \frac{\lambda_2 z_2}{2^2} + \dots + \frac{\lambda_m z_m}{2^m} + \frac{\lambda_{m+i_1} z}{2^{m+i_1}} + \dots + \frac{\lambda_{m+i_1} z}{2^{m+i_l}} \\ &= f(T) + \frac{\lambda_{m+i_1}}{2^m}(\frac{1}{2^{i_1}} + \dots + \frac{1}{2^{i_l}})z. \end{aligned} \tag{4}$$

In particular, if Z is of different type of Z_m, by (a) we have that $f(S) = f(S_k) = f(T) + \frac{\lambda_{m+i_k}}{2^m} rz = f(T) + \frac{\beta \lambda_m}{2^m} rz$ ($\lambda_{m+i_1} = \dots = \lambda_{m+i_k} = \lambda_m \beta$ since there is exactly one change of type from m to $m + i_1$).

Observe that Lemma 2.4 implies that

$$\begin{aligned} f(TS_1) &= f(T(Z_0 Z_1 \dots Z_m W_1 \dots W_{i_1 - 1}) \cup (Z_0 Z_1 \dots Z_m W_1 \dots W_{i_l - 1}) S_1) \\ &= f(T(Z_0 Z_1 \dots Z_m W_1 \dots W_{i_1 - 1})) \cup f((Z_0 Z_1 \dots Z_m W_1 \dots W_{i_l - 1}) S_1) \\ &= \{f(T)\} \cup f(Z_0 Z_1 \dots Z_m W_1 \dots W_{i_l - 1}) f(S_1) = f(T) f(S_1). \end{aligned}$$

By (4), this arc is the set $J_1 = \{f(T) + t \frac{\lambda_{m+i_1} z}{2^m}(\frac{1}{2^{i_1}}) : t \in [0,1]\}$. Similarly, $f(S_1 S_2) = f(S_1) f(S_2)$ and by (4), this arc is the set $J_2 = \{f(T) + \frac{\lambda_{m+i_1} z}{2^m}(\frac{1}{2^{i_1}}) + t \frac{\lambda_{m+i_2} z}{2^m}(\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}}) : t \in [0,1]\}$. Since $J_1 \cap J_2 = \{f(T) + \frac{\lambda_{m+i_1} z}{2^m}(\frac{1}{2^{i_1}})\} = \{f(S_1)\}$, we conclude that $f(TS_2) = f(TS_1) \cup f(S_1 S_2) = J_1 \cup J_2 = f(T) f(S_2)$.

Inductively, the proof of (b) can be completed. □

We have described the elements of $\mathcal{D}_4^*$ in (1) and we defined f with the expression in (3). We see how they are related.

First, we show how to associate a finite sequence of elements of $\mathcal{B}_C$ to an element of the form rz, where $r \in \mathcal{D} \setminus \{0,1\}$ and $z \in \mathcal{B}'_L$. Let $Z \in \mathcal{B}'_C$ be the element associated to z. Suppose that $r = \frac{k}{2^n}$, where k is odd. We write r using dyadic notation, that is, we write $r = \frac{s_1}{2^1} + \dots + \frac{s_n}{2^n}$, where $s_n = 1$ and for each $i \in \{1, \dots, n-1\}$, $s_i \in \{0,1\}$. Observe that $g(r) = n$. We define the sequence $Z_1 \dots Z_n$ by making $Z_i = D$, if $s_i = 0$; and $Z_i = Z$, if $s_i = 1$. Observe that $Z_n = Z$.

Given an element of the form tz, where $t \in (0,1]$ and $z \in \mathcal{B}'_L$, we associate to tz a sequence $Z_1 Z_2 \dots$ of elements in the set $\{D, Z\}$ in a similar way. That is, we start writing $t = \frac{s_1}{2^1} + \cdots$ and we define $Z_i = Z$ if $s_i = 1$, otherwise $Z_i = 0$ ($i \geq 1$). In the case that t has two dyadic representations, we simply choose the finite one (the one with a tail of zeros).

Lemma 2.6. *Let $r \in \mathcal{D} \setminus \{0,1\}$, $z \in \mathcal{B}'_L$ and $Z_1 \dots Z_n$ be the sequence associated to rz. Then $z_n = z$ and $rz = \frac{z_1}{2^1} + \dots + \frac{z_n}{2^n}$.*

Proof. We have observed that $Z_n = Z$, so $z_n = z$. As before, we write $r = \frac{s_1}{2^1} + \dots + \frac{s_n}{2^n}$. Given $i \in \{1, \dots, n\}$, if $s_i = 0$, then $Z_i = D$, so $(0,0) = d = z_i$, and $z_i = 0z = s_i z$; if

$s_i = 1$, then $Z_i = Z$, so $z_i = z = s_i z$. In both cases, $z_i = s_i z$. Therefore $\frac{z_1}{2^1} + \frac{z_2}{2^2} + \cdots + \frac{z_n}{2^n} = \frac{s_1 z}{2^1} + \frac{s_2 z}{2^2} + \cdots + \frac{s_n z}{2^n} = rz$. □

Lemma 2.7. *Let $r_1, \ldots, r_m$ in $\mathcal{D} \setminus \{0, 1\}$ and $z_1, \ldots, z_m$ in $\mathcal{B}'_L$. For each $k \in \{1, \ldots, m\}$, let $Z_1^{(k)} \ldots Z_{j_k}^{(k)}$ be the sequence in $\mathcal{B}_C$ associated to $r_k z_k$. Suppose that for each $k \in \{1, \ldots, m-1\}$, z_{k+1} is of distinct type than z_k. Let $T = Z_0 Z_1^{(1)} \ldots Z_{j_1}^{(1)} \ldots Z_1^{(m)} \ldots Z_{j_m}^{(m)}$. Then*

(a) $f(T) = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \cdots + j_{m-1}}}$, where $j_i = g(z_i)$, for each i,

(b) for each $k \in \{1, \ldots, m\}$, the contribution of the subsequence $Z_1^{(k)} \ldots Z_{j_k}^{(k)}$ to the sum that defines $f(T)$ is the term $\frac{\beta^{k-1} r_k z_k}{2^{j_1 + \cdots + j_{k-1}}}$,

(c) if $\lambda_1, \ldots, \lambda_{j_1 + \cdots + j_m}$ is the sequence associated to the vertex T, then $\lambda_{j_1} = \beta^0$, $\lambda_{j_1 + j_2} = \beta^1, \ldots, \lambda_{j_1 + \cdots + j_m} = \beta^{m-1}$,

(d) the number of terms in the sum that defines $f(T)$ in (3), equivalently, the number of terms in the sequence T, is equal to $j_1 + \cdots + j_m + 1 = g(r_1) + \cdots + g(r_m) + 1$,

(e) let $S = Y_0 Y_1 \ldots Y_n$ be a vertex of X and $R = Y_0 Y_1 \ldots Y_n Z_1^{(1)} \ldots Z_{j_1}^{(1)} \ldots Z_1^{(m)} \ldots Z_{j_m}^{(m)}$. Suppose that Y_n and Z_1 are of distinct type and $Y_n \neq D$. Let $\{\lambda_1, \ldots \lambda_n\}$ be the set of λ's defined for the sequence S and $\gamma = \frac{\beta \lambda_n}{2^n}$. Then

$$f(R) = f(S) + \gamma\left(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \cdots + j_{m-1}}}\right),$$

(f) let S and R be as in (e). Then $f(SR) = f(S) f(R)$.

Proof. Let $i \in \{1, \ldots, j_1\}$. Since $\{Z_1^{(1)}, \ldots, Z_{j_1}^{(1)}\} \subset \{D, Z_1\}$, by definition: $\lambda_i = 0$, if $Z_i^{(1)} = D$; and $\lambda_i = 1$ (there are not changes of types), if $Z_i^{(1)} = Z_1$. In the first case, since $d = (0, 0)$, we conclude that $\frac{\lambda_i z_i^{(1)}}{2^i} = \frac{\lambda_i (0,0)}{2^i} = \frac{z_i^{(1)}}{2^i}$. In the second case, $\frac{\lambda_i z_i^{(1)}}{2^i} = \frac{z_i^{(1)}}{2^i}$. Thus, by Lemma 2.6, $\frac{\lambda_1 z_1^{(1)}}{2^1} + \cdots + \frac{\lambda_{j_1} z_{j_1}^{(1)}}{2^{j_1}} = \frac{z_1^{(1)}}{2^1} + \cdots + \frac{z_{j_1}^{(1)}}{2^{j_1}} = r_1 z_1$.

Given $i \in \{1, \ldots, j_2\}$. Since $\{Z_1^{(2)}, \ldots, Z_{j_2}^{(2)}\} \subset \{D, Z_2\}$, by definition of $f(T)$: $\lambda_{j_1 + i} = 0$, if $Z_i^{(2)} = D$, and $\lambda_{j_1 + i} = \beta$ (there is exactly one change of type), if $Z_i^{(2)} = Z_2$. In the first case, since $d = (0, 0)$, we have that $\frac{\lambda_{j_1+i} z_i^{(2)}}{2^{j_1+i}} = \frac{\lambda_{j_1+i}(0,0)}{2^{j_1+i}} = \frac{\beta z_i^{(2)}}{2^{j_1+i}}$. In the second case, $\frac{\lambda_{j_1+i} z_i^{(2)}}{2^{j_1+i}} = \frac{\beta z_i^{(2)}}{2^{j_1+i}}$. Thus, by Lemma 2.6, $\frac{\lambda_{j_1+1} z_1^{(2)}}{2^{j_1+1}} + \cdots + \frac{\lambda_{j_1+j_2} z_{j_2}^{(2)}}{2^{j_1+j_2}} = \frac{\beta^1}{2^{j_1}}\left(\frac{z_1^{(2)}}{2^1} + \cdots + \frac{z_{j_2}^{(2)}}{2^{j_2}}\right) = \frac{\beta^1 r_2 z_2}{2^{j_1}}$.

The proofs of (a) and (b) can be completed continuing in this way.

Properties (c) and (d) are easy to show.

We prove (e). The case $m = 1$ was proved in Lemma 2.5 (c). We prove the case $m = 2$. Suppose that $\lambda_1, \ldots \lambda_{n+j_1}$ are the λ 's defined for the sequence $Y_1 \ldots Y_n Z_1^{(1)} \ldots Z_{j_1}^{(1)}$. Observe that since each λ_i depends only on the first i terms, $\lambda_1 \ldots \lambda_n$ are the λ's defined for $Y_1 \ldots Y_n$. Since there is exactly one change of type among the terms $Y_n Z_1^{(1)} \ldots Z_{j_1}^{(1)}$, we have that $\lambda_{n+j_1} = \lambda_n \beta$. By Lemma 2.5 (c), $f(Y_0 Y_1 \ldots Y_n Z_1^{(1)} \ldots Z_{j_1}^{(1)} Z_1^{(2)} \ldots Z_{j_2}^{(2)}) = f(Y_0 Y_1 \ldots Y_n Z_1^{(1)} \ldots Z_{j_1}^{(1)}) + \frac{\beta \lambda_{n+j_1}}{2^{n+j_1}} r_2 z_2 = f(S) + \gamma \frac{r_1 z_1}{2^0} + \frac{\beta^2 \lambda_n}{2^{n+j_1}} r_2 z_2 = f(S) + \gamma\left(\frac{r_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}}\right)$. The rest of (e) can be proved in a similar way.

We prove (f) by induction. The case $m = 1$ follows from Lemma 2.5 (b). Now, suppose that (f) holds for $m-1 \geq 1$. Let $R' = Y_0Y_1 \dots Y_n Z_1^{(1)} \dots Z_{j_1}^{(1)} \dots Z_1^{(m-1)} \dots Z_{j_{m-1}}^{(m-1)}$. Using the induction hypothesis and (e), we obtain that

$$\begin{aligned} f(SR) &= f((Y_0Y_1 \dots Y_n)R) = f((Y_0Y_1 \dots Y_n)R' \cup R'R) \\ &= f((Y_0Y_1 \dots Y_n)R') \cup f(R'R) = f(S)f(R') \cup f(R')f(R) \\ &= f(S)(f(S) + \gamma(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}})) \cup \end{aligned}$$

$$(f(S) + \gamma(\tfrac{r_1 z_1}{2^0} + \beta \tfrac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \tfrac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}}))(f(S) + \gamma(\tfrac{r_1 z_1}{2^0} + \beta \tfrac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-1} \tfrac{r_m z_m}{2^{j_1 + \dots + j_{m-1}}})).$$

Observe that the arc in D_4 joining the points $f(S) + \gamma(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}})$ and $f(S) + \gamma(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \dots + j_{m-1}}})$ is the set

$$L = \{f(S) + \gamma(\tfrac{r_1 z_1}{2^0} + \beta \tfrac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \tfrac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}} + t\beta^{m-1} \tfrac{r_m z_m}{2^{j_1 + \dots + j_{m-1}}}) : t \in [0, 1]\} = f(R')f(R),$$

and the intersection of L with the arc $L_0 = f(S)(f(S) + \gamma(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}})) = f(S)f(R')$ is the point $f(S) + \gamma(\frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}}) = f(R')$. Then $L \cup L_0 = f(R')f(R) \cup f(S)f(R')$ is the arc joining $f(S)$ and $f(R)$. Therefore $f(SR) = f(S)f(R)$. $\square$

Lemma 2.8. *$f(X) = D_4$.*

Proof. Let $r_1, \dots, r_m$ in $\mathcal{D} \setminus \{0, 1\}$, $z_1, \dots, z_m$ in $\mathcal{B}'_L$ and for each $k \in \{1, \dots, m-1\}$, z_{k+1} is of distinct type than z_k. By Lemma 2.7, each element $q \in R(D_4)$,

$$q = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \dots + \beta^{m-1} \frac{r_m z_m}{2^{g(r_1) + \dots + g(r_{m-1})}},$$

and any arc vq in D_4 is contained in $\mathrm{Im}(f)$. We obtain that $R(D_4) \subset f(\bigcup_{m=1}^{\infty} X_m) \subset D_4$. Since $X = \mathrm{cl}_X(\bigcup_{m=1}^{\infty} X_m)$ is compact and $R(D_4)$ is dense in D_4, we obtain that $f(X) = D_4$. $\square$

Lemma 2.9. *Let $T = Z_0 Z_1^{(1)} \dots Z_{j_1}^{(1)} \dots Z_1^{(m)} \dots Z_{j_m}^{(m)}$ and*

$$q = f(T) = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \dots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \dots + j_{m-1}}}$$

be as in Lemma 2.7. Let $k = j_1 + \dots + j_m$. Write the sequence T in the form $T = Y_0 Y_1 \dots Y_k$. Let $t \in \mathcal{D} \setminus \{0, 1, r_m\}$ be such that $\frac{1}{2^{g(r_m)}} > |r_m - t|$ and let

$$q_t = v + \frac{r_1 z_1}{2^0} + \dots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \dots + j_{m-2}}} + \beta^{m-1} \frac{t z_m}{2^{j_1 + \dots + j_{m-1}}}.$$

Then there exist $n \in \mathbb{N}$, $Y'_k \in \mathcal{B}'_C$, and $Y_{k+1}, \dots, Y_{k+n} \in \{D, Y'_k\}$ such that Y'_k is of the same type than $Y_k = Z_{j_m}^{(m)}$, and the vertex $T_{k+n} = Y_0 Y_1 \dots Y_k \dots Y_{k+n}$ has the following properties $f(T_{k+n}) = q_t$, $f(TT_{k+n}) = qq_t$, $g(r_m) + n = g(t)$ and $\lambda_{k+n} = \beta^{m-1}$ (where $\lambda_1, \dots, \lambda_{k+n}$ is the sequence defined for the vertex T_{k+n}).

Proof. We suppose that $z_m = a$, the rest of the cases (that is, z_m is one of the points $\{b, c, e\}$) are similar. We consider two cases.

Case 1. $t < r_m$.

We take the dyadic representation of the number $2^{g(r_m)}(r_m - t) \in \mathcal{D} \setminus \{0, 1\}$, to be:

$$2^{g(r_m)}(r_m - t) = \frac{s_1}{2^1} + \cdots + \frac{s_n}{2^n},$$

where $\{s_1, \ldots, s_n\} \subset \{0, 1\}$ and $s_n = 1$.

Since Y'_k is of the same type than Y_k, $t < r_m$ and $z_m = a$, we have that $z_{m+1} = -z_m = b$.

Let $r' = 2^{g(r_m)}(r_m - t)$, $Y_{k+1} \ldots Y_{k+n}$ be the sequence associated to $r'b = r'(-a) = r'(-z_m)$. Then $Y_{k+n} = -Z_m = B$, $\{Y_{k+1}, \ldots, Y_{k+n}\} \subset \{D, B\}$ and

$$T_{k+n} = Y_0 Z_1^{(1)} \ldots Z_{j_1}^{(1)} \ldots Z_1^{(m)} \ldots Z_{j_m}^{(m)} Y_{k+1} \ldots Y_{k+n} = Y_0 Y_1 \ldots Y_k Y_{k+1} \ldots Y_{k+n}.$$

Observe that $g(r') = n$. By Lemma 2.2 (a), $g(r_m) + n = g(r_m) + g(r') = g(r_m - \frac{r'}{2^{g(r_m)}}) = g(t)$. Thus $g(r_m) + n = g(t)$.

Since $\{Y_{k+1}, \ldots, Y_{k+n}\} \subset \{D, B\}$ and $B \in \{Y_{k+1}, \ldots, Y_{k+n}\}$, by Lemma 2.5 (b), we have that $f(TT_{k+n}) = f(T)f(T_{k+n}) = qf(T_{k+n})$. We prove that $f(T_{k+n}) = q_t$.

By definition,

$$f(T_{k+n}) = v + \frac{\lambda_1 y_1}{2^1} + \cdots + \frac{\lambda_k y_k}{2^k} + \frac{\lambda_{k+1} y_{k+1}}{2^{k+1}} + \cdots + \frac{\lambda_{k+n} y_{k+n}}{2^{k+n}}.$$

Since for each $i \in \{1, \ldots, k\}$, the definition of a number λ_i, depends only on the sequence $Y_0 \ldots Y_i$, we have that λ_i also is the one used in the definition of $f(T)$. Then

$$\begin{aligned} f(T) &= v + \frac{\lambda_1 y_1}{2^1} + \cdots + \frac{\lambda_k y_k}{2^k} \\ &= v + \frac{\lambda_1 z_1^{(1)}}{2^1} + \cdots + \frac{\lambda_{j_1} z_{j_1}^{(1)}}{2^{j_1}} + \cdots + \frac{\lambda_{j_1+\cdots+j_{m-1}+1} z_1^{(m)}}{2^{j_1+\cdots+j_{m-1}+1}} + \cdots + \frac{\lambda_{j_1+\cdots+j_m} z_{j_m}^{(m)}}{2^{j_1+\cdots+j_m}}. \end{aligned}$$

By Lemma 2.7 (a) and (c), the last sum is equal to

$$v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1+\cdots+j_{m-1}}}$$

and $\lambda_k = \beta^{m-1}$.

Thus

$$v + \frac{\lambda_1 y_1}{2^1} + \cdots + \frac{\lambda_k y_k}{2^k} = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1+\cdots+j_{m-1}}}.$$

Since y_k and b are of the same type, in fact, $b = -a = -z_m = -z_{j_m}^{(m)} = -y_k$, we have that for each $i \in \{1, \ldots, n\}$, $\beta^{m-1} = \lambda_k = \lambda_{k+i}$, if $Y_{k+i} = B$ (equivalently, $s_i = 1$); and $\lambda_{k+i} = 0$, if $Y_{k+i} = D$ (equivalently, $s_i = 0$). Then $y_{k+i} = s_i b$, and $\lambda_{k+i} y_{k+i} = \lambda_k y_{k+i} = \beta^{m-1} s_i b$. Therefore

$$\begin{aligned} f(T_{k+n}) &= v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \cdots + j_{m-1}}} + \frac{\beta^{m-1} s_1 b}{2^{k+1}} + \cdots + \frac{\beta^{m-1} s_n b}{2^{k+n}} \\ &= v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-1} \frac{r_m z_m}{2^{j_1 + \cdots + j_{m-1}}} - \frac{\beta^{m-1} z_m}{2^{j_1 + \cdots + j_{m-1}} 2^{j_m}} \left(\frac{s_1}{2^1} + \cdots + \frac{s_n}{2^n}\right) \\ &= v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{j_1}} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{j_1 + \cdots + j_{m-2}}} + \beta^{m-1} \frac{z_m}{2^{j_1 + \cdots + j_{m-1}}} \left(r_m - \frac{2^{j_m}(r_m - t)}{2^{j_m}}\right) \\ &= q_t. \end{aligned}$$

Hence, $f(T_{k+n}) = q_t$.

Case 2. $r_m < t$.

The proof in this case is similar to the proof of Case 1, using the dyadic representation of the number $r'' = 2^{g(r_m)}(t - r_m)$ and the sequence associated to $r''a$. □

Theorem 2.10. *The function f is weakly confluent.*

Proof. Take a subcontinuum B of D_4. We are going to show that there exists a subcontinuum A of X such that $f(A) = B$. By 2.8, we suppose that B is non-degenerate. Let $q_0 \in B$ be such that q_0 is the first point in B when we walk from v to B. That is, q_0 is the only point in B with the property that for each $q \in B$, $q_0 \in vq$ (equivalently, $vq_0 \subset vq$). Then $B \neq \{q_0\}$. So q_0 is not an end-point of D_4. So either $q_0 = v$ or q_0 can be written as in (1).

Case A. $q_0 \neq v$.

In this case

$$q_0 = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1) + \cdots + g(r_{m-2})}} + t^* \beta^{m-1} \frac{r_m z_m}{2^{g(r_1) + \cdots + g(r_{m-1})}} \tag{5}$$

where $t^* > 0$.

Let $w = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1) + \cdots + g(r_{m-2})}}$, $t_0 = t^* r_m$ and $z = \beta^{m-1} \frac{z_m}{2^{g(r_1) + \cdots + g(r_{m-1})}}$. Then

$$q_0 = w + t_0 z.$$

Consider the arc $L = \{w + tz \in D_4 : t \in [0, 1]\}$. We know that (see Definition 2.3)

$$vq_0 = \{v\} \cup L_1(q_0) \cup \cdots \cup L_{m-1}(q_0) \cup L_m(q_0).$$

where $L_m(q_0) = \{w + st_0 z : s \in (0, 1]\}$. Then for each $s < 1$, $w + st_0 z \notin B$. Thus $t_0 = \min\{t \in [0, 1] : w + tz \in B\}$. Since $B \cap L$ is a subcontinuum of D_4 there exists $t_2 \in [t_0, 1]$ such that $B \cap L = \{w + tz \in D_4 : t \in [t_0, t_2]\}$.

Case 1. $t_0 < t_2$.

By Lemma 2.2 (b), there exists a unique element $r \in (t_0, t_2) \cap (\mathcal{D} \setminus \{0, 1\})$ with minimum degree. Set

$$q_1 = w + rz.$$

Then
$$q_1 = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \beta^{m-1} \frac{r z_m}{2^{g(r_1)+\cdots+g(r_{m-2})+g(r_{m-1})}}.$$

Since $r \in \mathcal{D}\setminus\{0,1\}$, by Lemma 2.7 (a) and (d), there exist $k \in \mathbb{N}$ and a sequence $Y_0, \ldots, Y_k$ in $\mathcal{B}_C$ such that the vertex

$$T_0 = Y_0 Y_1 \ldots Y_k$$

of X satisfies $f(T_0) = q_1$ and $k = g(r_1) + \cdots + g(r_{m-1}) + g(r)$.

Claim 1. *Let $q \in (B \setminus \{q_0\}) \cap R(D_4)$. Then there exists an arc J_q in X such that $T_0 \in J_q$ and $q \in f(J_q) \subset B$.*

We prove Claim 1. We start writing q as in (2)
$$q = v + \frac{r'_1 z'_1}{2^0} + \beta \frac{r'_2 z'_2}{2^{g(r'_1)}} + \cdots + \beta^{m'-2} \frac{r'_{m'-1} z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1} \frac{r'_{m'} z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}}.$$

Since $q \in R(D_4)$, $r'_{m'} \in \mathcal{D} \setminus \{0,1\}$. Let $L_1(q), \ldots, L_m(q)$ be as in Definition 2.3. Since $\{v\} \cup L_1(q_0) \cup \cdots \cup L_{m-1}(q_0) \subset vq_0 \subset vq$, the uniqueness of arcs in D_4 implies that $L_1(q_0) = L_1(q), \ldots, L_{m-1}(q_0) = L_{m-1}(q)$, $m \leq m'$ and $z_m = z'_m$. Then $r_1 = r'_1, \ldots, r_{m-1} = r'_{m-1}$; and $z_1 = z'_1, \ldots, z_m = z'_m$. Thus

$$\begin{aligned} q &= v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-3} \frac{r_{m-2} z_{m-2}}{2^{g(r_1)+\cdots+g(r_{m-3})}} + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \\ &\quad \beta^{m-1} \frac{r'_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}} + \cdots + \beta^{m'-2} \frac{r'_{m'-1} z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1} \frac{r'_{m'} z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}} \\ &= w + r'_m z + \beta^m \frac{r'_{m+1} z'_{m+1}}{2^{g(r_1)+\cdots+g(r_{m-1})+g(r'_m)}} + \cdots + \beta^{m'-2} \frac{r'_{m'-1} z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1} \frac{r'_{m'} z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}} \end{aligned} \tag{6}$$

For each $i \in \{1, \ldots, m'\}$, let $W_1^{(i)}, \ldots, W_{j_i}^{(i)}$ be the sequence in $\mathcal{B}_C$ associated to $r'_i z'_i$. Let $k'' = g(r_1) + \cdots + g(r_{m'})$. Observe that by Lemma 2.7, if $V_0, \ldots, V_{k''} \in \mathcal{B}_C$ satisfies that the sequence

$$V = V_0 V_1 \ldots V_{k''}$$

is the sequence $V_0 W_1^{(1)} \ldots W_{g(r'_1)}^{(1)} \ldots W_1^{(m')} \ldots W_{g(r'_{m'})}^{(m')}$, then $f(V) = q$. Moreover,

$$V_0 V_1 \ldots V_{g(r'_1)+\cdots+g(r'_m)} = V_0 W_1^{(1)} \ldots W_{g(r'_1)}^{(1)} \ldots W_1^{(m)} \ldots W_{g(r'_m)}^{(m)}.$$

Then

$$V_{g(r'_1)+\cdots+g(r'_m)+1} \ldots V_{k''} = W_1^{(m+1)} \ldots W_{g(r'_{m+1})}^{(m+1)} \ldots W_1^{(m')} \ldots W_{g(r'_{m'})}^{(m')}.$$

Subcase 1.1. *$m < m'$.*

Take the natural order $<$ for the arc vq for which $v < q$. Since $q_0 \in L \cap vq$ and $w + r'_m z$ is the last point of vq in L, we have that $q_0 \leq w + r'_m z \leq q$. Then $w + r'_m z \in q_0 q \cap L \subset B \cap L$. Thus $r'_m \in [t_0, t_2]$ and $w + r'_m z \in B$.

1.1.1. Suppose that $r \neq r'_m$.

If $g(r) > 1$, by Lemma 2.2 (b) we have that $\frac{1}{2^{g(r)}} > \max\{t_2 - r, r - t_0\} \geq |r - r'_m|$; and if $g(r) = 1$, then $r = \frac{1}{2}$, since $r'_m \in (0,1)$, we conclude that $\frac{1}{2^{g(r)}} = \frac{1}{2} > |r'_m - r|$. Thus we can apply Lemma 2.9 to T_0, q_1 and $w + r'_m z$ to obtain that there exist $n \in \mathbb{N}$ and $Y_{k+1}, \dots, Y_{k+n} \in \mathcal{B}_{\mathcal{C}}$, such that the vertex $T_{k+n} = Y_0 Y_1 \dots Y_k \dots Y_{k+n}$ satisfies $f(T_{k+n}) = w + r'_m z$, $f(T_0 T_{k+n}) = q_1(w + r'_m z) = \{w + tz : t$ is in the subinterval of $[0,1]$ joining r and $r'_m\} \subset \{w + tz : t \in [t_0, t_2]\} \subset B$, $g(r) + n = g(r'_m)$ and $\lambda_{k+n} = \beta^{m-1}$ (where $\lambda_1, \dots \lambda_{k+n}$ are defined for the vertex T_{k+n}).

Since $k = g(r_1) + \cdots + g(r_{m-1}) + g(r)$, we obtain $k + n = g(r_1) + \cdots + g(r_{m-1}) + g(r'_m) = g(r'_1) + \cdots + g(r'_{m-1}) + g(r'_m)$. Therefore

$$k + n + 1 = g(r'_1) + \cdots + g(r'_{m-1}) + g(r'_m) + 1$$

Observe that

$$\begin{aligned} f(T_{k+n}) &= w + r'_m z \\ &= v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \beta^{m-1} \frac{r'_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}}. \end{aligned} \tag{7}$$

Note that $f(T_{k+n})$ coincides with the first terms in the equality (6). Define

$$Z^* = Y_0 Y_1 \dots Y_{k+n} V_{k+n+1} \dots V_{k''} = Y_0 Y_1 \dots Y_{k+n} W_1^{(m+1)} \dots W_{j_{m+1}}^{(m+1)} \dots W_1^{(m')} \dots W_{j_{m'}}^{(m')}.$$

We claim that $f(Z^*) = q$, $T_0 \in T_0 Z^*$, $f(T_0 Z^*) \subset B$.

Observe that $z_{m+1} \in \{v_{k+n+1}, \dots, v_{k+n+j_{m+1}}\} \subset \{d, z_{m+1}\}$, $Y_{k+n} = Z_m$ and Z_{m+1} are of different type, $k + n = g(r'_1) + \cdots + g(r'_{m-1}) + g(r'_m)$ and $\lambda_{k+n} = \beta^{m-1}$, by Lemma 2.7 (e) we have that

$$\begin{aligned} f(Z^*) &= f(T_{n+k}) + \beta^m \frac{r'_{m+1} z'_{m+1}}{2^{g(r'_1)+\cdots+g(r'_{m-1})+g(r'_m)}} + \cdots + \beta^{m'-2} \frac{r'_{m'-1} z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1} \frac{r'_{m'} z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}} \\ &= w + r'_m z + \beta^m \frac{r'_{m+1} z'_{m+1}}{2^{g(r_1)+\cdots+g(r_{m-1})+g(r'_m)}} + \cdots + \beta^{m'-2} \frac{r'_{m'-1} z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1} \frac{r'_{m'} z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}}. \end{aligned}$$

Therefore $f(Z^*) = q$. Moreover, by Lemma 2.7 (f), $f(T_{k+n} Z^*) = f(T_{k+n}) f(Z^*)$.

Set $J_q = T_0 Z^*$. Then $T_0 \in J_q$ and $q = f(Z^*) \in f(J_q)$. Since $f(T_{k+n}), f(Z^*) \in B$, we have that $f(J_q) = f(T_0 Z^*) \subset f(T_0 T_{k+n}) \cup f(T_{k+n} Z^*) \subset B \cup f(T_{k+n}) f(Z^*) \subset B$. Therefore $f(J_q) \subset B$. This completes the analysis of the case $r \neq r'_m$.

1.1.2. Suppose that $r = r'_m$.

In this case define $Z^* = Y_0 \dots Y_k W_1^{(m+1)} \dots W_{j_{m+1}}^{(m+1)} \dots W_1^{(m')} \dots W_{j_{m'}}^{(m')}$. Since $f(Y_0 \dots Y_k) = f(T_0) = q_1 = w + rz = w + r'_m$, by Lemma 2.7 (e) $f(Z^*) = f(Y_0 \dots Y_k) + \beta^m \frac{r'_{m+1} z'_{m+1}}{2^{g(r'_1)+\cdots+g(r'_{m-1})+g(r'_m)}} +$

$\cdots + \beta^{m'-2}\frac{r'_{m'-1}z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1}\frac{r'_{m'}z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}} = q$. Hence, $f(Z^*) = q$. Moreover, since $f(T_0) = w + r'_m \in B$, by Lemma 2.7 (f), $f(T_0Z^*) = f(T_0)f(Z^*) \subset B$.

Set $J_q = T_0Z^*$. Then $T_0 \in J_q$, $q = f(Z^*) \in f(J_q) \subset B$.

Subcase 1.2. $m = m'$.

In this subcase,

$$q = v + \frac{r_1z_1}{2^0} + \beta\frac{r_2z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2}\frac{r_{m-1}z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \beta^{m-1}\frac{r'_mz_m}{2^{g(r_1)+\cdots+g(r_{m-1})}}$$
$$= w + r'_mz.$$

In the case that $r \neq r'_m$, $q \in L \cap B$, so $r'_m \in [t_0, t_2]$. As at the beginning of subcase 1.1.1., we conclude that $\frac{1}{2^{g(r)}} > |r'_m - r|$, so we can apply Lemma 2.9 to T_0, q_1 and $w + r'_mz$ to obtain that there exist $M \in \mathbb{N}$ and $Y_{k+1}, \ldots, Y_{k+M} \in \mathcal{B}_C$, such that the vertex $T_{k+M} = Y_0Y_1 \ldots Y_k \ldots Y_{k+M}$ satisfies $f(T_{k+M}) = w + r'_mz = q$ and $f(T_0T_{k+M}) = q_1q = \{w + tz : t$ is in the subinterval of $[0, 1]$ joining r and $r'_m\} \subset \{w + tz : t \in [t_0, t_2]\} \subset B$. Set $S_0 = T_{k+M}$. In the case that $r = r'_m$, we have that $q_1 = q$. Set $S_0 = T_0$. In both cases, $T_0 \in T_0S_0$, $f(T_0S_0) \subset B$ and $q_1q = f(T_0S_0)$. In this case, define $J_q = T_0S_0$.

This completes the proof of Claim 1.

Hence, we have shown that for each $q \in (B \setminus \{q_0\}) \cap R(D_4)$, there exists an arc J_q in X such that $T_0 \in J_q$ and $q \in f(J_q) \subset B$.

Define $A = \mathrm{cl}_X(\bigcup\{J_q : q \in (B \setminus \{q_0\}) \cap R(D_4)\})$. Then A is a subcontinuum of X such that $f(A) \subset B$. Since $(B \setminus \{q_0\}) \cap R(D_4)$ is dense in B, $(B \setminus \{q_0\}) \cap R(D_4) \subset f(A)$ and $f(A)$ is compact, we have that $f(A) = B$.

Case 2. $t_0 = t_2$.

In this case, $B \cap L = \{q_0\}$.

Take an element $q \in (B \setminus \{q_0\}) \cap R(D_4)$. We write q as in (2):

$$q = v + \frac{r'_1z'_1}{2^0} + \beta\frac{r'_2z'_2}{2^{g(r'_1)}} + \cdots + \beta^{m'-2}\frac{r'_{m'-1}z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1}\frac{r'_{m'}z'_{m'}}{2^{g(r'_1)} + \cdots + 2^{g(r'_{m'-1})}}.$$

Since $q_0 \in vq$, proceeding as at the beginning of the proof of Claim 1, we obtain that $m \leq m'$, $r_1 = r'_1, \ldots, r_{m-1} = r'_{m-1}$; and $z_1 = z'_1, \ldots, z_m = z'_m$. Thus

$$q = v + \frac{r_1z_1}{2^0} + \beta\frac{r_2z_2}{2^{g(r_1)}} + \cdots + \beta^{m-2}\frac{r_{m-1}z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \beta^{m-1}\frac{r'_mz_m}{2^{g(r_1)+\cdots+g(r_{m-1})}} +$$
$$\beta^m\frac{r'_{m+1}z'_{m+1}}{2^{g(r'_1)+\cdots+g(r'_m)}} + \cdots + \beta^{m'-2}\frac{r'_{m'-1}z'_{m'-1}}{2^{g(r'_1)+\cdots+g(r'_{m'-2})}} + \beta^{m'-1}\frac{r'_{m'}z'_{m'}}{2^{g(r'_1)+\cdots+g(r'_{m'-1})}}.$$

Let $L_1(q), \ldots, L_{m'}(q)$ be as in Definition 2.3. Since $L \cap (\mathrm{cl}_{D_4}(L_{m+1}(q)) \cup \cdots \cup L_{m'}(q)) = \{w + r'_mz\}$, we have that the first point of the arc vq, going from q to v that belongs to L is $w + r'_mz$. Since $q_0 \in L$, we infer that $w + r'_mz \in q_0q$. Then $w + r'_mz \in L \cap B$. Therefore $q_0 = w + r'_mz = w + t_0z$ and $r'_m = t_0$. In particular, $t_0 \in \mathcal{D}$ and $q_0 \in R(D_4)$.

For each $i \in \{1, \ldots, m'\}$, let $W_1^{(i)}, \ldots, W_{j_i}^{(i)}$ be the sequence in $\mathcal{B}_C$ associated to $r'_i z'_i$. Let $k = j_1 + \cdots + j_m$ and $k' = j_1 + \cdots + j_{m'}$.

Observe that by Lemma 2.7, if $V_0, \ldots, V_{k'} \in \mathcal{B}_C$ satisfies that the sequence $Z = Z_0 \ldots Z_k$ (respectively, $Z' = Z_0 \ldots Z_k \ldots Z_{k'}$) is the sequence $Z_0 W_1^{(1)} \ldots W_{j_1}^{(1)} \ldots W_1^{(m)} \ldots W_{j_m}^{(m)}$ (respectively, $Z_0 W_1^{(1)} \ldots W_{j_1}^{(1)} \ldots W_1^{(m')} \ldots W_{j_{m'}}^{(m')}$) then $f(Z) = q_0$ and $f(Z') = q$. Observe that the sequence $W_1^{(i)}, \ldots, W_{j_m}^{(m)}$ depends on $r'_m z'_m = t_0 z_m$. This implies that the sequence Z depends on $r_1 z_1, \ldots, r_{m-1} z_{m-1}, t_0 z_m$. Thus Z depends only on q_0, therefore Z does not depend on q.

Note that $Z' = Z_0 \ldots Z_k W_1^{(m+1)} \ldots W_{j_{m+1}}^{(m+1)} \ldots W_1^{(m')} \ldots W_{j_{m'}}^{(m')}$. By Lemma 2.7 (f), $f(ZZ') = f(Z) f(Z') = q_0 q \subset B$.

Set $J_q = ZZ'$. Then $Z \in J_q$, $q = f(Z') \in f(J_q) \subset B$. Hence, we have shown that for each $q \in (B \setminus \{q_0\}) \cap R(D_4)$, there exists an arc J_q in X such that $Z \in J_q$ and $q \in f(J_q) \subset B$.

Define $A = \mathrm{cl}_X(\bigcup\{J_q : q \in (B \setminus \{q_0\}) \cap R(D_4)\})$. Then A is a subcontinuum of X such that $f(A) \subset B$. Since $(B \setminus \{q_0\}) \cap R(D_4)$ is dense in B, $(B \setminus \{q_0\}) \cap R(D_4) \subset f(A)$ and $f(A)$ is compact, we have that $f(A) = B$.

This completes the proof of the existence of A in the case that $b_0 \neq v$.

Case B. $q_0 = v$, equivalently, $v \in B$.

Given $q \in (B \setminus \{q_0\}) \cap R(D_4)$, write q as in (2). Then

$$q = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^1} + \cdots + \beta^{m-2} \frac{r_{m-1} z_{m-1}}{2^{g(r_1)+\cdots+g(r_{m-2})}} + \beta^{m-1} \frac{r_m z_m}{2^{g(r_1)+\cdots+g(r_{m-1})}}.$$

For each $k \in \{1, \ldots, m\}$, let $Z_1^{(k)} \ldots Z_{j_k}^{(k)}$ be the sequence in $\mathcal{B}_C$ associated to $r_k z_k$. Let $T_k = Z_1^{(1)} \ldots Z_{j_1}^{(1)} \ldots Z_1^{(k)} \ldots Z_{j_k}^{(k)}$ and

$$q_k = v + \frac{r_1 z_1}{2^0} + \beta \frac{r_2 z_2}{2^1} + \cdots + \beta^{k-1} \frac{r_k z_k}{2^{g(r_1)+\cdots+g(r_{k-1})}}.$$

By Lemma 2.7, $f(T_k) = q_k$.

Let $L_1(q), \ldots, L_m(q)$ be as in Definition 2.3. Then $vq = \{v\} \cup L_1(q) \cup \cdots \cup L_m(q)$. Since $vq \subset B$ and for each $k \in \{1, \ldots m\}$, $q_k \in L_k(q)$, we obtain that $q_k \in B$.

Given $k \in \{1, \ldots, m\}$, since $\{Z_1^{(k)}, \ldots, Z_{j_k}^{(k)}\} \subset \{D, Z_k\}$, we can apply Lemma 2.5 (c), to obtain that $f(T_{k-1} T_k) = f(T_{k-1}) f(T_k) = q_{k-1} q_k \subset B$. Therefore $f(VT_m) = f(VT_1 \cup T_1 T_2 \cup \cdots \cup T_{m-1} T_m) = f(VT_1) \cup f(T_1 T_2) \cup \cdots \cup f(T_{m-1} T_m) \subset B$.

Let $J_q = VT_m$. Then J_q is an arc in X such that $v = f(V) \in f(J_q)$, $q = q_m = f(T_m) \in f(J_q)$ and $f(J_q) \subset B$. Define $A = \mathrm{cl}_X(\bigcup\{J_q : q \in (B \setminus \{q_0\}) \cap R(D_4)\})$. Proceeding as before, we conclude that $f(A) = B$. This finishes the proof that f is weakly confluent. □

3. The Characterization

Theorem 3.1. *Let X be a dendrite such that $E(X)$ is at most countable. Then the Gehman dendrite G_3 is not a weakly confluent image of X.*

Proof. Suppose to the contrary that there exists a weakly confluent map $f : X \to G_3$. Fix a point $v \in G_3$ such that $\mathrm{ord}(v, G_3) = 2$. Recall that, $E(G_3)$ is homeomorphic to the Cantor set [5, p. 21]. Given $q \in E(G_3)$ consider the arc $B_q = vq$. Let A_q be a subcontinumm of X such that $f(A_q) = B_q$. Fix $a_q \in A_q$ such that $f(a_q) = q$. Fix a point $u \in X$. Observe that $X = \bigcup\{ue \subset X : e \in E(X)\}$. Since $R(X)$ and $E(X)$ are at most countable [4, Theorem 1.5 (d)] and $\{a_q \in X : q \in E(G_3)\}$ is uncountable, there exists $e_0 \in E(X)$ such that the set $D = (ue_0 \setminus (R(X) \cup \{u, e_0\})) \cap \{a_q : q \in E(G_3)\}$ is uncountable.

Given $a_q \in D$, since $a_q \notin R(X) \cup \{u, e_0\}$, we have that $A_q \cap ue_0$ is an arc. We identify the arc ue_0 with the interval $[0, 1]$, so we write $A_q \cap ue_0 = [s_q, t_q]$, where $s_q < t_q$. Since D is uncountable, there exists $\varepsilon > 0$ such that $2\varepsilon < t_q - s_q$ for uncountably many points $a_q \in D$. Since $a_q \in [s_q, t_q]$, we may assume that $t_q - a_q > \varepsilon$ for uncountably many points $a_q \in D$. Thus there exist $a_{q_1}, a_{q_2} \in D$ such that $[a_{q_1}, t_{q_1}] \cap [a_{q_2}, t_{q_2}] \neq \emptyset$ and $q_1 \neq q_2$. Thus we may assume that $a_{q_2} \in [a_{q_1}, t_{q_1}]$. Hence $a_{q_2} \in A_{q_1}$, $q_2 = f(a_{q_2}) \in f(A_{q_1}) = B_{q_1} = vq_1$. Therefore $q_2 \in vq_1$, a contradiction. This finishes the proof of the theorem. □

Denote by

$$\mathcal{M}(\mathcal{D}) = \{D : D \text{ is a dendrite and } E \leq_{\mathcal{W}} D \text{ for each dendrite } E\}.$$

Observe that $\mathcal{M}(\mathcal{D})$ denotes the family of dendrites that are maximum elements with respect to the preorder $\leq_{\mathcal{W}}$. By [5, Fact 5.22 and Theorem 5.27], all the universal dendrites D_n ($n \in \mathbb{N} \cup \{\omega\}$) belong to $\mathcal{M}(\mathcal{D})$. By Theorem 2.1, each Gehman dendrite G_n ($n \geq 3$) also belongs to $\mathcal{M}(\mathcal{D})$. In the following theorem we characterize the elements of $\mathcal{M}(\mathcal{D})$.

Theorem 3.2. *A dendrite X belongs to $\mathcal{M}(\mathcal{D})$ if and only if $E(X)$ is uncountable.*

Proof. The necessity is proved in Theorem 3.1. Now, suppose that $E(X)$ is uncountable. By [10, Theorem 1] X contains a dendrite G homeomorphic to G_3. By [5, Theorem 4.16], $G \leq_{\mathcal{M}} X$, so $G_3 \leq_{\mathcal{W}} X$ and $X \in \mathcal{M}(\mathcal{D})$. □

4. Another answer

In [5, Question 5.12], it was asked if the existence of a weakly confluent map from a dendrite X onto a dendrite Y implies the existence of a confluent map from X onto Y. The following example answers this question in the negative.

Example 4.1. *D_3 is a weakly confluent image of G_3, but D_3 is not a confluent image of G_3.*

We show the assertions in Example 4.1. By Theorem 2.1, there exists a weakly confluent map from G_3 onto D_3. In order to show that D_3 is not the confluent image of G_3, suppose to the contrary that $D_3 \leq_{\mathcal{C}} G_3$. By [5, Corollary 5.7], $D_3 \leq_{\mathcal{M}} G_3$. Since $G_3 \leq_{\mathcal{M}} D_3$ [3, Corollary 6.5], $G_3 \simeq_{\mathcal{M}} D_3$. By [5, Theorem 5.27], G_3 contains a copy of the dendrite L_0 constructed in [5, 5.6, p. 16]. This is a contradiction since L_0 contains sequences of ramification points converging to points of order ≥ 2 and, in G_3, each limit of ramification points is an end-point. Therefore, D_3 is not a confluent image of G_3.

A simpler example that answers Question 5.12 in [5], is the following. Let

$$X = ([-1, 1] \times \{0\}) \cup (\bigcup(\{\tfrac{1}{n}\} \times [0, \tfrac{1}{n}] : n \in \mathbb{N}\}).$$

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We can prove that X is a dendrite such that X is a weakly confluent image of G_3, but X is not a confluent image of G_3.

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