On Entire Solutions of Non-linear Binomial Differential Equations *

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Abstract: In this paper, we analyze the growth behavior of entire solutions of the non-linear binomial differential equation

$$Af^{m_0}(f')^{m_1}(f'')^{m_2}\cdots(f^{(p)})^{m_p}+Bf^{n_0}(f')^{n_1}(f'')^{n_2}\cdots(f^{(k)})^{n_k}=H,$$

where A, B are polynomials and H is an entire function. By applying this result and Cartan's second main theorem, we obtain the zero distribution of entire solutions in the case when H has the particular form

$$H(z)=H_0(z)+H_1(z)e^{\omega_1 z^q}+H_2(z)e^{\omega_2 z^q}+\cdots+H_m(z)e^{\omega_m z^q},$$

where $\omega_1,\ \cdots,\ \omega_m$ are distinct non-zero complex numbers, $H_0,\ H_1,\ \cdots,\ H_m$ are entire functions of order less than q with $H_1\cdots H_m\not\equiv 0$. Some examples are given to show the existence of solutions.

Key words: Entire solution; Non-linear differential equation; binomial differential equation; Nevanlinna theory.

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1 Introduction and Main Results

A function f is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. In what follows, we assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory on meromorphic functions (see e.g., [4], [13]). By $\rho(f)$ and $\lambda(f)$ we will denote the order and the exponent of convergence of zeros of f, respectively. According to a famous result due to Titchmarsh [10], the non-linear differential equation

$$f(z)f''(z)=-\sin^2 z \tag{1.1}$$

has no real finite order entire solutions, other than $f(z)=\pm\sin z$.

Later, Li, Lü and Yang [6] showed that any entire solution of Eq. (1.1) must be real and of finite order. Furthermore, they investigated the following differential equation

$$f(z)f''(z)=p(z)\sin^2 z, \tag{1.2}$$

where $p(z)\not\equiv 0$ is a polynomial with real coefficients and real zeros, they proved that if f is an entire solution of Eq. (1.2), then p must be a non-zero constant, and $f(z)=a\sin z$, where a is a constant satisfying $a^2=-p$.

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In connection to the classical trigonometric identity $2\sin z\cos z = \sin 2z$, Zhang and Yi [14] proved that all entire solutions of the differential equation

$$f(z)f'(z) = \frac{1}{2}\sin 2z \tag{1.3}$$

have only the four forms $f(z) = \pm\sin z, \pm i\cos z$.

Naturally, Eq. (1.1)-Eq. (1.3) can be classified into the following form

$$f(z)f^{(k)}(z) = H(z), \tag{1.4}$$

where $k \geq 1$ and $H(z)$ is an entire function with $H(z) \not\equiv 0$. It is interesting to consider the general differential equations of the form (1.4) and even more complicated ones

$$f^{n_0}(f')^{n_1}(f'')^{n_2}\cdots(f^{(k)})^{n_k} = H, \tag{1.5}$$

where H is entire with $H \not\equiv 0$, $k \geq 1$, $n_0 \geq 1$ and $n_k \geq 1$.

Very recently, Gundersen, Lü, Ng and Yang [3] proved the following double inequality for the growth of entire solutions of Eq. (1.5).

Theorem 1.1 ([3]). *If f is an entire solution of a monomial differential equation* (1.5), *then we have*

$$\frac{1}{q}T(r,H) + S(r,f) \leq T(r,f) \leq \frac{1}{n_0}T(r,H) + S(r,f),$$

where $q = n_0 + n_1 + \cdots + n_k$. Hence, $\rho(f) = \rho(H)$.

After giving the growth of all entire solutions of the differential equation (1.5), Gundersen, Lü, Ng and Yang [3] considered the following non-linear binomial differential equation

$$Af^{m_0}(f')^{m_1}(f'')^{m_2}\cdots(f^{(p)})^{m_p} + Bf^{n_0}(f')^{n_1}(f'')^{n_2}\cdots(f^{(k)})^{n_k} = H, \tag{1.6}$$

under the assumption:

(a): $p, k \geq 0$ are integers, A, B, H are entire functions with $ABH \not\equiv 0$, $m_i(i = 0, \cdots, p), n_j(j = 0, \cdots, k)$ are non-negative integers with

$$\max\{m_0, m_p\} \geq 1, \quad \max\{n_0, n_k\} \geq 1, \quad \max\{m_0, n_0\} \geq 1, \quad \max\{m_p, n_k\} \geq 1,$$

and where it is assumed that the left-hand side of Eq. (1.6) does not reduce to $(A + B)f^{m_0}(f')^{m_1}(f'')^{n_2}\cdots(f^{(p)})^{m_p}$.

First observe that an analogous result to Theorem 1.1 cannot hold for non-linear binomial equations of the form (1.6), since $f(z) = \sin z$ satisfies $f^2 + (f')^2 = 1$.

Our first result is to find some comparatively relaxed conditions for (1.6). Now, we state our result in accordance with $m_0 > \sum_{j=0}^{k} n_j$, which in some sense can be seen as corresponding slight improvement to Theorem 1.1.

Theorem 1.2. *Suppose that $m_0 > N$, A, B are polynomials, and assume (a). If f is an entire solution of Eq.* (1.6), *then we have*

$$\frac{1}{M}T(r,H) + S(r,f) \leq T(r,f) \leq \frac{1}{m_0 - N}T(r,H) + S(r,f),$$

where $M = m_0 + m_1 + \cdots + m_p, N = n_0 + n_1 + \cdots + n_k$. Hence, $\rho(f) = \rho(H)$.

From Theorem 1.2, the growth of all entire solutions of the differential equation (1.6) is clear. Hence, we will consider differential equations such that $H(z)$ has a special form. Motivated by the consideration of transcendental exponential polynomials as in [5, 7, 9, 11, 15], a natural question follows:

Question 1.3. *When $m_0 > N$, can we characterize all entire solutions f of Eq.* (1.6) *if $H(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + H_2(z)e^{\omega_2 z^q} + \cdots + H_m(z)e^{\omega_m z^q}$?*

In order to answer this question, we will use Cartan's second main theorem and Nevanlinna's theorem concerning a group of meromorphic functions to investigate the non-linear binomial differential equation

$$\begin{aligned} &Af^{m_0}(f')^{m_1}(f'')^{m_2}\cdots(f^{(p)})^{m_p} + Bf^{n_0}(f')^{n_1}(f'')^{n_2}\cdots(f^{(k)})^{n_k} \\ &= H_0 + H_1e^{\omega_1 z^q} + H_2e^{\omega_2 z^q} + \cdots + H_me^{\omega_m z^q}. \end{aligned} \tag{1.7}$$

We arrive at the following conclusion:

Theorem 1.4. *Suppose that $m_0 > N$, $m, q \geq 1$ are integers, and assume (a). Let $\omega_1, \cdots, \omega_m$ be distinct non-zero complex numbers and let $H_0, H_1, \cdots, H_m$ be entire functions of order less than q such that $H_1 \cdots H_m \not\equiv 0$. If Eq.* (1.7) *admits an entire solution f, then the following assertions hold.*

(1) *When $H_0 \equiv 0$, we have two possibilities:*

(i) $f(z) = \gamma_0(z)e^{\frac{\omega_j}{M}z^q}$ and $m = 2$, where $A\gamma_0^{m_0}\gamma_1^{m_1}\cdots\gamma_p^{m_p} = H_j$, $\gamma_i = \gamma_{i-1}' + \frac{\omega_j}{M}q\gamma_{i-1}z^{q-1}$ and $N\omega_j = M\omega_t$ ($\{j,t\} = \{1,2\},\{2,1\}$).

(ii) $\lambda(f) = \rho(f) = q$ and $m_0 \leq m + N$.

(2) *When $H_0 \not\equiv 0$, we have $\lambda(f) = \rho(f) = q$ and $m_0 \leq m + N + 1$.*

The following examples show the existence of entire solutions satisfying Theorem 1.4.

Example 1.5. *Yang and Li [12] showed that all solutions of the equation $f^3(z) + \frac{3}{4}f''(z) = -\frac{1}{4}\sin 3z$ satisfy $\lambda(f) = \rho(f) = 1$. Here $m_0 = m + N$. This example also shows that the case $\lambda(f) = \rho(f) = q$ in Theorem 1.4 may happen although $m = 2$.*

Example 1.6. *The equation $f^3(z) - 3f'(z) = e^{3z} - e^{-3z} - 6e^z$ has an entire solution $f(z) = e^z - e^{-z}$. Here $m_0 < m + N$, $\lambda(f) = \rho(f) = 1$.*

Example 1.7. *The equation $f^2(z) - 2zf'(z) = e^{2z} + z^2 - 2z$ has an entire solution $f(z) = e^z + z$. Here $m_0 < m + N + 1$, $\lambda(f) = \rho(f) = 1$.*

2 Some Lemmas

In this section, we will introduce some lemmas used to prove our main results in the present paper. In the following, let E_1 (or E_2) denote the set of finite linear measure (or finite logarithmic measure) respectively.

Lemma 2.1 (see, e.g., [13]). *Let $f_1, f_2, \cdots, f_n$ be linearly independent meromorphic functions such that $\sum_{j=1}^n f_j \equiv 1$. Then for $1 \leq j \leq n$, we have*

$$\begin{aligned} T(r, f_j) &\leq \sum_{k=1}^n N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) - \sum_{k=1}^n N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r) \\ &\leq \sum_{k=1}^n N\left(r, \frac{1}{f_k}\right) + (n-1)\sum_{k=1}^n \overline{N}(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r), \end{aligned}$$

where D is the Wronskian determinant $W(f_1, f_2, \cdots, f_n)$,

$$S(r) = o(T(r)) \quad (r \to \infty, r \notin E_1), \quad T(r) = \max_{1\leq k\leq n}\{T(r, f_k)\}.$$

For introducing the following lemma, we denote by $n_p\left(r, \frac{1}{f}\right)$ the number of zeros of f in $|z| \le r$ where a zero of multiplicity l is counted l times if $l \le p$ and p times if $l > p$. Then, we let $N_p\left(r, \frac{1}{f}\right)$ denote the corresponding integrated counting function(cf. [2], Definition 2.1).

Lemma 2.2 (Cartan's theorem, see, e.g., [1, 2]). *Let $f_1, f_2, \cdots, f_p$ be linearly independent entire functions. Assume that for each complex number z, $\max\{|f_1(z)|, \cdots, |f_p(z)|\} > 0$. For $r > 0$, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta - u(0), \quad u(z) = \sup_{1\le j\le p} \log|f_j(z)|.$$

Set $f_{p+1} = f_1 + \cdots + f_p$. Then

$$T(r) \le \sum_{j=1}^{p+1} N_{p-1}\left(r, \frac{1}{f_j}\right) + S(r) \le (p-1)\sum_{j=1}^{p+1} \overline{N}\left(r, \frac{1}{f_j}\right) + S(r),$$

where $S(r)$ is a quantity satisfying $S(r) = O(\log T(r)) + O(\log r)(r \to \infty, r \notin E_1)$. If at least one of the quotients f_j/f_m is a transcendental function, then $S(r) = o(T(r))(r \to \infty, r \notin E_1)$, while if all the quotients f_j/f_m are rational functions, then $S(r) \le -\frac{1}{2}k(k-1)\log r + O(1)(r \to \infty, r \notin E_1)$.

Lemma 2.3 (see, e.g., [1, 2]). *Assume that the hypotheses of Lemma 2.2 hold. Then for any j and m, we have*

$$T(r, f_j/f_m) = T(r) + O(1) \quad (r \to \infty),$$

and for any j, we have

$$N(r, 1/f_j) = T(r) + O(1) \quad (r \to \infty).$$

Lemma 2.4 (see, e.g., [8]). *Let m, q be positive integers, $\omega_1, \cdots, \omega_m$ be distinct non-zero complex numbers, and $A_0, A_1, \cdots, A_m$ be meromorphic functions of order less than q such that $A_j \not\equiv 0 (1 \le j \le m)$. Set $\varphi(z) = A_0(z) + \sum_{j=1}^m A_j(z)e^{\omega_j z^q}$, then the following results hold.*

(i) *There exist two positive numbers $d_1 < d_2$, such that for sufficiently large r,*

$$d_1 r^q \le T(r, \varphi) \le d_2 r^q.$$

(ii) *If $A_0 \not\equiv 0$, then $m\left(r, \frac{1}{\varphi}\right) = o(r^q) \quad (r \to \infty)$.*

Lemma 2.5 (see, e.g., [13]). *Let f be a non-constant meromorphic function, and k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + o(T(r, f)) \quad (r \to \infty, r \notin E_1).$$

Lemma 2.6. *Under the conditions of Theorem 1.4, if f is an entire solution of* (1.7), *then the following results hold.*

(i) *There exist two positive numbers $\tau_1 < \tau_2$, such that,*

$$\tau_1 r^q \le T(r, f) \le \tau_2 r^q \quad (r \to \infty).$$

(ii) *If $H_0 \not\equiv 0$, then $N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f)$.*

Proof. Let f be an entire solution of (1.7). By Theorem 1.2 and Lemma 2.4, we have

$$T(r,f) \geq \frac{1}{M}T(r,H) + S(r,f) \geq \frac{d_1}{M}r^q,$$

and

$$(1-o(1))T(r,f) \leq \frac{1}{m_0 - N}T(r,H) \leq \frac{d_2}{m_0-N}r^q,$$

which leads to $\tau_1 r^q \leq T(r,f) \leq \tau_2 r^q, (r \to \infty)$, where τ_1, τ_2 are positive numbers such that $\tau_1 < \tau_2$. The result (i) is thus proved.

Rewriting (1.7) in the form

$$\frac{1}{H_0 + \sum_{j=1}^{m} H_j e^{\omega_j z^q}} \left(\frac{Af^{m_0}\cdots(f^{(p)})^{m_p}}{f^M} + \frac{Bf^{n_0}\cdots(f^{(k)})^{n_k}}{f^N}\frac{1}{f^{M-N}} \right) = \frac{1}{f^M}.$$

If $H_0 \not\equiv 0$, then by Lemma 2.4, we get

$$Mm\left(r,\frac{1}{f}\right) \leq (M-N)m\left(r,\frac{1}{f}\right) + S(r,f) + o(r^q) \quad (r\to\infty),$$

which implies $m\left(r,\frac{1}{f}\right) = S(r,f)$. Hence, the result (ii) follows.

3 Proof of Theorem 1.2

Let f be an entire solution of the binomial differential equation (1.6). Note that $M = m_0 + m_1 + \cdots + m_p, N = n_0 + n_1 + \cdots + n_k$, and by the assumption $m_0 > N$, we have $M \geq m_0 > N$. Combining (1.6) and the logarithmic derivative lemma, we get

$$\begin{aligned}
T(r,H) &= T\left(r, Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p} + Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}\right)\\
&= m\left(r, Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p} + Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}\right)\\
&= m\left(r, f^N\left(\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{f^M}f^{M-N} + \frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{f^N}\right)\right) \qquad (3.1)\\
&\leq Nm(r,f) + (M-N)m(r,f) + S(r,f)\\
&= MT(r,f) + S(r,f).
\end{aligned}$$

We now give an estimate in another direction. By the logarithmic derivative lemma and the first fundamental theorem, we get

$$\begin{aligned}
&T\left(r, Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}\right) = T\left(r, \frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{f^M}f^M\right)\\
&\geq MT(r,f) - T\left(r, \frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{f^M}\right)\\
&= MT(r,f) - N\left(r, \frac{A(f')^{m_1}\cdots(f^{(p)})^{m_p}}{f^{M-m_0}}\right) - m\left(r, \frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{f^M}\right) \qquad (3.2)\\
&\geq MT(r,f) - (M-m_0)N\left(r,\frac{1}{f}\right) - S(r,f)\\
&\geq MT(r,f) - (M-m_0)T(r,f) - S(r,f)\\
&= m_0 T(r,f) - S(r,f).
\end{aligned}$$

By $m_0 > N$ and (3.2), we have

$$\begin{aligned} T(r,H) &= T\left(r, Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p} + Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}\right) \\ &\geq T\left(r, Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}\right) - T\left(r, Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}\right) \\ &\geq m_0T(r,f) - m\left(r, \frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{f^N}f^N\right) - S(r,f) \\ &\geq m_0T(r,f) - Nm(r,f) - S(r,f) \\ &= (m_0 - N)T(r,f) - S(r,f). \end{aligned} \tag{3.3}$$

Now, combining (3.1) and (3.3) yields

$$\frac{1}{M}T(r,H) + S(r,f) \leq T(r,f) \leq \frac{1}{m_0 - N}T(r,H) + S(r,f).$$

The proof of Theorem 1.2 is now completed.

4 Proof of Theorem 1.4

Let f be an entire solution of (1.7). By Lemma 2.6, we deduce that

$$\rho(f) = q, \quad S(r,f) = o(r^q). \tag{4.1}$$

Now we consider the following two cases.
Case 1. $H_0 \equiv 0$. Rewriting (1.7) in the form

$$\sum_{j=1}^{m} \frac{H_j e^{\omega_j z^q}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}} - \frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}} \equiv 1. \tag{4.2}$$

Subcase 1.1. $m_0 \geq m + N + 1$.
Using the similar argument to that of (3.3), and by (4.1), there exists a constant $\tau > 0$, such that

$$\begin{aligned} &T\left(r, \frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right) = T\left(r, \frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}\right) + O(1) \\ &\geq T\left(r, f^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}\right) - T\left(r, f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}\right) - S(r,f) \\ &\geq (m_0 - N)T(r,f) - S(r,f) \\ &\geq (m_0 - N - o(1))\tau r^q \quad (r \to \infty). \end{aligned} \tag{4.3}$$

Subcase 1.1.1. Suppose that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}, \quad H_1e^{\omega_1 z^q}, \quad H_2e^{\omega_2 z^q}, \quad \cdots, \quad H_me^{\omega_m z^q}$$

are $m+1$ linearly independent entire functions.

Let $\Pi_1(z)$ denote the canonical product (or the polynomial) formed by the common zeros $\{a_k\}_{k=1}^{u}$ of $Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}$, $Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$, $H_1e^{\omega_1 z^q}$, $\cdots$, $H_me^{\omega_m z^q}$, each common zero a_k is counted $\min\{s_k, t_k, l_{kj} : j = 1, \cdots, m\}$ times, where $u = \infty$ (or finite integer), s_k, t_k, l_{k1}, $\cdots$, l_{km} denote the respective multiplicities of the zero of $Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}$, $Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$, $H_1e^{\omega_1 z^q}$, $\cdots$, $H_me^{\omega_m z^q}$ at point a_k. Then by (4.1), we have

$$N\left(r, \frac{1}{\Pi_1}\right) \leq N\left(r, \frac{1}{H_1}\right) = o(r^q) \tag{4.4}$$

as $r \to \infty, r \notin E_2$.

Dividing both sides of (1.7) by Π_1

$$\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_1}=\sum_{j=1}^{m}\frac{H_je^{\omega_jz^q}}{\Pi_1}-\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{\Pi_1}, \tag{4.5}$$

we deduce that $\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_1}$, $\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{\Pi_1}$, $\frac{H_1e^{\omega_1z^q}}{\Pi_1}$, $\cdots$, $\frac{H_me^{\omega_mz^q}}{\Pi_1}$ are all entire functions without common zeros, and by (4.3), we know that $\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{\Pi_1}/\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_1}$ is transcendental.

Since f is an entire function and A is a polynomial, we describe the following two facts:

$$N\left(r,\frac{1}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)=m_0N\left(r,\frac{1}{f}\right)+N\left(r,\psi\right), \tag{4.6}$$

and

$$N_m\left(r,\psi\right)-N\left(r,\psi\right)\le 0. \tag{4.7}$$

where $\psi=\frac{1}{A(f')^{m_1}\cdots(f^{(p)})^{m_p}}$, for simplicity.

Then by (4.1), (4.4)-(4.7), Lemmas 2.2, 2.3, 2.5, we get

$$\begin{aligned}
m_0N\left(r,\frac{1}{f}\right)&=N\left(r,\frac{1}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)-N\left(r,\psi\right)\\
&\le N\left(r,\frac{\Pi_1}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)-N\left(r,\psi\right)+o(r^q)\\
&\le T_1(r)-N\left(r,\psi\right)+o(r^q)\\
&\le \sum_{j=1}^{m}N_m\left(r,\frac{\Pi_1}{H_je^{\omega_jz^q}}\right)+N_m\left(r,\frac{\Pi_1}{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}\right)\\
&\quad +N_m\left(r,\frac{\Pi_1}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)-N\left(r,\psi\right)+o(T_1(r))+o(r^q)\\
&\le \sum_{j=0}^{k}n_kN\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+N_m\left(r,\psi\right)\\
&\quad -N\left(r,\psi\right)+o(T_1(r))+o(r^q)\\
&\le (N+m)N\left(r,\frac{1}{f}\right)+o(T_1(r))+o(r^q)\quad (r\to\infty, r\notin E_1),
\end{aligned} \tag{4.8}$$

where

$$T_1(r)=\frac{1}{2\pi}\int_0^{2\pi}u_1(re^{i\theta})d\theta-u_1(0),$$

$$u_1(z)=\sup\left\{\log\left|\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{\Pi_1}\right|,\log\left|\frac{H_je^{\omega_jz^q}}{\Pi_1}\right|:1\le j\le m\right\}.$$

Combining (4.8) and Lemma 2.6, we obtain

$$(m_0-m-N)N\left(r,\frac{1}{f}\right)\le o(r^q)+o(T_1(r))\le o(r^q)\quad (r\to\infty, r\notin E_1), \tag{4.9}$$

this together with the assumption $m_0\ge m+N+1$ give us

$$N\left(r,\frac{1}{f}\right)=o(r^q)\quad (r\to\infty, r\notin E_1). \tag{4.10}$$

From (4.1), (4.10) and Lemma 2.5, we see that

$$\sum_{j=1}^{m}N\left(r,\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{H_je^{\omega_jz^q}}\right)+N\left(r,\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}\right)=o(r^q), \tag{4.11}$$

and

$$\sum_{j=1}^{m}\overline{N}\left(r,\frac{H_je^{\omega_jz^q}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)+\overline{N}\left(r,\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)=o(r^q),\quad (4.12)$$

where $r\to\infty, r\notin E_1$.

Set

$$T_f(r)=\max\left\{T\left(r,\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right),T\left(r,\frac{H_je^{\omega_jz^q}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right):\right.$$
$$\left. j=1,\cdots,m\right\}.$$

From Lemma 2.1, (4.2), (4.11) and (4.12), it follows that

$$(1-o(1))T_f(r)=o(r^q)\quad (r\to\infty, r\notin E_1),$$

it implies

$$T\left(r,\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)\le T_f(r)=o(r^q)\quad (r\to\infty, r\notin E_1),\quad (4.13)$$

which contradicts (4.3).

Subcase 1.1.2. Suppose that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},\quad H_1e^{\omega_1z^q},\quad H_2e^{\omega_2z^q},\quad \cdots,\quad H_me^{\omega_mz^q}$$

are $m+1$ linearly dependent entire functions.

From the fact that $H_1e^{\omega_1z^q},\cdots,H_me^{\omega_mz^q}$ are linearly independent, there exist constants $d_1,\cdots,d_m$, at least one of them is not zero, such that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}=\sum_{j=1}^{m}d_jH_je^{\omega_jz^q}.\quad (4.14)$$

Substituting (4.14) into (1.7), we get

$$Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}=\sum_{j=1}^{m}(1-d_j)H_je^{\omega_jz^q}.\quad (4.15)$$

(a) Suppose that there exist at least two of $1-d_1,\cdots,1-d_m$, say $1-d_1$ and $1-d_2$, such that $1-d_1\neq 0$ and $1-d_2\neq 0$. Then by rewriting (4.15), we have

$$\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{e^{\omega_1z^q}}=(1-d_1)H_1+\sum_{j=2}^{m}(1-d_j)H_je^{(\omega_j-\omega_1)z^q}.\quad (4.16)$$

Denote $\varphi_1=\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{e^{\omega_1z^q}}$, then by Lemma 2.4 and the first fundamental theorem, there exists a positive number D_1, such that for sufficiently large r,

$$N\left(r,\frac{1}{\varphi_1}\right)=T(r,\varphi_1)-m\left(r,\frac{1}{\varphi_1}\right)-O(1)\ge D_1r^q,$$

then from Lemma 2.5, we find

$$N\left(r,\frac{1}{f}\right)\ge\frac{1}{M}N\left(r,\frac{1}{\varphi_1}\right)-O(\log r)\ge\frac{D_1}{M}r^q-O(\log r).\quad (4.17)$$

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On the other hand, by dividing Π_2 on both sides of (4.15), we get

$$\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_2}=\sum_{\lambda_j\in\Lambda}\frac{l_{\lambda_j}H_{\lambda_j}e^{\omega_{\lambda_j}z^q}}{\Pi_2}, \tag{4.18}$$

where Λ is a subset of $\{1,\cdots,m\}$ such that $l_{\lambda_j}=1-d_{\lambda_j}\neq 0$, Π_2 is defined as Π_1, such that $\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_2}$, $\frac{l_{\lambda_j}H_{\lambda_j}e^{\omega_{\lambda_j}z^q}}{\Pi_2}$ are all entire functions without common zeros, and $N\left(r,\frac{1}{\Pi_2}\right)\leq N\left(r,\frac{1}{H_{\lambda_j}}\right)=o(r^q)(r\to\infty)$. Then by (4.18), Lemmas 2.2 and 2.3, we get

$$\begin{aligned}
m_0N\left(r,\frac{1}{f}\right)&=N\left(r,\frac{1}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)-N(r,\psi)\\
&\leq N\left(r,\frac{\Pi_2}{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right)-N(r,\psi)+o(r^q)\\
&\leq T_2(r)-N(r,\psi)+o(r^q)\\
&\leq \sum_{\lambda_j\in\Lambda}N_{m-1}\left(r,\frac{\Pi_2}{l_{\lambda_j}H_{\lambda_j}e^{\omega_{\lambda_j}z^q}}\right)+N_{m-1}\left(r,\frac{\Pi_2}{f^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}\right) \qquad (4.19)\\
&\quad -N(r,\psi)+o(T_1(r))+o(r^q)\\
&\leq (m-1)N\left(r,\frac{1}{f}\right)+N_{m-1}(r,\psi)-N(r,\psi)+o(T_1(r))+o(r^q)\\
&\leq (m-1)N\left(r,\frac{1}{f}\right)+o(T_1(r))+o(r^q)\quad (r\to\infty, r\notin E_1),
\end{aligned}$$

where

$$T_2(r)=\frac{1}{2\pi}\int_0^{2\pi}u_2(re^{i\theta})d\theta-u_2(0),\quad u_2(z)=\sup\left\{\log\left|\frac{l_{\lambda_j}H_{\lambda_j}e^{\omega_{\lambda_j}z^q}}{\Pi_2}\right|:\lambda_j\in\Lambda\right\}.$$

So we deduce from (4.19) and Lemma 2.6 that

$$(m_0-m+1)N\left(r,\frac{1}{f}\right)\leq o(r^q)\quad (r\to\infty, r\notin E_1),$$

which contradicts (4.17).

(b) Suppose that there exists one and only one of $1-d_1,\cdots,1-d_m$ is non-zero, say $1-d_1\neq 0$. Then $d_2=\cdots=d_m=1$. We now write (4.15) as

$$Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}=(1-d_1)H_1e^{\omega_1z^q}, \tag{4.20}$$

which implies that

$$N\left(r,\frac{1}{f}\right)\leq\frac{1}{m_0}N\left(r,\frac{1}{H_1}\right)=o(r^q)\quad (r\to\infty). \tag{4.21}$$

By (4.1), (4.20) and Hadamard's factorization theorem, we get

$$f(z)=\gamma_0(z)e^{\frac{\omega_1}{m_0+\cdots+m_p}z^q},\quad f^{(i)}(z)=\gamma_i(z)e^{\frac{\omega_1}{m_0+\cdots+m_p}z^q}, \tag{4.22}$$

where $\gamma_0(z),\gamma_i(z)$ satisfy the recurrence formulas $\gamma_0^{m_0}\cdots\gamma_p^{m_p}=(1-d_1)H_1$, $\gamma_i=\gamma_{i-1}'+\frac{\omega_1}{M}q\gamma_{i-1}z^{q-1}(i=1,\cdots,p)$.

On the other hand, we claim that the set $\{d_1,d_2=1,\cdots,d_m=1\}$ has also only one non-zero element, i.e. $d_1=0,m=2$. Otherwise, suppose that there exist at least two of $d_1,\cdots,d_m$, say d_1 and d_2, such that $d_1\neq 0$ and $d_2\neq 0$. By rewriting (4.14), we have

$$\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{e^{\omega_1z^q}}=d_1H_1+\sum_{j=2}^{m}H_je^{(\omega_j-\omega_1)z^q}.$$

Denote $\varphi_2 = \frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{e^{\omega_1 z^q}}$, then by Lemma 2.4, Lemma 2.5 and the first fundamental theorem, there exists a positive number D_2, such that for sufficiently large r,

$$\begin{aligned}\sum_{j=0}^{k} n_j N\left(r,\frac{1}{f}\right) &\geq N\left(r,\frac{1}{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}\right) - O(\log r)\\ &= N\left(r,\frac{1}{\varphi_2}\right) - O(\log r)\\ &= T(r,\varphi_2) - m\left(r,\frac{1}{\varphi_2}\right) - O(\log r)\\ &\geq D_2 r^q - O(\log r) \quad (r\to\infty, r\notin E_1),\end{aligned}$$

which contradicts (4.21). Thus (4.14) reduces to

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k} = H_2 e^{\omega_2 z^q}. \tag{4.23}$$

Substituting (4.22) into (4.23) results in

$$B\gamma_0^{n_0}\cdots\gamma_k^{n_k} e^{\frac{N}{M}\omega_1 z^q} = H_2 e^{\omega_2 z^q}. \tag{4.24}$$

Moreover, combining (4.22) with (4.20) yields

$$A\gamma_0^{m_0}\cdots\gamma_p^{m_p} e^{\omega_1 z^q} = H_1 e^{\omega_1 z^q}.$$

Thus, we have the following result

$$m=2,\quad f(z)=\gamma_0(z)e^{\frac{\omega_1}{M}z^q},\quad \frac{\omega_2}{\omega_1}=\frac{N}{M},\quad A\gamma_0^{m_0}\cdots\gamma_p^{m_p}=H_1. \tag{4.25}$$

Subcase 1.2. $m_0 \leq m+N$. By (4.1), we get

$$\lambda(f)\leq\rho(f)=q.$$

Furthermore, if $\lambda(f)<q$, we can obtain $N\left(r,\frac{1}{f}\right)=o(r^q)$.

If $Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}, H_1e^{\omega_1 z^q}, H_2e^{\omega_2 z^q},\cdots,H_me^{\omega_m z^q}$ are linearly independent, then using the similar argument to that of Subcase 1.1.1, we get a contradiction, thus $\lambda(f)=\rho(f)=q$, the result (1)-(ii) is proved.

If $Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}, H_1e^{\omega_1 z^q}, H_2e^{\omega_2 z^q},\cdots,H_me^{\omega_m z^q}$ are linearly dependent, then by using the same argument as in the proof of Subcase 1.1.2, we have (4.25), so the result (1)-(i) is thus proved.

Case 2. $H_0\not\equiv 0$. By Lemma 2.6, we conclude

$$\lambda(f)=\rho(f)=q,\quad N\left(r,\frac{1}{f}\right)=T(r,f)+o(r^q)\quad (r\to\infty, r\notin E_1). \tag{4.26}$$

Suppose that $m_0 > m+N+1$, we proceed to prove the following two subcases by contradiction.

Subcase 2.1. Suppose that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},\quad H_0,\quad H_1e^{\omega_1 z^q},\quad H_2e^{\omega_2 z^q},\quad \cdots,\quad H_me^{\omega_m z^q}$$

are $m+2$ linearly independent entire functions.

Let $\Pi_3(z)$ denote the canonical product (or the polynomial) formed by the common zeros $\{z_k\}_{k=1}^{v}$ of $Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}$, $Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$, $H_0, H_1e^{\omega_1 z^q},\cdots,H_me^{\omega_m z^q}$, each common zero z_k is counted $\min\{s_k,t_k,l_{kj}:j=0,1,\cdots,m\}$ times, where $v=\infty$ (or finite integer), s_k, t_k, l_{k1}, $\cdots$, l_{km} denote the respective multiplicities of the zero of

$Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p},\ Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},\ H_0,\ H_1e^{\omega_1 z^q},\ \cdots,\ H_m e^{\omega_m z^q}$ at point z_k. Then by (4.1), we have

$$N\left(r,\frac{1}{\Pi_3}\right)\leq N\left(r,\frac{1}{H_1}\right)=o(r^q)\quad (r\to\infty, r\notin E_1). \tag{4.27}$$

By dividing Π_3 on two sides of (1.7), we have

$$\frac{Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}}{\Pi_3}=\sum_{j=0}^{m}\frac{H_j e^{\omega_j z^q}}{\Pi_3}-\frac{Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}}{\Pi_3},\quad (\omega_0=0). \tag{4.28}$$

Then using the similar argument to that of Subcase 1.1.1, by (4.28), (4.1), (4.27), Lemmas 2.2, 2.3 and 2.5, we get

$$(m_0-m-N-1)N\left(r,\frac{1}{f}\right)\leq o(r^q)\quad (r\to\infty, r\notin E_1).$$

From this, (4.1) and (4.26), we get $T(r,f)=S(r,f)$. This is a contradiction.

Subcase 2.2. Suppose that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},\quad H_0,\quad H_1e^{\omega_1 z^q},\quad H_2e^{\omega_2 z^q},\quad \cdots,\quad H_m e^{\omega_m z^q}$$

are $m+2$ linearly dependent entire functions.

From the fact that $H_0, H_1e^{\omega_1 z^q},\cdots,H_m e^{\omega_m z^q}$ are linearly independent, there exist constants $l_0, l_1,\cdots,l_m$, at least one of them is not zero, such that

$$Bf^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}=l_0H_0+\sum_{j=1}^{m}l_jH_je^{\omega_j z^q}. \tag{4.29}$$

Substituting (4.29) into (1.7) yields

$$Af^{m_0}(f')^{m_1}\cdots(f^{(p)})^{m_p}=(1-l_0)H_0+\sum_{j=1}^{m}(1-l_j)H_je^{\omega_j z^q}. \tag{4.30}$$

From (4.26) and (4.30), it follows that there exist at least two of $1-l_0, 1-l_1,\cdots,1-l_m$ are not zero. Then using the similar argument to that of Subcase 1.1.2 (a), by (4.30), Lemmas 2.2 and 2.3, we get

$$(m_0-m)N\left(r,\frac{1}{f}\right)\leq o(r^q)\quad (r\to\infty, r\notin E_1). \tag{4.31}$$

From (4.31) and (4.26), we obtain $T(r,f)=S(r,f)$. This is a contradiction.

Thus we have $m_0\leq m+N+1$. The result (2) is thus proved.

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