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SOME OSCILLATION CRITERIA FOR NONLOCAL FRACTIONAL PROPORTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the oscillation of a class of generalized proportional fractional integrodifferential equations with forcing term. We present sufficient conditions to prove some oscillation criteria in both of the Riemann-Liouville and Caputo cases. Besides, we present some numerical examples for applicability of our results.

1. Introduction

Fractional calculus, dealing with derivatives and integrals to an arbitrary order, has been applied successfully in the modelling of many problems in science and engineering. For the advantages of the fractional differential equations over the models of integer order, we refer the reader to [1, 2]. Up to a recent time, when we take into account the high importance of oscillation theory, the number of published works about fractional differential and difference equations is still limited, see for example [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. There is also very little research on the oscillation theory of fractional integro-differential equations, see [5, 6, 7, 8]. This work investigates the oscillatory behavior of solutions to a fractional Volterra integro-differential equation using the method introduced in [5]. We believe that our study will inspire further research on fractional integro-differential equations.

To the best of our knowledge, the results of Grace et al. in [9] are considered as the first about the study of oscillation theory for fractional differential equations, the results in [16] are the first in the frame of discrete fractional calculus, and the article in [17] is the first in the q-fractional case.

In [4], Jarad et al. have introduced a nonlocal fractional proportional derivative or generalized proportional fractional (GPF) derivatives in the both Riemann-Liouville and Caputo senses. The GPF derivatives and integrals possess kernels involving exponential functions. The advantage of such newly defined derivatives is that their corresponding proportional fractional integrals possess a semi-group property in the fractional index α used to replace the iterated number n, and they result in the existing Riemann-Liouville and Caputo fractional derivatives for the particular case $\nu = 1$.

In this paper, motivated by [5], we study the oscillation of GPF integro-differential equation of the form

$$\begin{cases} D_a^{\alpha,\nu} x(t) = r(t) - \int_a^t \Psi(t,s)\Lambda(s,x(s))\,ds,\ t \geq a \geq 0, 0 < \alpha < 1, 0 < \nu \leq 1, \\ \lim_{t\to a^+}\ I_a^{1-\alpha,\nu} x(t) = b_1\,, \end{cases} \tag{1}$$

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where r (the forcing term), Ψ, and Λ are continuous functions, $b_1 \in \mathbb{R}$, and $D_a^{\alpha,\nu}$ and $I_a^{1-\alpha,\nu}$ denotes the left GPF derivative and integral operators in the Riemann-Liouville setting, respectively.

Throughout this article, we only consider those solutions of Eq. (1) which are nontrivial and continuable in any neighborhood of infinity. Such a solution is said to be oscillatory if it has arbitrarily large zeros on $(0,\infty)$; otherwise, it is called nonoscillatory. Eq. (1) itself is said to be oscillatory if all of its solutions are oscillatory. Or simply, the solution is called nonoscillatory, if it does not change its sign after some time.

2. Preliminaries

In this section, we recall some definitions and essential lemmas that will be used to proceed in proving the main results in this paper.

Definition 2.1. [4] For $\nu \in (0,1]$, $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, the left GPF integral of θ is defined by

$$I_a^{\alpha,\nu}\theta(t) := \frac{1}{\nu^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}\theta(s)ds = \nu^{-\alpha} e^{\frac{\nu-1}{\nu}t}\, I_a^\alpha\left(e^{\frac{1-\nu}{\nu}t}\theta(t)\right), \tag{2}$$

where I_a^α is the Riemann-Liouville fractional integral operator (see [1]).

Definition 2.2. [4] For $\nu \in (0,1]$, $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$, the left GPF derivative of Riemann-Liouville type of θ of order α is defined by

$$\begin{aligned} D_a^{\alpha,\nu}\theta(t) &:= D^{n,\nu} I_a^{n-\alpha,\nu}\theta(t) \\ &= \frac{D_t^{n,\nu}}{\nu^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{n-\alpha-1}\theta(s)ds, \end{aligned} \tag{3}$$

where $n = [Re(\alpha)] + 1$.

Definition 2.3. [4] For $\nu \in (0,1]$, $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$, the left derivative of Caputo type of θ of order α is defined by

$${}^C D_a^{\alpha,\nu}\theta(t) := \frac{1}{\nu^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{n-\alpha-1} D^{n,\nu}\theta(s)ds, \tag{4}$$

where $n = [Re(\alpha)] + 1$.

Lemma 2.4. [4] *Let $Re(\alpha) > 0$, $n = -[-Re(\alpha)]$, $\theta \in L_1(a,b)$, $I_a^{\alpha,\nu}\theta(t) \in AC^n[a,b]$, and $\nu \in (0,1]$. Then*

$$I_a^{\alpha,\nu} D_a^{\alpha,\nu}\theta(t) = \theta(t) - e^{\frac{\nu-1}{\nu}(t-a)} \sum_{j=1}^{n} \frac{(I_a^{j-\alpha,\nu}\theta)(a^+)}{\nu^{\alpha-j}\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}. \tag{5}$$

Lemma 2.5. [4] *For $\nu \in (0,1]$ and $n = [Re(\alpha)] + 1$, we have*

$$I_a^{\alpha,\nu}\, {}^C D_a^{\alpha,\nu}\theta(t) = \theta(t) - \sum_{j=0}^{n-1} \frac{D^{j,\nu}\theta(a)}{\nu^j j!}(t-a)^j e^{\frac{\nu-1}{\nu}(t-a)}. \tag{6}$$

Proposition 2.6. [4] Let $\alpha,\rho \in \mathbb{C}$ such that $Re(\alpha) \geq 0$ and $Re(\rho) > 0$. Then, for any $\nu \in (0,1]$, we have

(a)

$$I_a^{\alpha,\nu}(e^{\frac{\nu-1}{\nu}t}(t-a)^{\rho-1}) = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)\nu^{\alpha}}e^{\frac{\nu-1}{\nu}t}(t-a)^{\alpha+\rho-1},\ Re(\alpha) > 0, \tag{7}$$

(b)

$$D_a^{\alpha,\nu}(e^{\frac{\nu-1}{\nu}t}(t-a)^{\rho-1}) = \frac{\nu^{\alpha}\Gamma(\rho)}{\Gamma(\rho-\alpha)}e^{\frac{\nu-1}{\nu}t}(t-a)^{\rho-1-\alpha},\ Re(\alpha) \geq 0. \tag{8}$$

Proposition 2.7. [4] Let $\alpha,\rho \in \mathbb{C}$ such that $Re(\alpha) > 0$ and $Re(\rho) > 0$. Then, for any $\nu \in (0,1]$ and $n = [Re(\alpha)]+1$, we have

$${}^{C}D_a^{\alpha,\nu}(e^{\frac{\nu-1}{\nu}t}(t-a)^{\rho-1}) = \frac{\nu^{\alpha}\Gamma(\rho)}{\Gamma(\rho-\alpha)}e^{\frac{\nu-1}{\nu}t}(t-a)^{\rho-1-\alpha},\ Re(\rho) > n. \tag{9}$$

Lemma 2.8. [3] *If S and T are nonnegative, then*

$$S^{\sigma} + (\sigma-1)T^{\sigma} - \sigma S T^{\sigma-1} \geq 0,\ \sigma > 1, \tag{10}$$

and

$$S^{\sigma} - (1-\sigma)T^{\sigma} - \sigma S T^{\sigma-1} \leq 0,\ \sigma < 1, \tag{11}$$

with equality holds if and only if $S = T$.

3. Oscillation criteria for the GPF integro-differential equations in the Riemann-Liouville setting

Throughout this paper, we assume that the following conditions are satisfied without further mention:

(O1) $r : (a,\infty) \to \mathbb{R}, \Psi : (a,\infty)\times(a,\infty) \to \mathbb{R}$ are continuous with $\Psi(t,s) \geq 0$ for $t > s$;

(O2) there exist $\xi_1,\xi_2 : (a,\infty) \to [0,\infty)$, which are continuous functions such that $\Psi(t,s) \leq \xi_1(t)\xi_2(s)$ for all $t \geq s$;

(O3) $\Lambda : (a,\infty)\times\mathbb{R} \to \mathbb{R}$ with $\Lambda(t,x) := g_1(t,x) - g_2(t,x)$ is continuous such that $g_1,g_2 : (a,\infty)\times\mathbb{R} \to \mathbb{R}$ are continuous and that $x g_i(t,x) > 0$, $(i = 1,2)$ for $t \geq a$ and $x \neq 0$;

(O4) there exist real constants ρ,ε and $q_1,q_2 : (a,\infty) \to (0,\infty)$ continuous such that

$$g_1(t,x) \geq q_1(t)x^{\rho} \text{ and } g_2(t,x) \leq q_2(t)x^{\varepsilon}, t \geq a\ ,\ x \neq 0.$$

Theorem 3.1. *Assume that conditions (O1)-(O3) are satisfied with $g_2 = 0$. If for every constant $k > 0$*

$$\limsup_{t\to\infty} I_a^{\alpha,\nu}[r(t) - k\xi_1(t)] = \infty \tag{12}$$

and

$$\liminf_{t\to\infty} I_a^{\alpha,\nu}[r(t) + k\xi_1(t)] = -\infty, \tag{13}$$

then every solution of Eq. (1) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Eq. (1) with $g_2 = 0$. Without loss of generality, let's say that $x(t) > 0$ for $t \geq T_1$ for some sufficiently large $T_1 > a$. Hence, (O3) implies that $g_1(t,x(t)) > 0$ for $t \geq T_1$. Now, from Eq. (1), we have

$$\begin{aligned} D_a^{\alpha,\nu}x(t) &= r(t) - \int_a^t \Psi(t,s)\Lambda(s,x(s))\,ds \\ &= r(t) - \int_a^{T_1} \Psi(t,s)g_1(s,x(s))\,ds - \int_{T_1}^t \Psi(t,s)g_1(s,x(s))\,ds. \end{aligned} \tag{14}$$

Letting $\kappa := \min\{\Lambda(t,x(t)) : t \in [a,T_1]\} \leq 0$ and $k := -\kappa \int_a^{T_1} \xi_2(s)\,ds \geq 0$, it follows from (14) that

$$D_a^{\alpha,\nu}x(t) \leq r(t) + k\xi_1(t).$$

Using the monotonicity property of $I_a^{\alpha,\nu}$, we see that

$$I_a^{\alpha,\nu}\, D_a^{\alpha,\nu}x(t) \leq I_a^{\alpha,\nu}[r(t) + k\xi_1(t)],$$

and hence, from (5),

$$x(t) \leq \frac{b_1}{\nu^{\alpha-1}\Gamma(\alpha)} e^{\frac{\nu-1}{\nu}(t-a)}(t-a)^{\alpha-1} + I_a^{\alpha,\nu}[r(t) + k\xi_1(t)]. \tag{15}$$

In view of (13), it follows from (15) that

$$\liminf_{t\to\infty} x(t) = -\infty,$$

which contradicts the assumption that $x(t) > 0$ eventually. The proof is similar if $x(t)$ is eventually negative. $\square$

Theorem 3.2. *Assume that conditions (O1)-(O4) are satisfied with $\rho > 1$ and $\varepsilon = 1$. If further, in addition to the conditions presented in Theorem 3.1, we assume that*

$$\int_a^\infty \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^\alpha\Gamma(\alpha)} \int_a^s \Psi(s,u)q_1^{\frac{1}{1-\rho}}(u)q_2^{\frac{\rho}{\rho-1}}(u)\,du\,ds < \infty, \tag{16}$$

then every solution of Eq. (1) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Eq. (1) with $x(t) > 0$ for $t \geq T_1$. From conditions (O3)-(O4) with $\rho > 1$ and $\varepsilon = 1$, we have

$$D_a^{\alpha,\nu}x(t) \leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)[q_2(s)x(s) - q_1(s)x^\rho(s)]\,ds,$$

for some $k > 0$. If in (10), we let $\sigma = \rho$, $S = q_1^{\frac{1}{\rho}}x$, and $T = \left(\frac{1}{\rho}q_2 q_1^{\frac{-1}{\rho}}\right)^{\frac{1}{\rho-1}}$, then we get

$$q_2 x - q_1 x^\rho \leq (\rho-1)\rho^{\frac{\rho}{1-\rho}} q_1^{\frac{1}{1-\rho}} q_2^{\frac{\rho}{\rho-1}}, \tag{17}$$

and hence

$$D_a^{\alpha,\nu}x(t) \leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)(\rho-1)\rho^{\frac{\rho}{1-\rho}} q_1^{\frac{1}{1-\rho}}(s)q_2^{\frac{\rho}{\rho-1}}(s)\,ds. \tag{18}$$

Applying the operator $I_a^{\alpha,\nu}$ to (18), we see that

$$\begin{aligned} x(t) &\leq \frac{b_1}{\nu^{\alpha-1}\Gamma(\alpha)} e^{\frac{\nu-1}{\nu}(t-a)}(t-a)^{\alpha-1} + I_a^{\alpha,\nu}[r(t)+k\xi_1(t)] \\ &+ \int_a^t \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^{\alpha}\Gamma(\alpha)} \int_{T_1}^s \Psi(s,u)(\rho-1)\rho^{\frac{\rho}{1-\rho}} q_1^{\frac{1}{1-\rho}}(u) q_2^{\frac{\rho}{\rho-1}}(u)\,duds. \end{aligned}$$

(19)

By applying the limit inferior on both sides of (19) as $t \to \infty$, and using (13) and (16), we get

$$\liminf_{t\to\infty} x(t) = -\infty,$$

which contradicts the assumption that $x(t) > 0$ eventually. This completes the proof. □

Theorem 3.3. *Assume that conditions (O1)-(O4) are satisfied with $\rho = 1$ and $\varepsilon < 1$. Further, if in addition to the conditions of Theorem 3.1, we suppose that*

$$\int_a^{\infty} \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^{\alpha}\Gamma(\alpha)} \int_a^s \Psi(s,u) q_1^{\frac{\varepsilon}{\varepsilon-1}}(u) q_2^{\frac{1}{1-\varepsilon}}(u)\,duds < \infty, \tag{20}$$

then every solution of Eq. (1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq. (1). Say that $x(t) > 0$ for $t \geq T_1$. From conditions (O3)-(O4) with $\rho = 1$ and $\varepsilon < 1$, we have

$$D_a^{\alpha,\nu} x(t) \leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)[q_2(s)x^{\varepsilon}(s) - q_1(s)x(s)]\,ds,$$

for some $k > 0$. If we take in (11), $\sigma = \varepsilon$, $S = q_2^{\frac{1}{\varepsilon}} x$, and $T = \left(\frac{1}{\varepsilon} q_1 q_2^{\frac{-1}{\varepsilon}}\right)^{\frac{1}{\varepsilon-1}}$, then we get

$$q_2 x^{\varepsilon} - q_1 x \leq (1-\varepsilon)\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} q_1^{\frac{\varepsilon}{\varepsilon-1}} q_2^{\frac{1}{1-\varepsilon}}, \tag{21}$$

and hence

$$D_a^{\alpha,\nu} x(t) \leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)(1-\varepsilon)\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} q_1^{\frac{\varepsilon}{\varepsilon-1}}(s) q_2^{\frac{1}{1-\varepsilon}}(s)\,ds.$$

The rest of the proof is similar to that in Theorem 3.2, and hence we omit it. □

Theorem 3.4. *Assume that conditions (O1)-(O4) are satisfied with $\rho > 1$ and $\varepsilon < 1$. Further, if in addition to the conditions of Theorem 3.1, we assume that there exists a continuous function $\varsigma : \mathbb{R} \to (0,\infty)$ such that*

$$\int_a^{\infty} \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^{\alpha}\Gamma(\alpha)} \int_a^s \Psi(s,u) q_1^{\frac{1}{1-\rho}}(u) \varsigma^{\frac{\rho}{\rho-1}}(u)\,duds < \infty \tag{22}$$

and

$$\int_a^{\infty} \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^{\alpha}\Gamma(\alpha)} \int_a^s \Psi(s,u) \varsigma^{\frac{\varepsilon}{\varepsilon-1}}(u) q_2^{\frac{1}{1-\varepsilon}}(u)\,duds < \infty, \tag{23}$$

then every solution of Eq. (1) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Eq. (1) with $x(t) > 0$ for $t \geq T_1$. Using the same procedure as above, from conditions (O3)-(O4) with $\rho > 1$ and $\varepsilon < 1$, we see that

$$\begin{aligned} D_a^{\alpha,\nu}x(t) &\leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)[\varsigma(s)x(s) - q_1(s)x^\rho(s)]\,ds \\ &+ \int_{T_1}^t \Psi(t,s)[q_2(s)x^\varepsilon(s) - \varsigma(s)x(s)]\,ds, \end{aligned}$$

for some $k > 0$. Taking $q_2(s) = \varsigma(s)$ in (17) and $q_1(s) = \varsigma(s)$ in (21), we get

$$\begin{aligned} D_a^{\alpha,\nu}x(t) &\leq r(t) + k\xi_1(t) + \int_{T_1}^t \Psi(t,s)(\rho-1)\rho^{\frac{\rho}{1-\rho}} q_1^{\frac{1}{1-\rho}}(s)\varsigma^{\frac{\rho}{\rho-1}}(s)\,ds \\ &+ \int_{T_1}^t \Psi(t,s)(1-\varepsilon)\varepsilon^{\frac{\varepsilon}{1-\varepsilon}}\varsigma^{\frac{\varepsilon}{\varepsilon-1}}(s)q_2^{\frac{1}{1-\varepsilon}}(s)\,ds. \end{aligned}$$

The rest of the proof is similar to that in Theorem 3.2. $\square$

The following example clarifies Theorem 3.1.

Example 3.5. Consider the integro-differential equation with Riemann-Liouville GPF derivative

$$\left\{\begin{array}{l} D_0^{1/3,1/2}x(t) = \dfrac{e^{-t}t^{2/3}}{\sqrt[3]{2}\,\Gamma(5/3)} - (t^3+2t^2+2t)e^{-t} + 2t - t\displaystyle\int_0^t sx(s)\,ds, \\ \lim_{t\to 0^+} I_0^{2/3,1/2}x(t) = 0. \end{array}\right. \tag{24}$$

Comparing with Eq. (1) with $g_2 = 0$, we have

$$\alpha = \frac{1}{3}, \nu = \frac{1}{2}, a = b_1 = 0, g_1(t,x) = x, r(t) = \frac{e^{-t}t^{2/3}}{\sqrt[3]{2}\,\Gamma(5/3)} - (t^3+2t^2+2t)e^{-t} + 2t, \Psi(t,s) = ts.$$

Conditions (O1)–(O3) are satisfied and condition (13) does not hold. We have

$$r(t) \geq \frac{e^{-t}t^{2/3}}{\sqrt[3]{2}\,\Gamma(5/3)} - (t^3+2t^2+2t)e^{-t},\ t \geq 0. \tag{25}$$

Applying the operator $I_0^{1/3,1/2}$ to (25), we see that

$$I_0^{1/3,1/2}r(t) \geq te^{-t} - \frac{6\sqrt[3]{2}}{\Gamma(13/3)}t^{10/3}e^{-t} - \frac{4\sqrt[3]{2}}{\Gamma(10/3)}t^{7/3}e^{-t} - \frac{2\sqrt[3]{2}}{\Gamma(7/3)}t^{4/3}e^{-t}. \tag{26}$$

Taking limit inferior on both sides of (26) as $t \to \infty$, one can easily see that the right hand side is zero, so we get

$$\liminf_{t\to\infty} I_0^{1/3,1/2}r(t) \geq 0. \tag{27}$$

Using Proposition 2.6 (b), it is easy to verify that $x(t) = te^{-t}$ is a nonoscillatory solution of Eq. (24). Here,

$$\lim_{t\to 0^+} I_0^{2/3,1/2}(e^{-t}t) = \lim_{t\to 0^+} \frac{\sqrt[3]{4}}{\Gamma(8/3)}t^{5/3}e^{-t} = 0.$$

Note that here $\kappa = k = 0$.

The following example clarifies Theorem 3.2.

Example 3.6. Consider the integro-differential equation with Riemann-Liouville GPF derivative

(28)
$$\begin{cases} D_0^{1/2,1/2}x(t) = \dfrac{4\sqrt{2}}{3\sqrt{\pi}}e^{-t}t^{3/2} + t\left(4+e^{-t}\left(-4-t\left(4+t\left(2+t\right)\right)\right)\right) - t\displaystyle\int_0^t s\left[x(s) - \frac{x(s)}{s}\right]ds, \\ \lim_{t\to 0^+} I_0^{1/2,1/2}x(t) = 0\,. \end{cases}$$

Comparing with Eq. (1), we have

$$\alpha = \nu = \frac{1}{2}, a = b_1 = 0, g_1(t,x) = x, g_2(t,x) = \frac{x}{t},$$

$$r(t) = \frac{4\sqrt{2}}{3\sqrt{\pi}}e^{-t}t^{3/2} + t\left(4+e^{-t}\left(-4-t\left(4+t\left(2+t\right)\right)\right)\right),$$

$$\Psi(t,s) = ts.$$

Conditions (O1)-(O4) are satisfied with $\varepsilon = 1$, $\rho = 2$ and $q_1(t) = t^{-3}$, $q_2(t) = t$. However, condition (16) is not satisfied since

$$\lim_{b\to\infty}\int_0^b \sqrt{\frac{2}{\pi}}\frac{e^{s-t}}{\sqrt{t-s}}\left(\int_0^s su^4du\right)ds = \lim_{b\to\infty}\frac{\sqrt{\frac{2}{\pi}}}{5}\int_0^b \frac{s^6e^{s-t}}{\sqrt{t-s}}ds = \infty.$$

Using Proposition 2.6 (b), it is easy to verify that $x(t) = t^2e^{-t}$ is a nonoscillatory solution of Eq. (28). Here,

$$\lim_{t\to 0^+} I_0^{1/2,1/2}(t^2e^{-t}) = \frac{16\sqrt{2}}{15\sqrt{\pi}}\lim_{t\to 0^+} t^{5/2}e^{-t} = 0.$$

4. Oscillation criteria for the GPF integro-differential equations in the Caputo setting

In this section, we study the oscillation of the GPF integro-differential equations in the Caputo setting of the form

(29)
$$\begin{cases} {}^CD_a^{\alpha,\nu}x(t) = r(t) - \int_a^t \Psi(t,s)\Lambda(s,x(s))\,ds, \\ D^{k,\nu}x(a) = b_k \in \mathbb{R},\ k = 0,1,\dots,n-1, \end{cases}$$

where $n = \lceil\alpha\rceil$, ${}^CD_a^{\alpha,\nu}$ is defined by Eq. (4), $D^{k,\nu} = \underbrace{D^\nu D^\nu \dots D^\nu}_{k\text{-times}}$, and D^ν is the proportional derivative.

Below, we provide corresponding results for Eq. (29). Since the arguments resemble the case of Riemann-Liouville, we will only prove the first of the following theorems.

Theorem 4.1. *Assume that conditions (O1)-(O3) are satisfied with $g_2 = 0$. If for every constant $k > 0$*

(30)
$$\limsup_{t\to\infty} t^{1-n}I_a^{\alpha,\nu}[r(t) - k\xi_1(t)] = \infty$$

and

(31)
$$\liminf_{t\to\infty} t^{1-n}I_a^{\alpha,\nu}[r(t) + k\xi_1(t)] = -\infty,$$

then every solution of Eq. (29) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Eq. (29) with $g_2 = 0$. Without loss of generality, assume that $x(t) > 0$ for $t \geq T_1$. Proceeding as in the proof of Theorem 3.1, we get

$$^{C}D_a^{\alpha,\nu}x(t) \leq r(t) + k\xi_1(t). \tag{32}$$

Applying the operator $I_a^{\alpha,\nu}$ to (32), we see from (6) that

$$\begin{aligned} t^{1-n}x(t) &\leq t^{1-n}e^{\frac{\nu-1}{\nu}(t-a)}\sum_{j=0}^{n-1}\frac{D^{j,\nu}x(a)}{\nu^j j!}(t-a)^j + t^{1-n}\ I_a^{\alpha,\nu}[r(t)+k\xi_1(t)] \\ &\leq e^{\frac{\nu-1}{\nu}(t-a)}\left(\frac{t-a}{t}\right)^{n-1}\sum_{j=0}^{n-1}\frac{|D^{j,\nu}x(a)|}{\nu^j j!}(t-a)^{j-n+1} + t^{1-n}\ I_a^{\alpha,\nu}[r(t)+k\xi_1(t)] \\ &\leq \sum_{j=0}^{n-1}\frac{|D^{j,\nu}x(a)|}{\nu^j j!}(T_2-a)^{j-n+1} + t^{1-n}\ I_a^{\alpha,\nu}[r(t)+k\xi_1(t)], \quad t \geq T_2 > T_1. \end{aligned} \tag{33}$$

Now, from (31), it follows that

$$\liminf_{t\to\infty} t^{1-n}x(t) = -\infty,$$

which is a contradiction to that $x(t) > 0$ eventually. Hence, the proof is complete. □

Theorem 4.2. *Assume that conditions (O1)-(O4) are satisfied with $\rho > 1$ and $\varepsilon = 1$. In addition to the conditions of Theorem 4.1, if*

$$\lim_{t\to\infty} t^{1-n}\int_a^t \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^\alpha\Gamma(\alpha)}\int_a^s \Psi(s,u)q_1^{\frac{1}{1-\rho}}(u)q_2^{\frac{\rho}{\rho-1}}(u)\,duds < \infty, \tag{34}$$

then every solution of Eq. (29) is oscillatory.

Theorem 4.3. *Assume that conditions (O1)-(O4) are satisfied with $\rho = 1$ and $\varepsilon < 1$. In addition to the conditions of Theorem 4.1, if*

$$\lim_{t\to\infty} t^{1-n}\int_a^t \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^\alpha\Gamma(\alpha)}\int_a^s \Psi(s,u)q_1^{\frac{\varepsilon}{\varepsilon-1}}(u)q_2^{\frac{1}{1-\varepsilon}}(u)\,duds < \infty, \tag{35}$$

then every solution of Eq. (29) is oscillatory.

Theorem 4.4. *Assume that conditions (O1)-(O4) are satisfied with $\rho > 1$ and $\varepsilon < 1$. In addition to the conditions of Theorem 4.1, assume that there exists a continuous function $\varsigma : \mathbb{R} \to (0,\infty)$ such that*

$$\lim_{t\to\infty} t^{1-n}\int_a^t \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^\alpha\Gamma(\alpha)}\int_a^s \Psi(s,u)q_1^{\frac{1}{1-\rho}}(u)\varsigma^{\frac{\rho}{\rho-1}}(u)\,duds < \infty \tag{36}$$

and

$$\lim_{t\to\infty} t^{1-n}\int_a^t \frac{e^{\frac{\nu-1}{\nu}(t-s)}(t-s)^{\alpha-1}}{\nu^\alpha\Gamma(\alpha)}\int_a^s \Psi(s,u)\varsigma^{\frac{\varepsilon}{\varepsilon-1}}(u)q_2^{\frac{1}{1-\varepsilon}}(u)\,duds < \infty, \tag{37}$$

then every solution of Eq. (29) is oscillatory.

The following example clarifies Theorem 4.1.

Example 4.5. Consider the integro-differential equation with Caputo GPF derivative

$$\begin{cases} {}^{C}D_0^{3/2,1/2}x(t) = \sqrt{\dfrac{2}{\pi}}e^{-t}\sqrt{t} - (t^4+3t^3+6t^2+6t)e^{-t} + 6t - t\displaystyle\int_0^t sx(s)\,ds, \\ x(0)=0, x'(0)=0. \end{cases} \tag{38}$$

Comparing with Eq. (29) with $g_2 = 0$, we have

$$\alpha = \frac{3}{2}, \nu = \frac{1}{2}, a = b_0 = b_1 = 0, r(t) = \sqrt{\frac{2}{\pi}}e^{-t}\sqrt{t} - (t^4+3t^3+6t^2+6t)e^{-t} + 6t,$$

$g_1(t,x) = x$, and $\Psi(t,s) = ts$. Conditions (O1)–(O3) are satisfied, and condition (31) does not hold. We have

$$r(t) \geq \sqrt{\frac{2}{\pi}}e^{-t}\sqrt{t} - (t^4+3t^3+6t^2+6t)e^{-t},\ t \geq 0. \tag{39}$$

Applying the operator $I_0^{3/2,1/2}$ to (39), we see that

$$\begin{aligned} I_0^{3/2,1/2}r(t) &\geq t^2e^{-t} - \frac{48\sqrt{2}}{\Gamma(13/2)}t^{11/2}e^{-t} - \frac{36\sqrt{2}}{\Gamma(11/2)}t^{9/2}e^{-t} - \frac{24\sqrt{2}}{\Gamma(9/2)}t^{7/2}e^{-t} - \frac{12\sqrt{2}}{\Gamma(7/2)}t^{5/2}e^{-t} \\ &= -\frac{\sqrt{\frac{2}{\pi}}e^{-t}t^2\left(1024t^{7/2}+4224t^{5/2}+12672t^{3/2}+22176t^{1/2}-3465\right)}{3465}, \end{aligned}$$

and hence

$$t^{-1}I_0^{3/2,1/2}r(t) \geq -\frac{\sqrt{\frac{2}{\pi}}e^{-t}t\left(1024t^{7/2}+4224t^{5/2}+12672t^{3/2}+22176t^{1/2}-3465\right)}{3465}. \tag{40}$$

If we apply limit inferior on both sides of (40) as $t \to \infty$, then we get

$$\liminf_{t\to\infty} t^{-1}I_0^{3/2,1/2}r(t) \geq 0. \tag{41}$$

Using Proposition 2.7, one can easily prove that $x(t) = t^2e^{-t}$ is a nonoscillatory solution of Eq. (38).

5. Conclusion

Local fractional proportional derivatives, say of order $\nu \in [0,1]$, were used in [4] to generate nonlocal fractional proportional derivatives by adding a second index α instead of the number n which represents the number of iterations in the fractionalizing process. The produced nonlocal fractional proportional operator ${}_aD^{\alpha,\nu}$, either in the Riemann-Liouville or the Caputo sense, includes the exponential function in the kernel. In this work, we have investigated and analyzed such a kernel to study the oscillation of certain nonlocal fractional proportional integro-differential equations. The case in which $\nu = 1$ reduces to the Caputo and Riemann-Liouville fractional operator ones and hence the results in [5] are recovered. We have presented some examples to illustrate the applicability of our results.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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