Properties of the weak solution of the fractional Laplacian system with different order *

Xinjing Wang†

(School of Mathematics and Statistics, Huanghuai University, Zhumadian, Henan, 463000, China)

Abstract: We prove a classical properties of the nonnegative weak solution of the fractional systems with the different order for fractional Laplacian operators. Use the moving plane combined with some integral inequality, we obtain the Liouville type theorems for the nonlinear fractional Laplacian system with the nonlinear right hand terms.

Keywords: Fractional Laplacian system, method of moving planes, Liouville type theorem.

Mathematics Subject Classification: 35A01, 35J57, 35D99

1 Introduction

We study the properties of weak solutions of the following fractional Laplacian system

$$\begin{cases} (-\Delta)^{\alpha} u = f(u, v), & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\beta} v = g(u, v), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta < 1$, $n > max\{2\alpha, 2\beta\}$, f, g are some nonlinear continuous functions. The fractional Laplacian operator $(-\Delta)^{\alpha}$ defined as

$$(-\Delta)^{\alpha} u = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} dy,$$

where $C_{n,\alpha}$ is positive constant, $P.V.$ stands for the Cauchy principal value. The fractional Laplacian operator $(-\Delta)^{\beta}$ has the same form of $(-\Delta)^{\alpha}$, so we omit the definition of it, and in the following use the $(-\Delta)^{\alpha}$ as the represents in the background knowledge(Section 1 and Section 2). By the difference quotients of the fractional Laplacian, one see that the fractional Laplacian is a nonlocal pseudo-differential

*This work is supported by the Key Scientific Research Project of the Colleges and Universities in Henan Province (No.22A110013) , the Key Specialized Research and Development Breakthrough Program in Henan Province (No.222102310265), and the Natural Science Foundation of Henan Province of China (No.222300420499).

†Corresponding author's Email: shingw@sina.com

10 Apr 2023 03:35:13 PDT
230410-Wang Version 1 - Submitted to Rocky Mountain J. Math.

operator. The nonlocality makes it difficult to study, to handle the nonlocality, Caffarelli and Silvestre [4] introduced the extension method which can reduce the nonlocal problem of $(-\Delta)^\alpha$ to a local one in higher dimensions. For a function $u : \mathbb{R}^n \to \mathbb{R}$, consider its extension function $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases} div(y^{1-2\alpha}\nabla U(x, y)) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x), & x \in \mathbb{R}^n, \end{cases}$$

then

$$(-\Delta)^\alpha u = -C_{n,\alpha} \lim_{y\to 0^+} y^{1-2\alpha} \frac{\partial U}{\partial y}.$$

In this paper, base on the extension result in [4], we investigate the extension boundary system of (1.1). Combining with some integral inequality to the method moving planes, we obtain the Liouville type results of system (1.1). For the special case, consider the single equation which forms as the equation of (1.1). The Liouville type theorems in $\mathbb{R}^n$ for Lane-Emden equation

$$-\Delta u = f(u) \tag{1.2}$$

have been studied by many famous authors. Caffarelli, Gidas and Spruck [3] studied the asymptotic symmetry and local behavior of the above semilinear elliptic equations, when the right item power is critical Sobelov growth. In [13], Gidas and Spruk proved that there was no positive classical solution of (1.2), which the spacial case that $f(u) = u^p$ ($0 < p < \frac{n+2}{n-2}$). And then Chen and Li [6] given a simple proof of the same result as [13] about the problem (1.2), the tool be used was the method of moving planes.

For the equation (1.2) with Neumann boundary

$$\begin{cases} -\Delta u(x, y) = 0, & (x, y) \in \mathbb{R}^{n+1}, \\ \dfrac{\partial u}{\partial \nu} = f(u), & x \in \mathbb{R}^n, \end{cases}$$

where Δ is the Laplacian operator in $\mathbb{R}^{n+1}$ and ν is the unit outward normal. In [16], Hu established nonexistence for positive solutions of the case that $f(u) = u^p$ ($1 \le p < \frac{n}{n-1}$). Ou [19] extended the conclusion of [16] to the range is $-\infty \le p < \frac{n}{n-1}$, which used the method of moving planes. Wan and Xiang [24] studied the following nonlinear Neumann problem

$$\begin{cases} div(y^a \nabla u(x, y)) = 0, & x \in \mathbb{R}^n, \ \ y > 0, \\ \lim_{y\to 0^+} y^a u_y(x, y) = -f(u(x, 0)), & x \in \mathbb{R}^n, \end{cases}$$

where $a \in (-1,1)$, $n \geq 1$, $\nabla = (\partial x_1, \cdots, \partial x_n, \partial y)$, f is nonnegative function. They proved a Liouville type theorem for nonnegative solutions of the above Neumann problem, and mentioned that this Neumann problem in $\mathbb{R}^n$ is relate to the fractional equation

$$(-\Delta)^{\alpha} u = f(u),$$

which is the extension result of [4], where $\alpha = \frac{1-a}{2}$. By using the extension method in [4], there are many fruitful results were obtained, can see [2, 5, 7, 25] and the references therein.

For the semilinear elliptic system

$$\begin{cases} -\Delta u = v^p, & \text{in } \mathbb{R}^n, \\ -\Delta v = u^q, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

If $p > 0$, $q > 0$, De Figueiredo and Felmer [10] conjectured that the hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{n}$$

is the dividing curve between existence and nonexistence of the solution to the semilinear elliptic system (1.3). Serrin and Zou [20, 21] given some positive answer when the solutions are radially of system (1.3). The authors in [10] also proved that this system has no positive solutions, when $0 < p, q \leq \frac{n+2}{n-2}$ and $(p,q) \neq (\frac{n+2}{n-2}, \frac{n+2}{n-2})$. Guo and Liu [14] given the Liouville type results for positive solutions of the following nonlinear elliptic system

$$\begin{cases} -\Delta u = f(u,v), & \text{in } \mathbb{R}^n, \\ -\Delta v = g(u,v), & \text{in } \mathbb{R}^n, \end{cases}$$

when $n \geq 3$. In [14] the method of moving planes combined with integral inequalities was fully utilized.

Recently, Wang [25] studied the properties for solutions of the fractional Laplacian system

$$\begin{cases} (-\Delta)^{\alpha} u = f(u,v), & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\alpha} v = g(u,v), & \text{in } \mathbb{R}^n, \end{cases}$$

where the $(-\Delta)^{\alpha}$ defined as above. The method of moving planes combined with integral inequalities also be used to prove the Liouville type theorems. For the case that different order of fractional Laplacain, Araujo, Faria, Leite and Miyagaki [1] established there is no positive solution for the following system

$$\begin{cases} (-\Delta)^{\alpha} u = u^{-p} v^{-q}, & \text{in } \Omega, \\ (-\Delta)^{\beta} v = u^{-r} v^{-\theta}, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

here Ω is a smooth bounded open subset in $\mathbb{R}^n$, where $0 < \alpha, \beta < 1$, $n \geq 2$, and $r, q > 0$, $p, \theta \geq 0$. Li and Ma [18] derived symmetry of the solutions to the fractional system

$$\begin{cases} (-\Delta)^{\alpha} u = f(v), & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\beta} v = g(u), & \text{in } \mathbb{R}^n, \\ u, v \geq 0, & \text{in } \mathbb{R}^n, \end{cases}$$

where f and g satisfy some continuity.

The extension method from [4] is a key tool in this paper, this method was commonly used in recent literature since it allows nonlocal problems to be written in a local way, which permits of variational techniques and tools for the problems involving of fractional operator, that were used in second elliptic problems. In this paper, we build the properties of solutions for system (1.1) which base on the extension method of [4]. So we consider the following elliptic system

$$\begin{cases} div(y^{1-2\alpha} \nabla U(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim_{y \to 0^+} y^{1-2\alpha} \dfrac{\partial U(x,y)}{\partial y} = -f(U(x,0), V(x,0)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \\ div(y^{1-2\beta} \nabla V(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim_{y \to 0^+} y^{1-2\beta} \dfrac{\partial V(x,y)}{\partial y} = -g(U(x,0), V(x,0)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \end{cases} \tag{1.4}$$

where $\mathbb{H} = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$, denote

$$\frac{\partial U}{\partial \nu^{\alpha}} = \lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial U(x,y)}{\partial y},$$

and

$$\frac{\partial U}{\partial \nu^{\beta}} = \lim_{y \to 0^+} y^{1-2\beta} \frac{\partial U(x,y)}{\partial y}.$$

We will get the Liouville type properties for the solutions of the fractional system (1.1) which following the results of (1.4).

Theorem 1.1. *For* $0 < \alpha,\ \beta < 1$, *let* $(u, v) \in H^{\alpha}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \times H^{\beta}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *be a nonnegative solution of problem*

$$\begin{cases} (-\Delta)^{\alpha} u = f(v), & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\beta} v = g(u), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.5}$$

Suppose that f *and* $g : [0, +\infty) \to \mathbb{R}^+$ *are continuous functions satisfying:*

(i) $f(t), g(t)$ *are nondecreasing in* $(0, +\infty)$,

(ii) $h(t) = \frac{f(t)}{t^{\frac{n+2\alpha}{n-2\beta}}}$, $k(t) = \frac{g(t)}{t^{\frac{n+2\beta}{n-2\alpha}}}$ *are nonincreasing in* $(0, +\infty)$.

Then either $(u, v) = (c_1, c_2)$ *for some constants* c_1 *and* c_2 *with* $f(c_2) = g(c_1) = 0$ *or there exist positive constants A and B such that* $h(t) = A$, $k(t) = B$ *and* (u, v) *is radially symmetric about some point.*

Theorem 1.2. *For* $0 < \alpha, \beta < 1$, *let* $(u, v) \in H^{\alpha}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \times H^{\beta}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *be a positive solution of problem (1.1). Suppose that f and g* : $[0, +\infty) \to \mathbb{R}^+$ *are continuous functions satisfying:*

(i) $f(t_1, t_2)$ *is nondecreasing in* t_2 *and* $f \geq 0$ *for* $t_1, t_2 > 0$,

(ii) there exist $p_1 \geq 0$, $q_1 > 0$, $(n - 2\alpha)p_1 + (n - 2\beta)q_1 = n + 2\alpha$, *such that* $\frac{f(t_1,t_2)}{{t_1}^{p_1}{t_2}^{q_1}}$ *is nonincreasing in* (t_1, t_2),

(iii) $g(t_1, t_2)$ *is nondecreasing in* t_1 *and* $g \geq 0$ *for* $t_1, t_2 > 0$,

(iv) there exist $p_2 > 0$, $q_2 \geq 0$, $(n - 2\alpha)p_2 + (n - 2\beta)q_2 = n + 2\beta$, *such that* $\frac{g(t_1,t_2)}{{t_1}^{p_2}{t_2}^{q_2}}$ *is nonincreasing in* (t_1, t_2).

Then either $(u, v) = (c_1, c_2)$ *for some constants* c_1 *and* c_2 *with* $f(c_1, c_2) = g(c_1, c_2) = 0$ *or there exist positive constants* $\tilde{A}$ *and* $\tilde{B}$ *such that* $f(t_1, t_2) = \tilde{A}{t_1}^{p_1}{t_2}^{q_1}$, $g(t_1, t_2) = \tilde{B}{t_1}^{p_2}{t_2}^{q_2}$ *and* (u, v) *is radially symmetric about some point.*

Theorem 1.3. *For* $0 < \alpha, \beta < 1$, *let* $(u, v) \in H^{\alpha}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \times H^{\beta}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *be a positive solution of problem (1.1). Suppose that f and g* : $[0, +\infty) \to \mathbb{R}^+$ *are* C^0 *functions satisfying:*

(i) $\frac{f(\gamma t_1,\gamma t_2)}{\gamma^{\frac{n+2\alpha}{n-2\alpha}}}$, $\frac{g(\gamma t_1,\gamma t_2)}{\gamma^{\frac{n+2\beta}{n-2\beta}}}$ *are nonincreasing in* γ, *and either*

(ii) $f(t_1, t_2)$ *is increasing and locally Lipschitz continuous in* t_2, *that is*

$$0 < f(t_1, t_2) - f(t_1, {t_2}') \leq L(m)(t_2 - {t_2}')$$

provided that $m \geq t_2 \geq {t_2}' > 0$, $m \geq t_1 \geq 0$, *and* $g(t_1, t_2)$ *is increasing and locally Lipschitz continuous in* t_1 *in the sense that*

$$0 < g(t_1, t_2) - g({t_1}', t_2) \leq L'(m)(t_1 - {t_1}'),$$

or

(ii)$'$ $f(t_1, t_2)$ *is increasing in* t_2 *and nondecreasing in* t_1; $g(t_1, t_2)$ *is increasing in* t_1 *and nondecreasing in* t_2.

Then either $(u, v) = (c_1, c_2)$ *for some constants* c_1 *and* c_2 *with* $f(c_1, c_2) = g(c_1, c_2) = 0$ *or there exist positive constants* $\tilde{A}$ *and* $\tilde{B}$ *such that* $f(t_1, t_2) = \tilde{A}{t_1}^{p_1}{t_2}^{q_1}$, $g(t_1, t_2) = \tilde{B}{t_1}^{p_2}{t_2}^{q_2}$ *and* (u, v) *is radially symmetric about some point.*

In this paper, the key tool is some integral inequalities. The idea comes from Terracini in [22, 23], and also can refer to [14, 24, 26] etc.. The main difficulty stems is the nonlinearities are coupled, based on the Kelvin's transform together with the method of moving plane, we use some integral inequality which make sure the start of the method of moving plane. Moreover, since we only assumed that f and g are continuous, therefore, the classical maximum principle can not work on the method of moving plane directly, so we use the method of moving planes combined with integral inequality. The results in this paper partially contain the conclusions in [1] and [17]. We extend from elliptic system in [14] to fractional Laplacian system, and generalize the order of the fractional operators in [25] from same to different. Be different from using the direct method of moving planes as [18], which the regularity of solution for the system was weakened. By the method of moving planes combined with integral inequality, we can handle the general right terms f and g of the nonlinear fractional Laplacian system. Be different from the process in [25], the more detailed and careful calculations should be given to the fractional Laplacian system with different order, and a similar Liouville type result (Theorem 1.3) is obtained for some more general conditions of f and g.

The paper is organized as follows. In Section 2, we collect some basic knowledge, and give the proof of some integral inequalities. Theorem 1.1 is proved in Section 3 which using the method of moving planes. Section 4 is devoted to obtain the Liouville type results (Theorem 1.2 and Theorem 1.3) with some assumptions of nonlinear terms f and g.

2 Preliminaries

In this section, we use some commonly notations to introduce the preliminaries. The index s can be chosen as α or β which without cause confusion. Di Nezza, Palatucci and Valdinoci [9] given the some relations of the fractional Sobolev spaces, which define $W^{s,p}$ as follows

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega)\}.$$

This fractional Sobolev spaces is a Banach space between the classical Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, which use the norm

$$||u||_{W^{s,p}(\Omega)} := (\int_\Omega |u(x)|dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy)^{\frac{1}{p}},$$

here

$$[u]_{W^{s,p}(\Omega)} := (\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy)^{\frac{1}{p}}$$

is the so called Gagliardo smeinorm.

The fractional Sobolev space $W^{s,p}$ is a Hilbert space, where $p = 2$, and its denoted as H^s. This space can also be defined by the Fourier transform

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^{2s})|\mathcal{F}u(\xi)|^2 d\xi < \infty\},$$

where $\mathcal{F}u$ is the Fourier transform of u.

In [9], authors also proved that

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi = 2C(n,s)^{-1}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)}.$$

In this paper, we don't care the concrete value of constant $C(n,s)$, so we use the notation as [24], that $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the quadratic form

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi = \|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)}.$$

In fact, the two definitions of fractional Sobolev spaces in [9] are equivalent. And the extension principle of [4] also works out in $\dot{H}^s(\mathbb{R}^n)$. In [24], authors studied the variational problem

$$S = \inf\{\int_{\mathbb{H}} y^a|\nabla\phi(x,y)|^2 dxdy : \phi \in C_0^\infty(\overline{\mathbb{H}}), \int_{\partial\mathbb{H}} |\phi(x,0)|^{\frac{2n}{n-1+a}}dx = 1\}. \tag{2.1}$$

The constant S of problem (2.1) is well defined, due to the trace inequality

$$\int_{\partial\mathbb{H}} |\phi(x,0)|^{\frac{2n}{n-1+a}}dx \le C_{n,a}\int_{\mathbb{H}} y^a|\nabla\phi(x,y)|^2dxdy, \qquad \forall\phi \in C_0^\infty(\overline{\mathbb{H}}), \tag{2.2}$$

where $C_{n,a}$ is a positive constant depending only on n and a, here $s = \frac{1-a}{2}$. This result can also be seen in Frank et al. [11, 12], which introduced the sharp trace inequality

$$\|(-\Delta)^{\frac{s}{2}}Tu\|_2^2 \le C_{n,a}\int_{\mathbb{H}} y^a|\nabla u(x,y)|^2dxdy, \tag{2.3}$$

where T is a trace operator, such that $Tu(x) = u(x,0)$.

Now we list the comparison principle in [24].

Lemma 2.1. *Let $\Omega \subset \mathbb{H}$ be an open set with a part of flat boundary $\Gamma \subset \partial\mathbb{H}$. Let $u \ge 0$, $u \not\equiv 0$, be classical solution to equation*

$$\begin{cases} div(y^a\nabla u(x,y)) = 0, & \text{in } \Omega, \\ \lim\limits_{y\to 0^+} y^a u_y(x,y) \le 0, & x \in \Gamma, \end{cases}$$

Then

$$u > 0 \qquad \text{on } \Omega \cup \Gamma.$$

Indeed, this conclusion also works on the system (1.4) on Ω, which is a strong maximum principle for the system (1.4) on $\mathbb{H}$.

Let (U, V) be nonnegative, we give their Kelvin transform in $\mathbb{H}$ which centered at zero as

$$w(X) = \frac{1}{|X|^{n-2\alpha}} U(\frac{X}{|X|^2}),\ z(X) = \frac{1}{|X|^{n-2\beta}} V(\frac{X}{|X|^2}),\ X = (x, y) \in \overline{\mathbb{H}} \setminus \{0\}. \tag{2.4}$$

Obviously, w and z are continuous and nonnegative in $\overline{\mathbb{H}} \setminus \{0\}$. It is easy to check that

Lemma 2.2. *Let $(U, V) \in W^{1,2}_{loc}(\mathbb{H}) \cap C(\mathbb{H}) \times W^{1,2}_{loc}(\mathbb{H}) \cap C(\mathbb{H})$ be a nonnegative (weak) solution of system:*

$$\begin{cases} div(y^{1-2\alpha}\nabla U(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial U(x,y)}{\partial y} = -f(V(x,0)), & \text{on } \partial\mathbb{H} \setminus \{0\}, \\ div(y^{1-2\beta}\nabla V(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\beta}\dfrac{\partial V(x,y)}{\partial y} = -g(U(x,0)), & \text{on } \partial\mathbb{H} \setminus \{0\}. \end{cases}$$

Then (w, z) satisfies (weakly) the following system:

$$\begin{cases} div(y^{1-2\alpha}\nabla w(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial w(x,y)}{\partial y} = -\dfrac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\beta} z(x)), & \text{on } \partial\mathbb{H} \setminus \{0\}, \\ div(y^{1-2\beta}\nabla z(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\beta}\dfrac{\partial z(x,y)}{\partial y} = -\dfrac{1}{|x|^{n+2\beta}} g(|x|^{n-2\alpha} w(x)), & \text{on } \partial\mathbb{H} \setminus \{0\}. \end{cases} \tag{2.5}$$

Moreover, w and z have decay at infinity as

$$\lim_{|X|\to\infty} |X|^{n-2\alpha} w(X) = U(0),\ \lim_{|X|\to\infty} |X|^{n-2\beta} z(X) = V(0), \tag{2.6}$$

and hence $w \in L^{\frac{q}{n-2\alpha}}(\Sigma_\lambda)$, $z \in L^{\frac{q}{n-2\beta}}(\Sigma_\lambda)$ for any $n + 1 < q \leq \infty$.

Let $\lambda \in \mathbb{R}$ and $X = (x_1, x_2, \cdots, x_n, y) \in \mathbb{H}$. Without loss of generality, we may assume that

$$T_\lambda = \{X \in \mathbb{H} : x_1 = \lambda\},$$

and

$$\Sigma_\lambda = \{X \in \mathbb{H} : x_1 > \lambda\},$$

and denote

$$p_\lambda = (2\lambda, 0, \cdots, 0, 0) \in \partial\mathbb{H},\ X_\lambda = (2\lambda - x_1, x_2, \cdots, x_n, y).$$

Define the reflected functions

$$w_\lambda(X) = w(X_\lambda),\ z_\lambda(X) = z(X_\lambda).$$

Let

$$W_\lambda(X) = w(X) - w(X_\lambda),\ Z_\lambda(X) = z(X) - z(X_\lambda).$$

The key integral inequalities will be shown in following which will be employed in the method of moving planes.

Lemma 2.3. *For any fixed $\lambda > 0$, $w \in L^{\frac{2n}{n-2\alpha}}(\Sigma_\lambda)$, $z \in L^{\frac{2n}{n-2\beta}}(\Sigma_\lambda)$, such that*

$$\int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX \le C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{2n}{n-2\beta}}(x)dx\Big)^{\frac{2\alpha+2\beta}{n}} \int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX, \tag{2.7}$$

$$\int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX \le C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} w^{\frac{2n}{n-2\alpha}}(x)dx\Big)^{\frac{2\alpha+2\beta}{n}} \int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX, \tag{2.8}$$

where $A_\lambda^1 = \{X \in \Sigma_\lambda : Z_\lambda(X) \ge 0\}$, $A_\lambda^2 = \{X \in \Sigma_\lambda : W_\lambda(X) \ge 0\}$, $W_\lambda^+ = \max\{W_\lambda,\ 0\}$, $Z_\lambda^+ = \max\{Z_\lambda,\ 0\}$, and $C_\lambda > 0$ is a constant which is bounded when λ is away from zero.

Proof. Here we only show the inequality (2.7), the proof of (2.8) is similar. For any fixed $\lambda > 0$, then w and $W_\lambda^+ \le w$ belong to space $L^{\frac{2n}{n-2\alpha}}(\Sigma_\lambda)$.

For $\varepsilon > 0$ small, we introduce a cut-off function $\eta_\varepsilon \in C_0^\infty(\mathbb{H})$, where $0 \le \eta_\varepsilon \le 1$,

$$\eta_\varepsilon(X) = 1,\ \text{ for } 2\varepsilon \le |X - p_\lambda| \le \varepsilon^{-1},$$

$$\eta_\varepsilon = 0,\ \text{ for } |X - p_\lambda| \le \varepsilon \text{ or } |X - p_\lambda| \ge 2\varepsilon^{-1},$$

and $|\nabla\eta_\varepsilon| \le C\varepsilon^{-1}$ for $\varepsilon \le |X - p_\lambda| \le 2\varepsilon$, $|\nabla\eta_\varepsilon| \le C\varepsilon$ for $\varepsilon^{-1} \le |X - p_\lambda| \le 2\varepsilon^{-1}$, here $C > 0$ is independent of ε. For $h(t) = \frac{f(t)}{t^{\frac{n+2\alpha}{n-2\beta}}}$, if $X \in \Sigma_\lambda$, we rewrite (2.5) as

$$\begin{cases} div(y^{1-2\alpha}\nabla w(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial w(x,y)}{\partial y} = -h(|x|^{n-2\beta}z(x))z^{\frac{n+2\alpha}{n-2\beta}}(x), & \text{on } \partial\mathbb{H}\setminus\{0\}, \end{cases} \tag{2.9}$$

and

$$\begin{cases} div(y^{1-2\alpha}\nabla w_\lambda(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial w_\lambda(x,y)}{\partial y} = -h(|x_\lambda|^{n-2\beta}z_\lambda(x))z_\lambda^{\frac{n+2\alpha}{n-2\beta}}(x), & \text{on } \partial\mathbb{H}\setminus\{p_\lambda\}. \end{cases} \tag{2.10}$$

Multiply (2.9) and (2.10) by $\phi_\varepsilon = W_\lambda^+ \eta_\varepsilon^2$, we get that

$$
\begin{aligned}
&\int_{\Sigma_\lambda \cap \{2\varepsilon \le |X-p_\lambda| \le \varepsilon^{-1}\}} y^{1-2\alpha} \left|\nabla W_\lambda^+\right|^2 dX \\
\le &\int_{\Sigma_\lambda} y^{1-2\alpha} \left|\nabla (W_\lambda^+ \eta_\varepsilon)\right|^2 dX \\
= &\int_{\Sigma_\lambda} y^{1-2\alpha} \nabla W_\lambda^+ \cdot \nabla \phi_\varepsilon dX + \int_{\Sigma_\lambda} y^{1-2\alpha} \left|W_\lambda^+\right|^2 |\nabla \eta_\varepsilon|^2 dX \\
= &I + I_\varepsilon.
\end{aligned} \tag{2.11}
$$

Firstly we estimate I_ε. Write $R_r = \{X \in \mathbb{H} : r \le |X - p_\lambda| \le 2r\}$ for $r > 0$, then

$$
\begin{aligned}
I_\varepsilon &\le C\varepsilon^{-2} \int_{\Sigma_\lambda \cap R_\varepsilon} y^{1-2\alpha} \left|W_\lambda^+\right|^2 dX + C\varepsilon^2 \int_{\Sigma_\lambda \cap R_{\varepsilon^{-1}}} y^{1-2\alpha} \left|W_\lambda^+\right|^2 dX \\
&\le C\varepsilon^{-2} \int_{\Sigma_\lambda \cap R_\varepsilon} y^{1-2\alpha} ((w-w_\lambda)^+)^2 dX + C\varepsilon^2 \int_{\Sigma_\lambda \cap R_{\varepsilon^{-1}}} y^{1-2\alpha} ((w-w_\lambda)^+)^2 dX \\
&\le C\varepsilon^{-2} \int_{R_\varepsilon} y^{1-2\alpha} w^2 dX + C\varepsilon^2 \int_{R_{\varepsilon^{-1}}} y^{1-2\alpha} w^2 dX,
\end{aligned}
$$

where $C > 0$ independent of ε. For $\varepsilon > 0$ sufficiently small, we derive from (2.6) that

$$
\varepsilon^{-2} \int_{R_\varepsilon} y^{1-2\alpha} w^2 dX \le C_\lambda \varepsilon^{-2} \int_{\{X \in \mathbb{H} \,:\, |X-p_\lambda| \le 2\varepsilon\}} y^{1-2\alpha} dX = O(\varepsilon^{n-2\alpha}),
$$

and

$$
\begin{aligned}
\varepsilon^2 \int_{R_{\varepsilon^{-1}}} y^{1-2\alpha} w^2 dX &\le C_\lambda \varepsilon^2 \int_{\{X \in \mathbb{H} \,:\, \varepsilon^{-1} \le |X-p_\lambda| \le 2\varepsilon^{-1}\}} y^{1-2\alpha} |X|^{2(2\alpha-n)} dX \\
&\le C_\lambda \varepsilon^{2+2(n-2\alpha)} \int_{\{X \in \mathbb{H} \,:\, |X-p_\lambda| \le 2\varepsilon^{-1}\}} y^{1-2\alpha} dX \\
&= O(\varepsilon^{n-2\alpha}),
\end{aligned}
$$

for some constants $C_\lambda > 0$. Therefore, as $\varepsilon \to 0$, we claim

$$
I_\varepsilon = O(\varepsilon^{n-2\alpha}) \to 0. \tag{2.12}
$$

And then we estimate I. Since $|x| > |x_\lambda|$, and h is nonincreasing, if $z(x) \ge z(x_\lambda) \ge 0$, then $-h(|x_\lambda|^{n-2\alpha} z_\lambda(x)) \ge -h(|x|^{n-2\alpha} z(x))$.

By (2.9) and (2.10), we deduce

$$
\begin{aligned}
I &= \int_{\Sigma_\lambda} y^{1-2\alpha}\nabla W_\lambda^+ \cdot \nabla\phi_\varepsilon dX \\
&= \int_{\partial(\Sigma_\lambda\cap \mathrm{supp}\eta_\varepsilon)} y^{1-2\alpha}\phi_\varepsilon \nabla W_\lambda^+ \cdot \nu dx \\
&= \int_{\{x\in\mathbb{R}^n\,:\,x_1>\lambda,\ \varepsilon\le|X-p_\lambda|\le 2\varepsilon^{-1}\}} (h(|x|^{n-2\beta}z(x))z^{\frac{n+2\alpha}{n-2\beta}}(x) - h(|x_\lambda|^{n-2\beta}z_\lambda(x))z_\lambda^{\frac{n+2\alpha}{n-2\beta}}(x))\phi_\varepsilon dx \\
&\le \int_{\{x\in\mathbb{R}^n\,:\,x_1>\lambda,\ \varepsilon\le|X-p_\lambda|\le 2\varepsilon^{-1}\}} h(|x|^{n-2\beta}z(x))(z^{\frac{n+2\alpha}{n-2\beta}}(x) - z_\lambda^{\frac{n+2\alpha}{n-2\beta}}(x))\phi_\varepsilon dx \\
&\le C'_\lambda \int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{n+2\alpha}{n-2\beta}-1}(x)(z-z_\lambda)^+(w-w_\lambda)^+ dx \\
&\le C'_\lambda \Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{\alpha+\beta}{n}}\Big(\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda}((z-z_\lambda)^+)^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{n-2\beta}{2n}}\Big(\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda}((w-w_\lambda)^+)^{\frac{2n}{n-2\alpha}}dx\Big)^{\frac{n-2\alpha}{2n}}
\end{aligned}
$$

where $C'_\lambda := \frac{2\alpha+2\beta}{n-2\beta}\sup_{x_1>\lambda} h(|x|^{n-2\beta}z(x))$. By the results of (2.6), if $|x|\to\infty$, then $|x|^{n-2\beta}z(x)\to v(0)$. So $C'_\lambda \to \frac{2\alpha+2\beta}{n-2\beta}h(v(0)) > 0$ as $\lambda\to\infty$, which implies that C'_λ is bounded for λ being away from zero. The trace inequality (2.2) shows that

$$
S\Big(\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda}((w-w_\lambda)^+)^{\frac{2n}{n-2\alpha}}dx\Big)^{\frac{n-2\alpha}{n}} \le \int_{\Sigma_\lambda} y^{1-2\alpha}(\nabla(w-w_\lambda)^+)^2 dX
$$

and

$$
S\Big(\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda}((z-z_\lambda)^+)^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{n-2\beta}{n}} \le \int_{\Sigma_\lambda} y^{1-2\beta}(\nabla(z-z_\lambda)^+)^2 dX,
$$

where S is the constant defined in (2.1). Hence,

$$
I \le C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{\alpha+\beta}{n}}\Big(\int_{\Sigma_\lambda} y^{1-2\beta}(\nabla W_\lambda^+)^2 dX\Big)^{\frac{1}{2}}\Big(\int_{\Sigma_\lambda} y^{1-2\alpha}(\nabla Z_\lambda^+)^2 dX\Big)^{\frac{1}{2}}, \tag{2.13}
$$

where C_λ is a positive constant which is bounded when λ is away from zero. By using dominated convergence, letting $\varepsilon\to 0$ in (2.11), combine (2.12) and (2.13), we know that (2.7) holds. □

3 Proof of Theorem 1.1

Step 1 Moving planes from the infinity.

Lemma 3.1. *There exist $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$, $W_\lambda(X)\le 0$ and $Z_\lambda(X)\le 0$ for any $X\in\Sigma_\lambda$.*

Proof. If $\lambda > 0$ is large enough, since $w\in L^{\frac{2n}{n-2\alpha}}(\Sigma_\lambda)$, $z\in L^{\frac{2n}{n-2\beta}}(\Sigma_\lambda)$, we have

$$
C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{2\alpha+2\beta}{n}} < 1, \quad \text{for all } \lambda\ge\lambda_0,
$$

and

$$C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} w^{\frac{2n}{n-2\alpha}}dx\Big)^{\frac{2\alpha+2\beta}{n}} < 1, \quad \text{for all } \lambda \geq \lambda_0.$$

By Lemma 2.3, we deduce that

$$\int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX = 0$$

and

$$\int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX = 0$$

for all $\lambda \geq \lambda_0$. For $\lambda > 0$ large enough, we obtain that $W_\lambda(X) \leq 0$ and $Z_\lambda(X) \leq 0$, for any $X \in \Sigma_\lambda$ □

Step 2 Step 1 provides a starting point, and then we can go on moving the planes to its limits position. Define

$$\Lambda = \inf\{\lambda > 0 \mid W_\mu(X) \leq 0,\ Z_\mu(X) \leq 0,\ \forall X \in \Sigma_\mu,\ \mu > \lambda\}. \tag{3.1}$$

Lemma 3.2. *If $\Lambda > 0$ then $W_\Lambda(X) \equiv 0$ and $Z_\Lambda(X) \equiv 0$ for any $X \in \Sigma_\Lambda$.*

Proof. By the continuity of W and Z, we have $W_\Lambda(X) \leq 0$ and $Z_\Lambda(X) \leq 0$ for any $X \in \Sigma_\Lambda$.

Suppose on the contrary, if $W_\Lambda(X) \not\equiv 0$ in Σ_Λ, then for any point $(x, 0) \in \partial\mathbb{H} \cap \partial\Sigma_\Lambda$, we have

$$\begin{aligned} h(|x|^{n-2\beta}z(x))z^{\frac{n+2\alpha}{n-2\beta}}(x) &= \frac{f(|x|^{n-2\beta}z(x))}{|x|^{n+2\alpha}} \\ &\leq \frac{f(|x|^{n-2\beta}z_\Lambda(x))}{|x|^{n+2\alpha}} \\ &\leq \frac{f(|x_\Lambda|^{n-2\beta}z_\Lambda(x))}{|x_\Lambda|^{n+2\alpha}} \\ &= h(|x_\Lambda|^{n-2\beta}z_\Lambda(x))z_\Lambda^{\frac{n+2\alpha}{n-2\beta}}(x). \end{aligned}$$

Applying Lemma 2.1 to $W_\Lambda(X)$, we claim that $W_\Lambda(X) \leq 0$. The strong maximum principle implies that $W_\Lambda(X) < 0$ in Σ_Λ. The strict inequality shows that the characteristic function $\mathcal{X}_{\partial A_\lambda^2} \to 0$ a.e. in $\mathbb{R}^n$ as $\lambda \to \Lambda$. The dominated convergence theorem indicates

$$\lim_{\lambda\to\Lambda} C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} w^{\frac{2n}{n-2\alpha}}dx\Big)^{\frac{2\alpha+2\beta}{n}} = 0,$$

and hence for any $\lambda \in (\Lambda - \delta, \Lambda)$

$$C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} z^{\frac{2n}{n-2\beta}}dx\Big)^{\frac{2\alpha+2\beta}{n}} C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} w^{\frac{2n}{n-2\alpha}}dx\Big)^{\frac{2\alpha+2\beta}{n}} < 1,$$

where δ is a sufficiently small positive constant. Recalling the previous argument, which implies that $W_\lambda(X) \leq 0$ and $Z_\lambda(X) \leq 0$ for any $X \in \Sigma_\lambda$, which against the definition of Λ in (3.1). □

If $\Lambda = 0$, for any $(x_1, x_2, \cdots, x_n, y) \in \Sigma_0$, we get

$$w(x_1, x_2, \cdots, x_n, y) \leq w(-x_1, x_2, \cdots, x_n, y)$$

and

$$z(x_1, x_2, \cdots, x_n, y) \leq z(-x_1, x_2, \cdots, x_n, y).$$

We also can moving the planes from the left to right, and obtain that

$$w(x_1, x_2, \cdots, x_n, y) \geq w(-x_1, x_2, \cdots, x_n, y)$$

and

$$z(x_1, x_2, \cdots, x_n, y) \geq z(-x_1, x_2, \cdots, x_n, y).$$

Hence, we have

$$w(x_1, x_2, \cdots, x_n, y) = w(-x_1, x_2, \cdots, x_n, y)$$

and

$$z(x_1, x_2, \cdots, x_n, y) = z(-x_1, x_2, \cdots, x_n, y).$$

Therefore, we know that $w(x, y) = w(|x|, y)$ and $z(x, y) = z(|x|, y)$.

Since any point can be chosen as the center point of Kelvin transform, then w and z must be independent of x. That is, U and V are only dependent of y. For u and v, we have that $(u, v) = (c_1, c_2)$ for some constants c_1 and c_2 with $f(c_2) = g(c_1) = 0$.

If $\Lambda > 0$, we see that $w = w_\Lambda$ and $z = z_\Lambda$. Those imply that w and z are regular at the origin, that is, U and V are regular at infinity. Since $w = w_\Lambda$ and $z = z_\Lambda$, from (2.9) and (2.10) we have

$$h(|x|^{n-2\beta} z(x)) = h(|x_\Lambda|^{n-2\beta} z_\Lambda(x)).$$

Noting that for any x, the inequality $|x| > |x_\Lambda|$ holds, and h is nonincreasing, it follows that the function h must be constant, i.e., there exist a positive constant A such that $h(t) = A$. And we also know that there exist a positive constant B such that $g(t) = b$. In this case, the above Lemmas show that w and z are symmetric about some point for the variable x, so does to U and V. Therefore, we obtain that (u, v) is radially symmetric about some point.

4 Proof of Theorem 1.2 and 1.3

We use the notations as in Section 2. Let $(U, V) \in W^{1,2}_{loc}(\mathbb{H}) \cap C(\mathbb{H}) \times W^{1,2}_{loc}(\mathbb{H}) \cap C(\mathbb{H})$ be a positive (weak) solution of system (1.4), then (w, z) satisfies (weakly) the following system:

$$\begin{cases} div(y^{1-2\alpha}\nabla w(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial w(x,y)}{\partial y} = -\dfrac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x)), & \text{on } \partial\mathbb{H}\setminus\{0\}, \\ div(y^{1-2\beta}\nabla z(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\beta}\dfrac{\partial z(x,y)}{\partial y} = -\dfrac{1}{|x|^{n+2\beta}} g(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x)), & \text{on } \partial\mathbb{H}\setminus\{0\}, \end{cases} \tag{4.1}$$

and then

$$\begin{cases} div(y^{1-2\alpha}\nabla w_\lambda(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\alpha}\dfrac{\partial w_\lambda(x,y)}{\partial y} = -\dfrac{1}{|x_\lambda|^{n+2\alpha}} f(|x_\lambda|^{n-2\alpha}w_\lambda(x), |x_\lambda|^{n-2\beta}z_\lambda(x)), & \text{on } \partial\mathbb{H}\setminus\{p_\lambda\}, \\ div(y^{1-2\beta}\nabla z_\lambda(x,y)) = 0, & \text{in } \mathbb{H}, \\ \lim\limits_{y\to 0^+} y^{1-2\beta}\dfrac{\partial z_\lambda(x,y)}{\partial y} = -\dfrac{1}{|x_\lambda|^{n+2\beta}} g(|x_\lambda|^{n-2\alpha}w_\lambda(x), |x_\lambda|^{n-2\beta}z_\lambda(x)), & \text{on } \partial\mathbb{H}\setminus\{p_\lambda\}. \end{cases} \tag{4.2}$$

Lemma 4.1. *For any fixed $\lambda > 0$, $w \in L^{\frac{2n}{n-2\alpha}}(\Sigma_\lambda)$, $z \in L^{\frac{2n}{n-2\beta}}(\Sigma_\lambda)$, such that*

$$\begin{aligned} \int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX \le & C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} \frac{1}{x^{2n}}dx\Big)^{\frac{2\alpha}{n}} \int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX \\ & + C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1\cap\partial A_\lambda^2} \frac{1}{x^{2n}}dx\Big)^{\frac{\alpha+\beta}{n}} \Big(\int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX\Big)^{\frac{1}{2}} \Big(\int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX \le & C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1} \frac{1}{x^{2n}}dx\Big)^{\frac{2\beta}{n}} \int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX \\ & + C_\lambda\Big(\int_{\partial\mathbb{H}\cap\partial A_\lambda^1\cap\partial A_\lambda^2} \frac{1}{x^{2n}}dx\Big)^{\frac{\alpha+\beta}{n}} \Big(\int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z_\lambda^+\right|^2 dX\Big)^{\frac{1}{2}} \Big(\int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W_\lambda^+\right|^2 dX\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.4}$$

where $A_\lambda^1 = \{X \in \Sigma_\lambda : Z_\lambda(X) \ge 0\}$, $A_\lambda^2 = \{X \in \Sigma_\lambda : W_\lambda(X) \ge 0\}$, $W_\lambda^+ = \max\{W_\lambda, 0\}$, $Z_\lambda^+ = \max\{Z_\lambda, 0\}$, $C_\lambda > 0$ is a constant which is bounded when λ is away from zero.

Proof. We also just prove (4.3), the proof of (4.4) is omit. For any fixed constant $\lambda > 0$, thus w and $W_\lambda^+ \le w$ belong to $L^{\frac{2n}{n-2\alpha}}(\Sigma_\lambda)$. For $\varepsilon > 0$ small, choose a suitable cut-off function $\eta_\varepsilon \in C_0^\infty(\mathbb{H})$ such that $0 \le \eta_\varepsilon \le 1$,

$$\eta_\varepsilon(X) = 1, \text{ for } 2\varepsilon \le |X - p_\lambda| \le \varepsilon^{-1},$$

$$\eta_\varepsilon = 0, \ \text{ for } \ |X - p_\lambda| \le \varepsilon \ \text{ or } \ |X - p_\lambda| \ge 2\varepsilon^{-1},$$

and $|\nabla \eta_\varepsilon| \le C\varepsilon^{-1}$ for $\varepsilon \le |X - p_\lambda| \le 2\varepsilon$, $|\nabla \eta_\varepsilon| \le C\varepsilon$ for $\varepsilon^{-1} \le |X - p_\lambda| \le 2\varepsilon^{-1}$, here $C > 0$ independent of ε. The test function $\phi_\varepsilon = W_\lambda^+ \eta_\varepsilon^2$ be used in (4.1) and (4.2). Hence one can assume $w \ge w_\lambda$. So that $|x|^{n+2\alpha} w \ge |x_\lambda|^{n+2\alpha} w_\lambda$ for any $\lambda > 0$.

If $z \le z_\lambda$, by assumptions of f in Theorem 1.2, we obtain that

$$\begin{aligned} f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z_\lambda(x)) &\ge f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z(x) \frac{w_\lambda(x)}{w(x)}) \\ &\ge f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) \frac{|x_\lambda|^{n+2\alpha}}{|x|^{n+2\alpha}} (\frac{w_\lambda(x)}{w(x)})^{p_1+q_1}, \end{aligned}$$

and then

$$\begin{aligned} -(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) &\le \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}}) \\ &\le \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) \frac{n+2\alpha}{n-2\alpha} (1 - \frac{w_\lambda(x)}{w(x)}) \\ &= \frac{n+2\alpha}{n-2\alpha} \frac{f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))}{|x|^{n-2\alpha} w(x)} \frac{1}{|x|^{4\alpha}} (w(x) - w_\lambda(x)) \\ &\le \frac{C_\lambda}{|x|^{4\alpha}} (w(x) - w_\lambda(x)), \end{aligned} \tag{4.5}$$

for some constant C_λ.

If $z \ge z_\lambda$, we know that

$$\begin{aligned} f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z_\lambda(x)) &\ge f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) (\frac{|x_\lambda|^{n-2\alpha} w_\lambda(x)}{|x|^{n-2\alpha} w(x)})^{p_1} (\frac{|x_\lambda|^{n-2\beta} z_\lambda(x)}{|x|^{n-2\beta} z(x)})^{q_1} \\ &= f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) \frac{|x_\lambda|^{n+2\alpha}}{|x|^{n+2\alpha}} (\frac{w_\lambda(x)}{w(x)})^{p_1} (\frac{z_\lambda(x)}{z(x)})^{q_1}, \end{aligned}$$

and then

$$\begin{aligned} -(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) \le& \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\beta} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{p_1} (\frac{z_\lambda(x)}{z(x)})^{q_1}) \\ \le& \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}} (\frac{z_\lambda(x)}{z(x)})^{\frac{n+2\beta}{n-2\beta}}) \\ \le& \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(\frac{n+2\alpha}{n-2\alpha}(1 - \frac{w_\lambda(x)}{w(x)}) + \frac{n+2\beta}{n-2\beta}(1 - \frac{z_\lambda(x)}{z(x)})) \\ =& \frac{n+2\alpha}{n-2\alpha} \frac{f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))}{|x|^{n-2s} w(x)} \frac{1}{|x|^{4\alpha}} (w(x) - w_\lambda(x)) \\ &+ \frac{n+2\beta}{n-2\beta} \frac{f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))}{|x|^{n-2\beta} z(x)} \frac{1}{|x|^{2\alpha+2\beta}} (z(x) - z_\lambda(x)) \\ \le& \frac{C_\lambda}{|x|^{4\alpha}} (w(x) - w_\lambda(x)) + \frac{C_\lambda}{|x|^{2\alpha+2\beta}} (z(x) - z_\lambda(x)). \end{aligned} \tag{4.6}$$

Hence, combine (4.5) and (4.6), for $w \geq w_\lambda$, we have

$$-(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) \leq \frac{C_\lambda}{|x|^{4\alpha}}(w(x) - w_\lambda(x))^+ + \frac{C_\lambda}{|x|^{2\alpha+2\beta}}(z(x) - z_\lambda(x))^+.$$

Here we also have (2.11), and the estimate of I_ε can refer to Lemma 2.3. Next, we give the estimate of I.

$$\begin{aligned}
I =& \int_{\Sigma_\lambda} y^{1-2s} \nabla W_\lambda^+ \cdot \nabla \phi_\varepsilon dX \\
=& \int_{\partial(\Sigma_\lambda \cap \mathrm{supp}\eta_\varepsilon)} y^{1-2s} \phi_\varepsilon \nabla W_\lambda^+ \cdot \nu dx \\
\leq& \int_{\{x\in\mathbb{R}^n \,:\, x_1>\lambda,\ \varepsilon\leq|X-p_\lambda|\leq 2\varepsilon^{-1}\}} (\frac{C_\lambda}{|x|^{4\alpha}}(w(x) - w_\lambda(x))^+ + \frac{C_\lambda}{|x|^{2\alpha+2\beta}}(z(x) - z_\lambda(x))^+)\phi_\varepsilon dx \\
\leq& C_\lambda \int_{\partial\mathbb{H}\cap\partial A_\lambda^2} (\frac{1}{|x|^{4\alpha}}(W_\lambda^+)^2 + \frac{1}{|x|^{2\alpha+2\beta}} W_\lambda^+ Z_\lambda^+) dx \\
\leq& C_\lambda (\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} \frac{1}{|x|^{2n}} dx)^{\frac{2\alpha}{n}} (\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda} (W_\lambda^+)^{\frac{2n}{n-2\alpha}} dx)^{\frac{n-2\alpha}{n}} \\
&+ C_\lambda (\int_{\partial\mathbb{H}\cap\partial A_\lambda^2\cap\partial A_\lambda^1} \frac{1}{|x|^{2n}} dx)^{\frac{\alpha+\beta}{n}} (\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda} (W_\lambda^+)^{\frac{2n}{n-2\alpha}} dx)^{\frac{n-2\alpha}{2n}} (\int_{\partial\mathbb{H}\cap\partial\Sigma_\lambda} (Z_\lambda)^+)^{\frac{2n}{n-2\beta}} dx)^{\frac{n-2\beta}{2n}}.
\end{aligned}$$

Therefore, by the trace inequality (2.2), we finish the proof. □

Lemma 4.2. *There exist $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $W_\lambda(X) \leq 0$ and $Z_\lambda(X) \leq 0$ for any $X \in \Sigma_\lambda$.*

Proof. Since $\frac{1}{|X|^{2n}} \in L^1(\Sigma_\lambda)$, as $\lambda \to +\infty$ then

$$\int_{\partial\mathbb{H}\cap\partial A_\lambda^2\cap\partial A_\lambda^1} \frac{1}{|x|^{2n}} \leq \int_{\Sigma_\lambda} \frac{1}{|X|^{2n}} \to 0.$$

It follows that there exists $\lambda_0 > 0$, for all $\lambda \geq \lambda_0$ such that

$$C_\lambda (\int_{\partial\mathbb{H}\cap\partial A_\lambda^2} \frac{1}{|x|^{2n}} dx)^{\frac{2\alpha}{n}} < 1,$$

and

$$C_\lambda (\int_{\partial\mathbb{H}\cap\partial A_\lambda^1\cap\partial A_\lambda^2} \frac{1}{|x|^{2n}} dx)^{\frac{\alpha+\beta}{n}} < 1.$$

By Lemma 4.2, we deduce

$$\int_{\Sigma_\lambda} y^{1-2\alpha} \left|\nabla W_\lambda^+\right|^2 dX = 0$$

and

$$\int_{\Sigma_\lambda} y^{1-2\beta} \left|\nabla Z_\lambda^+\right|^2 dX = 0$$

for all $\lambda \geq \lambda_0$. Thus for $\lambda \geq \lambda_0$ and any $X \in \Sigma_\lambda$, $W_\lambda(X) \leq 0$ and $Z_\lambda(X) \leq 0$ hold. □

As the definition of Λ in Section 3, we get the following lemma.

Lemma 4.3. *If $\Lambda > 0$ then $W_\Lambda(X) \equiv 0$ and $Z_\Lambda(X) \equiv 0$ for any $X \in \Sigma_\Lambda$.*

Proof. By the continuity, we claim that if $w = w_\Lambda$ at some point $X_0 \in \Sigma_\Lambda$, then it holds in a neighborhood of X_0, and hence $w = w_\Lambda$ in Σ_Λ.

In fact, by continuity of W and Z, we see $W_\Lambda(X) \le 0$ and $Z_\Lambda(X) \le 0$ for $X \in \Sigma_\Lambda$. Since $w(X_0) = w_\Lambda(X_0)$, we have $|X|^{n-2\alpha} w(X_0) > |X_\Lambda|^{n-2\alpha} w_\Lambda(X_0)$ for $X \in B_r(X_0)$ a neighborhood of X_0. Using the same arguments as Lemma 4.1 and the fact that if $t_1 > t_1'$, $t_2 > t_2'$, we know

$$f(t_1', t_2') \ge f(t_1, t_2)(\frac{t_1'}{t_1})^{p_1}(\frac{t_2'}{t_2})^{q_1},$$

and

$$g(t_1', t_2') \ge g(t_1, t_2)(\frac{t_1'}{t_1})^{p_2}(\frac{t_2'}{t_2})^{q_2}.$$

And then

$$\begin{aligned} f(|x_\Lambda|^{n-2\alpha} w_\Lambda(x), |x_\Lambda|^{n-2\beta} z(x)) &\ge f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(\frac{|x_\Lambda|^{n-2\alpha} w_\Lambda(x)}{|x|^{n-2\alpha} w(x)})^{p_1}(\frac{|x_\Lambda|^{n-2\beta} z(x)}{|x|^{n-2\beta} z(x)})^{q_1} \\ &= f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) \frac{|x_\Lambda|^{n+2\alpha}}{|x|^{n+2\alpha}}(\frac{w_\Lambda(x)}{w(x)})^{p_1}, \end{aligned}$$

therefore,

$$\begin{aligned} -(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\Lambda}{\partial \nu^\alpha}) &= \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\Lambda|^{n+2\alpha}} f(|x_\Lambda|^{n-2\alpha} w_\Lambda(x), |x_\Lambda|^{n-2\beta} z_\Lambda(x)) \\ &\le \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\Lambda|^{n+2\alpha}} f(|x_\Lambda|^{n-2\alpha} w_\Lambda(x), |x_\Lambda|^{n-2\beta} z(x)) \\ &\le \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\Lambda(x)}{w(x)})^{p_1}) \\ &\le -C(w(x) - w_\Lambda(x)), \quad \text{in } B_r(X_0), \end{aligned}$$

for some constant $C > 0$ which depending on X_0 and r. Hence,

$$\begin{cases} div(y^{1-2\alpha} \nabla W_\Lambda(x, y)) = 0 \\ \lim\limits_{y \to 0^+} y^{1-2\alpha} \dfrac{\partial W_\Lambda(x, y)}{\partial y} + C W_\Lambda \le 0 \\ W_\Lambda \le 0, W_\Lambda(X_0) = 0 \quad \text{in } B_r(X_0). \end{cases}$$

The strong maximum principle shows that $W_\Lambda(X) \equiv 0$ in $B_r(X_0)$.

Now we claim that $W_\Lambda(X) \equiv 0$ implies $Z_\Lambda(X) \equiv 0$. In fact, by the equations (4.1) and (4.2), we see that

$$\begin{aligned}\frac{1}{|x|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x)) &= \frac{1}{|x_\Lambda|^{n+2\alpha}}f(|x_\Lambda|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x))\\ &\geq \frac{1}{|x_\Lambda|^{(n-2\beta)q_1}|x|^{(n-2\alpha)p_1}}f(|x|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x))\\ &> \frac{1}{|x|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x)).\end{aligned}$$

Since $f(t_1,t_2)$ is nondecreasing in t_2, by the above inequality, we deduce that

$$|x|^{n-2\beta}z(x) > |x_\Lambda|^{n-2\beta}z_\Lambda(x). \tag{4.7}$$

By the assumption of $f(t_1,t_2)$,

$$\begin{aligned}\frac{1}{|x|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x)) &= \frac{1}{|x_\Lambda|^{n+2\alpha}}f(|x_\Lambda|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x))\\ &\geq \frac{1}{|x_\Lambda|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z(x))\\ &\geq \frac{1}{|x|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x)),\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{|x|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x)) &= \frac{1}{|x_\Lambda|^{n+2\alpha}}f(|x_\Lambda|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x))\\ &= \frac{1}{|x_\Lambda|^{n+2\alpha}}f(|x|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z(x)),\end{aligned} \tag{4.8}$$

hence

$$\frac{f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x))}{(|x|^{n-2\alpha}w(x))^{p_1}(|x|^{n-2\beta}z(x))^{q_1}} = \frac{f(|x_\Lambda|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z(x))}{(|x_\Lambda|^{n-2\alpha}w(x))^{p_1}(|x_\Lambda|^{n-2\beta}z(x))^{q_1}}. \tag{4.9}$$

It follows from (4.7), that

$$|x|^{n-2\alpha}w(x) \geq |x_\Lambda|^{n-2\alpha}w(x),$$

and

$$|x|^{n-2\beta}z(x) \geq |x_\Lambda|^{n-2\beta}z_\Lambda(x) \geq |x_\Lambda|^{n-2\beta}z(x).$$

By (4.9) and assumption (ii) of Theorem 1.2, we have

$$\frac{f(|x|^{n-2\alpha}w(x),|x|^{n-2\beta}z(x))}{(|x|^{n-2\alpha}w(x))^{p_1}(|x|^{n-2\beta}z(x))^{q_1}} = \frac{f(|x_\Lambda|^{n-2\alpha}w(x),|x_\Lambda|^{n-2\beta}z_\Lambda(x))}{(|x_\Lambda|^{n-2\alpha}w(x))^{p_1}(|x_\Lambda|^{n-2\beta}z_\Lambda(x))^{q_1}}. \tag{4.10}$$

As a consequence of (4.8) and (4.10), $z^{q_1} = z_\Lambda^{q_1}$ is holds, and hence $z = z_\Lambda$ since $q_1 > 0$.

Suppose that $W_\Lambda(X) \not\equiv 0$ and $Z_\Lambda(X) \not\equiv 0$ in Σ_Λ, then $w(X) < w_\Lambda(X)$ and $z(X) < z_\Lambda(X)$ in Σ_Λ. Let χ_S be the characteristic function of set S. Then $\frac{1}{|x|^{2n}}\chi_{\partial A^2_\lambda}$ converges pointwisely to zero as $\lambda \to \Lambda$ in $\mathbb{H} \setminus (T_\Lambda \cup \{p_\Lambda\})$. Hence, if $0 < \Lambda - \delta < \Lambda$ (here δ sufficiently small), then $\frac{1}{|x|^{2n}}\chi_{\partial A^2_\lambda} \le \frac{1}{|X|^{2n}}\chi_{\Sigma_{\Lambda-\delta}} \in L^1(\Sigma_\lambda)$. By the dominate convergence, as $\lambda \to \Lambda$, we have

$$\int_{\partial\mathbb{H}\cap\partial A^2_\lambda} \frac{1}{|x|^{2n}} \to 0,$$

then

$$C_\lambda(\int_{\partial\mathbb{H}\cap\partial A^2_\lambda} \frac{1}{|x|^{2n}}dx)^{\frac{2\alpha}{n}} < 1$$

and

$$C_\lambda(\int_{\partial\mathbb{H}\cap\partial A^1_\lambda\cap\partial A^2_\lambda} \frac{1}{|x|^{2n}}dx)^{\frac{\alpha+\beta}{n}} < 1$$

for all $\lambda \in (\Lambda - \delta, \Lambda)$, which implies that

$$C_\lambda(\int_{\partial\mathbb{H}\cap\partial A^1_\lambda} \frac{1}{|x|^{2n}}dx)^{\frac{2\beta}{n}} < 1, \ \ \forall\, \lambda \in (\Lambda - \delta, \Lambda).$$

By Lemma 4.1, we claim that

$$\int_{\Sigma_\lambda} y^{1-2\alpha}\left|\nabla W^+_\lambda\right|^2 dX = 0$$

and

$$\int_{\Sigma_\lambda} y^{1-2\beta}\left|\nabla Z^+_\lambda\right|^2 dX = 0$$

for all $\lambda \ge \Lambda - \delta$, which implies that

$$W_\lambda \le 0 \ \text{ and } \ Z_\lambda \le 0 \ \text{ in } \ \Sigma_\lambda \ \text{ for any } \ \lambda \ge \Lambda - \delta,$$

this contradicts with the definition of Λ. □

Proof of Theorem 1.2: By the assumption that (u, v) is a positive solution of (1.1). Take the Kelvin transform about point $p \in \mathbb{H}$ and definition Λ in Section 3. If $\Lambda = 0$, for all p, we know that (w, z) is a radially symmetric with respect to all p, then (w, z) must to be constant, so does to (u, v). If $\Lambda > 0$, we have (w, z) is radially symmetric about some point, that is (u, v) should be radially symmetric about some point. □

Proof of Theorem 1.3: We claim that the key estimates in the proof of Lemma 4.1 also hold. The rest of the processes are as same as the Theorem 1.2 with some necessary modifications. In fact, the condition (ii)$'$ holds. For any fixed $\lambda > 0$, $|x|^{n+2\alpha} \geq |x_\lambda|^{n+2\alpha}$, it follows that if $z \leq z_\lambda$, and following the nonincreasing condition $\gamma = \frac{w_\lambda(x)}{w(x)}(\frac{|x_\lambda|}{|x|})^{n-2\alpha} \leq 1$

$$
\begin{aligned}
-(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) &= \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\lambda|^{n+2\alpha}} f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z_\lambda(x)) \\
&\leq \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\lambda|^{n+2\alpha}} f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z(x) \frac{w_\lambda(x)}{w(x)}) \\
&\leq \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}}) \\
&\leq \frac{C_\lambda}{|x|^{4\alpha}} (w(x) - w_\lambda(x)),
\end{aligned}
$$

If $z \geq z_\lambda$,

$$
\begin{aligned}
-(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) &= \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\lambda|^{n+2\alpha}} f(|x_\lambda|^{n-2\alpha} w_\lambda(x), |x_\lambda|^{n-2\beta} z_\lambda(x)) \\
&\leq \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x)) - \frac{1}{|x_\lambda|^{n+2\alpha}} f(|x_\lambda|^{n-2\alpha} \frac{w_\lambda(x) z_\lambda(x)}{z(x)}, |x_\lambda|^{n-2\beta} \frac{w_\lambda(x) z_\lambda(x)}{w(x)}) \\
&\leq \frac{1}{|x|^{n+2\alpha}} f(|x|^{n-2\alpha} w(x), |x|^{n-2\beta} z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}} (\frac{z_\lambda(x)}{z(x)})^{\frac{n+2\beta}{n-2\beta}}) \\
&\leq \frac{C_\lambda}{|x|^{2\alpha+2\beta}} [(w(x) - w_\lambda(x)) + (z(x) - z_\lambda(x)],
\end{aligned}
$$

where we use the condition with $\gamma = \frac{w_\lambda(x)}{w(x)}(\frac{|x_\lambda|}{|x|})^{n-2\alpha} \frac{z_\lambda(x)}{z(x)}(\frac{|x_\lambda|}{|x|})^{n-2\beta} \leq 1$.

The condition (ii) holds. For any fixed $\lambda > 0$, $|x|^{n+2\alpha} \geq |x_\lambda|^{n+2\alpha}$ always right, if $z \leq z_\lambda$, the proof must be same as (ii)$'$.

If $z \geq z_\lambda$,

$$
\begin{aligned}
-(\frac{\partial w}{\partial \nu^\alpha} - \frac{\partial w_\lambda}{\partial \nu^\alpha}) &= \frac{1}{|x|^{n+2\alpha}}(f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x)) - \frac{1}{|x_\lambda|^{n+2\alpha}}f(|x_\lambda|^{n-2\alpha}w_\lambda(x), |x_\lambda|^{n-2\beta}z_\lambda(x)) \\
&\leq \frac{1}{|x|^{n+2\alpha}}(f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x)) - f(|x|^{n-2\alpha}w_\lambda(x), |x|^{n-2\beta}z_\lambda(x))) \\
&= \frac{1}{|x|^{n+2\alpha}}(f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x)) - f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z_\lambda(x))) \\
&\quad + \frac{1}{|x|^{n+2\alpha}}(f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z_\lambda(x)) - f(|x|^{n-2\alpha}w_\lambda(x), |x|^{n-2\beta}z_\lambda(x))) \\
&\leq \frac{1}{|x|^{n+2\alpha}}[L(m)|x|^{n-2\beta}(z(x) - z_\lambda(x)) + f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z_\lambda(x)) \\
&\quad - f(|x|^{n-2\alpha}w_\lambda(x), |x|^{n-2\beta}z_\lambda(x)\frac{w_\lambda(x)}{w(x)})] \\
&\leq \frac{1}{|x|^{n+2\alpha}}[L(m)|x|^{n-2\beta}(z(x) - z_\lambda(x)) + f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z_\lambda(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}})] \\
&\leq \frac{1}{|x|^{n+2\alpha}}[L(m)|x|^{n-2\beta}(z(x) - z_\lambda(x)) + f(|x|^{n-2\alpha}w(x), |x|^{n-2\beta}z(x))(1 - (\frac{w_\lambda(x)}{w(x)})^{\frac{n+2\alpha}{n-2\alpha}})] \\
&\leq \frac{C_\lambda}{|x|^{2\alpha+2\beta}}[(w(x) - w_\lambda(x)) + (z(x) - z_\lambda(x)].
\end{aligned}
$$

The same arguments as in Lemma 4.1, we know that the key inequalities are hold. □

References

[1] A. Araujo, L. Faria, E. Leite, O. Miyagaki, Positive Solutions for Non-Variational Fractional Elliptic Systems with Negative Exponents, Z. Ana. Anwend., 40(1)(2021), 111-129.

[2] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez, A concaveconvex elliptic problem involving the fractional Laplacian, Proc. Royal Soc. Edinburgh, 143 (2013), 39-71.

[3] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure App. Math., 42(3)(1989), 271-297.

[4] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Part. Diff. Eq., 2 (2007), 1245-1260.

[5] E. Cinti, J. Tan, A nonlinear Liouville theorem for fractional equations in the Heisenberg group, J. Math. Anal. Appl., 433(2016), 434-454.

10 Apr 2023 03:35:13 PDT
230410-Wang Version 1 - Submitted to Rocky Mountain J. Math.

[6] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (3) (1991), 615-622.

[7] W. Chen, J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, J. Differential Equations, 260(5) (2014), 4758-4785.

[8] C. Cheng, Z. Lu, Y. Lu, A direct method of moving planes for the system of the fractional Laplacian, Pac. J. Math., 290(2)(2017), 301-320.

[9] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. math., 136 (2012), 521-573.

[10] D.G. De Figueiredo, P.L. Felmer, A Liouville type theorem for Elliptic systems, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 21(1994),387-397.

[11] R.L. Frank, E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in R, Acta Math., 210 (2013), 261-318.

[12] R.L. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, Comm. Pure Appl. Math., 69 (2016), 1671-1726.

[13] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., 35 (1982), 528-598.

[14] Y.X. Guo, J.Q. Liu, Liouville type theorems for positive solutions of elliptic system in $\mathbb{R}^n$. Comm. Part. Diff. Eq., 33(2)(2008), 263-284.

[15] L.L. Helms, Introduction to Potential Theory, Pure and Applied Mathematics 22, Wiley-Interscience, New York, London, Sydney (1969).

[16] B. Hu, Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition, Diff. Int. Eq., 7(1994), 301-313.

[17] D. Li, Z. Li, A radial symmetry and Liouville theorem for systems involving fractional Laplacian, Front. Math. China, 12(2)(2017),389-402.

[18] Y. Li, M. Pei, Symmetry of solutions for a fractional system, Sci. China Math., 60(2017), 1805-1824.

[19] B. Ou, Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition, Diff. Int. Eq., 9(1996), 1157-1164.

[20] J. Serrin, H. Zou, Existence of positive solutions of Lane-Emden system, Atti Sem. Mat. Fis. Univ. Modena. Sippl., 46(1998), 369-380.

[21] J. Serrin, H. Zou, The existence of positive entire solutions of elliptic Hamiltonian system, Comm. Part. Diff. Eq., 23(1998), 577-599.

[22] S. Terracini, Symmetry properties of positive solutions to some elliptic equa- tions with nonlinear boundary conditions, Diff. Int. Eq., 8(8)(1995), 1911-1922.

[23] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Diff. Eq., 1(2)(1996), 241-264.

[24] Y. Wan, C.L. Xiang, Uniqueness of positive solutions to some Nonlinear Neumann Problems, J. Math. Anal. Appl., 455(2)(2017), 1835-1847.

[25] X. Wang, Liouville type theorem for Fractional Laplacian system, Commun. Pure Appl. Anal., 19(11)(2020), 5253-5268.

[26] X. Yu, Liouville type theorems for two mixed boundary value problems with general nonlinearities, J. Math. Anal. Appl., 462(1)(2018), 305-322.