S-asymptotically (omega, xi) periodic mild solutions for some differential equations with delay and applications

Mounir Ben Salah *, Mohsen Miraoui† and Amine Rekik ‡

Abstract:
The aim behind this document is to present the $\vartheta-$pseudo S-asymptotically $(\omega, \mathcal{E})$ periodic of class r and its applications. Firstly, this document will introduce the new notion of $\vartheta-$pseudo S-asymptotically $(\omega, \mathcal{E})$ periodic functions. In addition, we study some of qualitative proprieties of this type, then we will be interested in the existence and uniqueness of $\vartheta-$pseudo-S-asymptotically $(\omega, \mathcal{E})$ type periodic mild solutions for some differential equations with finite delay. In Section 5 , an application is presented to demonstrate the effectiveness of the results. I end this work with a Conclusion in Section 6.
Keywords:
Differential equation with finite delay; pseudo S-asymptotically $(\omega, \mathcal{E})$ periodic solutions.

MSC CLASSIFICATION: 34K14; 35B15.

1 Introduction

It is well known that existence problem of bounded solutions has been one of the most attractive topics in qualitative theory of ordinary or functional differential equations due to its significance for physical sciences. Functional differential equations have many applications in population dynamics. They can be used to describe the evolution of many phenomena over the course of time.
After its definition by the physicist Bloch, the type of periodical functions bearing his name found increasing interest. Indeed, applications of this type of periodicity can be found in many fields such as solid state physics, condensed matter and quantum mechanics. It should be mentioned that periodicity and anti-periodicity are two special cases of this kind. Periodic events are commonplace in nature. When the earth orbits the sun and the moon orbits the earth, it is in a periodic movement. Then, does the motion of the moon in relation to the sun turn periodically again. In mathematics, is the sum of the two periodical functions always a periodical function? The answer is generally no. The sum of two periodic functions is generally no longer periodic functions, but very close to the periodic function, which is called the almost periodic function. Danish mathematician Harald Bohr revealed this for the first time in the mid-twenties. After this, people made various generalizations to the almost periodic function, like almost automorphy and pseudo almost periodic and automorphic functions with measure(see [1, 2, 13, 14, 15]). Delayed differential equation arise in some models, of which the state at one given instant, is a function that depends on its past.
We recall that delay equations were introduced to model phenomena in which there is a time lag between the action on the system and the response of the system to this action, for example, in the birth processes

*Department of Mathematics, IPEIK,Kairouan University, Kairouan, Tunisia. E-mail:mounir.bensalah01@gmail.com
†Corresponding author.Mohsen Miraoui, Department of Mathematics, IPEIK,Kairouan University, Kairouan, Tunisia. E-mail:miraoui.mohsen@yahoo.fr
‡Department of Mathematics, FSS,Sfax University, Tunisia. E-mail:aminerekik91@gmail.com

of biological populations(cells, bacteria,....), which require certain threshold to reach before the system is activated.
Many phenomena encountered in physics, biology, chemistry, study of networks neurons...
We call delay equation any equation in which the value of derivative at an instant of solution also depends on the values taken before this instant.
A general class of delayed differential equations was originally introduced by V.Volterra(1928)[17], he studied the predator-prey model. In the second half of the last century, the theory of delay equations experienced a great development notably, we find Bellman and Cooke (1963)[4], El'sgol'ts and Norkin(1973)[8], Lunel and Walther(1995)[7]... In practice, some models may depend on certain parameters: temperature, voltage, resistance,....
Latterly, Hasler ans N'Gurkata introduced the concept of Bloch type periodicity in [11] which comprise the ω periodicity, ω anti-periodicity as special cases.
In their work [16], Oueama-Guengai and N'Gurkata studied the existence and uniqueness of Bloch type periodic mild solutions to semi-linear fractional differential equation in Banach space.
On the other hand, S-asymptotic ω -periodicity [10] is a significant extension of classical periodicity, for more details about S-asymptotic ω periodicity, we refer to [12] and references cited therein.
In this document and after seeing the work of Miraoui et al. [9], who introduced the notion of doubly weighted pseudo-almost periodicity, we investigated a new notion of $\vartheta-$ pseudo-S-asymptotically Bloch type periodic functions (see [3]), denoted by $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$ defined by for given $\omega\in\mathbb{R}$ and $q\geq 0$

$$\lim_{l\to+\infty}\frac{1}{\vartheta([-l,l])}\int_{-l}^{l}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda)=0.$$

The paper [6] is mainly concerned with the existence of pseudo S-asymptotically Bloch type periodic solutions to damped evolution equations in Banach spaces. Some existence results for classical Cauchy conditions and nonlocal Cauchy conditions are established through properties of pseudo S-asymptotically Bloch type periodic functions and regularized families. In [5], the obtained results are applied to investigate the existence and uniqueness of pseudo S-asymptotically Bloch type periodic mild solutions for some semi-linear integro-differential equations and semi-linear fractional differential equations in Banach spaces.

The purpose behind this work is to develop some results on ϑ pseudo S-asymptotically (or$(\omega,\mathcal{E})$) type periodic functions of class $q\geq 0$ and give an application to differential equation with finite delay. In this paper, we consider a generalization of the space of pseudo S-asymptotically Bloch type periodic functions by the space of ϑ-Pseudo S-asymptotically $(\omega,\mathcal{E})$ periodic functions. If we take $\mathcal{E}(x)=e^{ixk}$ where $x,\ k\in\mathbb{R}$, the two spaces coincide.

Here, we selected the following differential equation:

$$\begin{cases}\frac{d}{dt}[u(t)-\mathcal{F}(t,u_t)]=\mathcal{J}[u(t)-\mathcal{F}(t,u_t)]+\mathcal{G}(t,u_t)\ \ t\geq 0,\\ u_0=\varphi\in\mathcal{C},\end{cases}\tag{1.1}$$

where $\mathcal{J}$ is the infinitesimal generator of an uniformly exponentially stable analytic semigroup $(\Phi(t))_{t\geq 0}$ on Banach space $\mathbb{L}$, $u_t\in\mathcal{C}$ is defined by $u_t(\alpha)=u(t+\alpha)$ for $\alpha\in[-r,0]$, r is a no-negative constant, $\mathcal{C}$ is clarified in the later.
The rest of this article is given by: In Section 2, we recall some preliminary results. In Section 3, the concepts of ϑ-Pseudo-S-asymptotically $(\omega,\mathcal{E})$ periodic function of class q is introduced and studied. In section 4, we prove the existence and uniqueness of ϑ-Pseudo S-asymptotically $(\omega,\mathcal{E})$ periodic solution of equation (1.1). In Section 5, we will use our theoretical results on the scalar reaction-diffusion equation with finite delay. I end this work with a Conclusion in Section 6.

2 Preliminaries

Let $(\mathbb{L}, \|.\|)$, $(\mathbb{W}, \|.\|)$ be two Banach spaces.
$\mathfrak{BC}(\mathbb{R}, \mathbb{L})$ denote the Banach space formed by all bounded continuous functions $h : \mathbb{R} \to \mathbb{L}$ with sup-norm

$$\|h\|_\infty = \sup_{\alpha \in \mathbb{R}} \|h(\alpha)\|.$$

$\mathfrak{BC}(\mathbb{R} \times \mathbb{L}, \mathbb{W})$ denote the Banach space formed of bounded continuous functions from $\mathbb{R} \times \mathbb{L}$ to $\mathbb{W}$. $\mathcal{C} = C([-r, 0], \mathbb{L})$ the space of continuous functions from $[-r, 0]$ to $\mathbb{L}$ with supremum norm.

Definition 2.1. [6] If we fixe $\omega, k \in \mathbb{R}$, a function $h \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$ is called to be Bloch (or (ω, k)) type periodic if $\forall \alpha \in \mathbb{R}, h(\alpha + \omega) = e^{ik\omega} h(\alpha)$.
We will symbolise this type of functions by $BP_{\omega,k}(\mathbb{R}, \mathbb{L})$.

Lemma 2.1. *[5] Let $f, g \in BP_{\omega,k}(\mathbb{R}, \mathbb{L})$. Then $BP_{\omega,k}(\mathbb{R}, \mathbb{L})$ is a Banach space under the supremum norm.*

Definition 2.2. [6] A function $f \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$ is said to be pseudo-S-asymptotically (or (ω, k)) type periodic, if for given $\omega, k \in \mathbb{R}$,

$$\lim_{l \to +\infty} \frac{1}{2l} \int_{-l}^{l} \|f(\alpha + \omega) - e^{ik\omega} f(\alpha)\| d\alpha = 0.$$

The type of its functions is denoted by $PSABP_{\omega,k}(\mathbb{R}, \mathbb{L})$.

3 (ϑ)-Pseudo-S-asymptoticall$(\omega, \mathcal{E})$ type periodic function

In this part , we define the new notion of ϑ pseudo-S- asymptotically ω periodic function.
we denote Δ the Lebesgue σ field of $\mathbb{R}$ and $\mathfrak{M}$ the set of all measures on Δ such that $\vartheta(\mathbb{R}) = +\infty$, moreover $\vartheta([a_1, a_2]) < +\infty$, $\forall (a_1, a_2) \in \mathbb{R}^2 (a_1 \leq a_2)$

Definition 3.1. Let $\vartheta, \chi \in \mathfrak{M}$, we call that measures ϑ and χ are equivalent $(\vartheta \sim \chi)$ if there exist $(\alpha, \lambda) \in \mathbb{R}^2$ and a bounded interval $J \subset \mathbb{R}$ (eventually $\emptyset$) such that:
$\alpha\vartheta(A) \leq \chi(A) \leq \lambda\vartheta(A)$, for all $A \in \Delta$ satisfying $A \cap J = \emptyset$.

For $\mu \in \mathfrak{M}$, $\tau \in \mathbb{R}$, we denote ϑ_τ the measure on $(\mathbb{R}, \Delta)$ defined by: $\vartheta_\tau(A) = \vartheta(a + \tau, a \in A)$, for all $A \in \Delta$.

Now, we need the hypothese below:

(M1) Let $\vartheta \in \mathfrak{M}$ such that for all $\tau \in \mathbb{R}$, there exist $\beta > 0$, and a bounded interval I such that: $\vartheta_\tau(A) \leq \beta\vartheta(A)$, if $A \in \Delta$ satisfies $A \cap J = \emptyset$.

Lemma 3.1. *[18] Let ϑ satisfies **(M1)** if and only if $(\vartheta \sim \vartheta_\tau)$ for any $\tau \in \mathbb{R}$.*

Lemma 3.2. *[18] If **(M1)** holds, then for all $\lambda > 0$,*

$$\limsup_{l \to +\infty} \frac{\vartheta([-l - \lambda, l + \lambda]}{\vartheta([-l, l])} < +\infty.$$

7 Aug 2023 12:58:12 PDT
230409-Miraoui Version 2 - Submitted to Rocky Mountain J. Math.

Definition 3.2. Let $\vartheta \in \mathfrak{M}, \omega \in \mathbb{R}$ and a function $\mathcal{E}$ from $\mathbb{R}$ to $\mathbb{C}$ such that $\mathcal{E}(\omega) \neq 0$, A function $f \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$ is said ϑ-pseudo-S-asymptotically (or $(\omega, \mathcal{E})$) type periodic, if for given $\omega \in \mathbb{R}$,

$$\lim_{l \to +\infty} \frac{1}{\vartheta([-l, l])} \int_{-l}^{l} \|f(t + \omega) - \mathcal{E}(\omega) f(t)\| d\vartheta(\lambda) = 0.$$

We will symbolise this type of functions by $PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta)$.

Definition 3.3. Let $\vartheta \in \mathfrak{M}$, $\omega \in \mathbb{R}$ and a function $\mathcal{E}$ from $\mathbb{R}$ to $\mathbb{C}$ such that $\mathcal{E}(\omega) \neq 0$. A function $f \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$ is called ϑ-pseudo-S-asymptotically (or $(\omega, \mathcal{E})$) type periodic of class $q \geq 0$ if

$$\lim_{l \to +\infty} \frac{1}{\vartheta([-l, l])} \int_{-l}^{l} \sup_{\lambda \in [t-q,t]} \|f(\lambda + \omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) = 0.$$

We will symbolise this type of functions by $PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$.

Theorem 3.3. *Let J be a bounded interval (eventually $J = \emptyset$). If $\phi \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$, then the statements below are equivalent :*

i) $\phi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$.

ii) We have

$$\lim_{l \to +\infty} \frac{1}{\vartheta([-l, l] \setminus J)} \int_{[-l,l] \setminus J} \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| d\vartheta(\lambda) = 0.$$

iii) $\forall\ \varepsilon > 0$, *we have*

$$\lim_{l \to +\infty} \frac{\vartheta(t \in [-l, l] \backslash J; \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| > \varepsilon)}{\vartheta([-l, l] \setminus J)} = 0.$$

Proof. *i)* $\Longleftrightarrow$ *ii)*
We denote $\mathbb{F} = \int_J (\sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\|) d\vartheta(\lambda)$ and $\mathbb{P} = \vartheta(J)$.

Since J is bounded and $\phi \in \mathfrak{BC}(\mathbb{R}, \mathbb{L})$ then $\mathbb{F}$ and $\mathbb{P}$ are finite let $l > 0$ such that $J \subset [-l, l]$ and $\vartheta([-l, l] \setminus J) > 0$, we have

$$\begin{aligned}
&\frac{1}{\vartheta([-l, l] \setminus J)} \int_{[-l,l] \setminus J} \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| d\vartheta(\lambda) \\
&= \frac{1}{\vartheta([-l, l]) - \mathbb{P}} (\int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| d\vartheta(\lambda) - \mathbb{F}) \\
&= \frac{1}{\vartheta([-l, l]) - \mathbb{P}} (\int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| d\vartheta(\lambda) - \mathbb{F}) \\
&= \frac{\vartheta([-l, l])}{\vartheta([-l, l]) - \mathbb{P}} \Big[\frac{1}{\vartheta([-l, l])} \int_{-l}^{l} \sup_{\lambda \in [t-q,t]} \|\phi(\lambda + \omega) - \mathcal{E}(\omega) \phi(\lambda)\| d\vartheta(\lambda) - \frac{\mathbb{F}}{\vartheta([-l, l])} \Big].
\end{aligned}$$

Due to $\vartheta(\mathbb{R}) = +\infty$, we deduce that ii) is equivalent to i).

$ii) \Longrightarrow iii)$
Given $\varepsilon > 0$, we can denote by A_l^ε and B_l^ε the following sets

$$A_l^\varepsilon = \{t \in [-l,l]\backslash J : \sup_{\lambda \in [t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| > \varepsilon\}.$$

$$B_l^\varepsilon = \{t \in [-l,l]\backslash J : \sup_{\lambda \in [t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| \leq \varepsilon\}.$$

We have

$$\begin{aligned}
\int_{[-l,l]\backslash J} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) &= \int_{A_l^\varepsilon} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \\
&+ \int_{B_l^\varepsilon} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \\
&\geq \varepsilon\vartheta(A_l^\varepsilon).
\end{aligned}$$

Then

$$\frac{1}{\vartheta([-l,l]\setminus J)} \int_{[-l,l]\backslash J} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \geq \frac{\varepsilon\vartheta(A_l^\varepsilon)}{\vartheta([-l,l]\setminus J)}.$$

For l large enough, we obtain iii).

$iii) \Longrightarrow ii)$
For l sufficiently large and $\epsilon > 0$, we have

$$\begin{aligned}
\frac{1}{\vartheta([-l,l]\setminus J)} \int_{[-l,l]\backslash J} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) &\leq \frac{1}{\vartheta([-l,l]\setminus J)} \int_{A_l^\varepsilon} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \\
&+ \frac{1}{\vartheta([-l,l]\setminus J)} \int_{B_l^\varepsilon} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \\
&\leq \frac{2\vartheta(A_l^\varepsilon)\|\phi\|_\infty}{\vartheta([-l,l]\setminus J)} + \frac{\vartheta(B_l^\varepsilon)\varepsilon}{\vartheta([-l,l]\setminus J)} \\
&\leq \frac{2\vartheta(A_l^\varepsilon)\|\phi\|_\infty}{\vartheta([-l,l]\setminus J)} + \varepsilon.
\end{aligned}$$

Due to $\vartheta(\mathbb{R}) = +\infty$, then for any $\varepsilon > 0$, we have

$$\limsup_{l\to+\infty} \frac{1}{\vartheta([-l,l]\setminus J)} \int_{[-l,l]\backslash J} \sup_{\lambda\in[t-q,t]} \|\phi(\lambda+\omega) - \mathcal{E}(\omega)\phi(\lambda)\| d\vartheta(\lambda) \leq \varepsilon.$$

Consequently ii) holds. □

Proposition 3.4. *Assume that **(M1)** holds. If $\vartheta_i \in M$ $(i = 1, 2)$ and $\vartheta_1 \sim \vartheta_2$, then*

$$PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta_1, q) = PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta_2, q)$$

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7 Aug 2023 12:58:12 PDT
230409-Miraoui Version 2 - Submitted to Rocky Mountain J. Math.

Proof. since $\vartheta_1 \sim \vartheta_2$ and Δ is the Lebesgue σ field, then for all $A \in \Delta$ satisfying $A \cap J = \emptyset$, by Definition 3.1, there exists $\rho > 0, \gamma > 0$ such that

For l sufficiently large, one has

$$\begin{aligned} & \frac{\rho}{\gamma} \frac{\vartheta_1(t \in [-l,l] \backslash J; \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega)f(\lambda)\| > \varepsilon)}{\vartheta_1([-l,l] \setminus J)} \\ \leq \quad & \frac{\vartheta_2(t \in [-l,l] \backslash J; \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega)f(\lambda)\| > \varepsilon)}{\vartheta_2([-l,l] \setminus J)} \\ \leq \quad & \frac{\gamma}{\rho} \frac{\vartheta_1(t \in [-l,l] \backslash J; \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega)f(\lambda)\| > \varepsilon)}{\vartheta_1([-l,l] \setminus J)}. \end{aligned}$$

Then by Theorem 3.3, $PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta_1, q) = PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta_2, q)$. □

Proposition 3.5. *If **(M1)** holds and* $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$, *then* $f(.-\xi) \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$, *for all* $\xi \in \mathbb{R}$.

Proof. For $\vartheta \in \mathfrak{M}$, inasmuch as $\vartheta(\mathbb{R}) = +\infty$, there exist $l_0 > 0$ such that $\vartheta([-l- \mid \xi \mid, l+ \mid \xi \mid]) > 0$, $\forall l > l_0$. We consider

$$\xi^+ = max(0, \xi); \ \xi^- = max(0, -\xi).$$

Then one has $\mid \xi \mid + \xi = 2\xi^+$, $\mid \xi \mid - \xi = 2\xi^-$; so $[-l- \mid \xi \mid +\xi, l+ \mid \xi \mid +\xi] = [-l - 2\xi^-, l + 2\xi^+]$ for $l > l_0$ and $\xi \in \mathbb{R}$, we have

$$\begin{aligned} & \frac{1}{\vartheta([-l,l])} \int_{-l}^{l} \sup_{\lambda \in [t-q,t]} \|f(\lambda - \xi + \omega) - \mathcal{E}(\omega)f(\lambda - \xi)\| d\vartheta(\lambda) \\ & \leq \frac{1}{\vartheta([-l,l])} \int_{[-l-2\xi^-, l+2\xi^+]} \sup_{\lambda \in [t-q,t]} \|f(\lambda - \xi + \omega) - \mathcal{E}(\omega)f(\lambda - \xi)\| d\vartheta(\lambda) \\ & \leq \frac{\vartheta([-l-2\xi^-, l+2\xi^+])}{\vartheta([-l,l])} \Upsilon_\xi(l) \\ & \leq \frac{\vartheta([-l-2\mid \xi \mid, l+2\mid \xi \mid])}{\vartheta([-l,l])} \Upsilon_\xi(l), \end{aligned}$$

where

$$\Upsilon_\xi(l) = \frac{1}{\vartheta([-l-2\xi^-, l+2\xi^+])} \int_{[-l-2\xi^-, l+2\xi^+]} \sup_{\lambda \in [t-q,t]} \|f(\lambda - \xi + \omega) - \mathcal{E}(\omega)f(\lambda - \xi)\| d\vartheta(\lambda).$$

Then we have

$$\begin{aligned} \Upsilon_\xi(l) &= \frac{1}{\vartheta([-l- \mid \xi \mid +\xi, l+ \mid \xi \mid +\xi])} \int_{[-l-|\xi|+\xi, l+|\xi|+\xi]} \sup_{\lambda \in [t-q,t]} \|f(\lambda - \xi + \omega) - \mathcal{E}(\omega)f(\lambda - \xi)\| d\vartheta(\lambda) \\ &= \frac{1}{\vartheta_\xi([-l- \mid \xi \mid, l+ \mid \xi \mid])} \int_{[-l-|\xi|+\xi, l+|\xi|+\xi]} \sup_{\lambda \in [t-\xi-q, t-\xi]} \|f(\lambda + \omega) - \mathcal{E}(\omega)f(\lambda)\| d\vartheta(\lambda) \\ &= \frac{1}{\vartheta_\xi([-l- \mid \xi \mid, l+ \mid \xi \mid])} \int_{[-l-|\xi|, l+|\xi|]} \sup_{\lambda \in [t-q,t]} \|f(\lambda + \omega) - \mathcal{E}(\omega)f(\lambda)\| d\vartheta_\xi(t). \end{aligned}$$

Note that $\vartheta \sim \vartheta_\xi$. By Lemma 3.1, we have $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta_\xi,q)$. By proposition 3.4, so $\lim_{l\to+\infty}\Upsilon_\xi(l)=0$. By Lemma 3.2, one has

$$\lim_{l\to+\infty}\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda-\xi+\omega)-\mathcal{E}(\omega)f(\lambda-\xi)\|d\vartheta(\lambda)=0,$$

so $f(.-\xi)\in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$ for all $\xi\in\mathbb{R}$.
□

Remark 3.6. $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,0)=PSABP_{\omega,k}(\mathbb{R},\mathbb{L},\vartheta)$.

Proposition 3.7. *If **(M1)** holds.*
i) $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)\subset PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta)$
ii) $(PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q),\|.\|_\infty)$ *is a Banach space .*

Proof.
i) According to this inequality

$$\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda)\le\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda),$$

we see that i) holds.

ii) Let $f_n\in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$ and $f_n\to f$ *in* $\mathfrak{BC}(\mathbb{R},\mathbb{L})$. Then

$$\begin{aligned}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda) &\le \int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-f_n(\lambda+\omega)\|d\vartheta(\lambda)\\ &+ int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda)-f_n(\lambda)\|d\vartheta(\lambda)\\ &+ \int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f_n(\lambda+\omega)-\mathcal{E}(\omega)f_n(\lambda)\|d\vartheta(\lambda).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda)\\ &\le\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-f_n(\lambda+\omega)\|d\vartheta(\lambda)\\ &+\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda)-f_n(\lambda)\|d\vartheta(\lambda)\\ &+\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f_n(\lambda+\omega)-\mathcal{E}(\omega)f_n(\lambda)\|d\vartheta(\lambda)\\ &\le 2\|f-f_n\|_\infty+\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f_n(\lambda+\omega)-\mathcal{E}(\omega)f_n(\lambda)\|d\vartheta(\lambda).\end{aligned}$$

Then we have

$$\lim_{l\to+\infty}\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega)-\mathcal{E}(\omega)f(\lambda)\|d\vartheta(\lambda)\le 2\|f-f_n\|_\infty,$$

for all $n \in \mathbb{N}$. Since $\|f - f_n\|_\infty \longrightarrow 0$, we have

$$\lim_{l \to +\infty} \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) = 0.$$

□

Proposition 3.8. *If* $(M1)$ *holds and* $q_1 > 0, q_2 > 0$ *then*

$$PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_1) = PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_2).$$

Proof. Let $q > 0$. Initially, we prove that

$$PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q) \subset PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, 2q).$$

For $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$, one has

$$\begin{aligned}
\frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-2q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) &\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-2q,t-q]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) \\
&+ \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega-q) - \mathcal{E}(\omega) f(\lambda-q)\| d\vartheta(\lambda) \\
&+ \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda).
\end{aligned}$$

By proposition 3.5, we have:

$$\lim_{l \to +\infty} \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-2q,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) = 0.$$

Then $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, 2q)$.

Now let $q_1 > q_2$, if $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, \chi, q_1)$, then

$\frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q_2,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda) \leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [t-q_1,t]} \|f(\lambda+\omega) - \mathcal{E}(\omega) f(\lambda)\| d\vartheta(\lambda)$,

so $f \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_2)$. Therefore

$$PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_1) \subset PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_2).$$

On the other hand since $q_1 > q_2$ there exists $d \in \mathbb{N}$ such that $2^d q_2 > q_1$, from the above we have

$$PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_2) \subset PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, 2^d q_2) \subset PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q_1).$$

The proof is complete. □

7 Aug 2023 12:58:12 PDT
230409-Miraoui Version 2 - Submitted to Rocky Mountain J. Math.

Proposition 3.9. *If $(M1)$ holds and $\phi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,,q)$, then $\phi_t \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathcal{C},\vartheta,q)$.*

Proof.
For $\vartheta \in \mathfrak{M}$, $q \geq 0$, since $\vartheta(\mathbb{R}) = +\infty$, there exist l_0 such that $\vartheta([-l-q,l+q]) > 0$ for all $l > l_0$. Hence for $l > l_0$, one has

$$\begin{aligned}
&\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\sup_{\varsigma\in[-q,0]}\|\phi(\lambda+\varsigma+\omega)-\mathcal{E}(\omega)\phi(\lambda+\varsigma)\|d\vartheta(\lambda)\\
&\leq \frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\zeta\in[t-2q,t-q]}\|\phi(\zeta+\omega)-\mathcal{E}(\omega)\phi(\zeta)\|d\vartheta(t)\\
&+\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\zeta\in[t-q,t]}\|\phi(\zeta+\omega)-\mathcal{E}(\omega)\phi(\zeta)\|d\vartheta(t).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-2q,t-q]}\|\phi(\lambda+\omega)-\mathcal{E}(\omega)\phi(\lambda)\|d\vartheta(\lambda)\\
&=\frac{1}{\vartheta([-l,l])}\int_{[-l-q,l-q]}\sup_{\lambda\in[t-q,t]}\|\phi(\lambda+\omega)-\mathcal{E}(\omega)\phi(\lambda)\|d\vartheta_q(t)\\
&=\frac{\vartheta_q([-l-q,l+q])}{\vartheta([-l,l])}\times\frac{1}{\vartheta_q([-l-q,l+q])}\int_{[-l-q,l+q]}\sup_{\lambda\in[t-q,t]}\|\phi(\lambda+\omega)-\mathcal{E}(\omega)\phi(\lambda)\|d\vartheta_q(t).
\end{aligned}$$

Since $\vartheta \sim \vartheta_q$, then by Lemma 3.1 we have $\phi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta_q,\vartheta_q,q)$. By proposition 3.4 so

$$\lim_{l\to+\infty}\frac{1}{\vartheta_q([-l-q,l+q])}\int_{[-l-q,l+q]}\sup_{\lambda\in[t-q,t]}\|\phi(\lambda+\omega)-\mathcal{E}(\omega)\phi(\lambda)\|d\vartheta_q(t)=0,$$

then

$$\lim_{l\to+\infty}\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-2q,t-q]}\|\phi(\lambda+\omega)-\mathcal{E}(\omega)\phi(\lambda)\|d\vartheta(\lambda)=0. \tag{3.1}$$

Since $\phi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$ and (3.1) hold, then $\phi_t \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathcal{C},\vartheta,q)$.

□

Theorem 3.10. *Let $f \in \mathfrak{BC}(\mathbb{R}\times\mathbb{L},\mathbb{W})$ satisfies the followings conditions*
1) $\forall\ (\varrho,z)\in\mathbb{R}\times\mathbb{L}$, $f(\varrho+\omega,z)=\mathcal{E}(\omega)f(\varrho,\mathcal{E}(\omega)^{-1}z)$.

2) $\exists\ L>0$ such that $\forall(z,y)\in\mathbb{L}^2$ and $\varrho\in\mathbb{R}$

$$\|f(\varrho,z)-f(\varrho,y)\|\leq L\|z-y\|.$$

Then $\forall\ \chi\in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$, we have $\varrho\longmapsto f(\varrho,\chi(\varrho))\in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{W},\vartheta,q)$.

Proof.
We have

$$\begin{aligned}
&\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|f(\lambda+\omega,\chi(\lambda+\omega))-\mathcal{E}(\omega)f(\lambda,\chi(\lambda))\|d\vartheta(\lambda)\\
&=\frac{1}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|\mathcal{E}(\omega)f(\lambda,\mathcal{E}(\omega)^{-1}\chi(\lambda+\omega))-\mathcal{E}(\omega)f(\lambda,\chi(\lambda))\|d\vartheta(\lambda)\\
&\leq\frac{L|\mathcal{E}(\omega)|}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|\mathcal{E}(\omega)^{-1}\chi(\lambda+\omega)-\chi(\lambda)\|d\vartheta(\lambda)\\
&\leq\frac{L}{\vartheta([-l,l])}\int_{[-l,l]}\sup_{\lambda\in[t-q,t]}\|\chi(\lambda+\omega)-\mathcal{E}(\omega)\chi(\lambda)\|d\vartheta(\lambda).
\end{aligned}$$

Since $\chi\in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,q)$, hence the proof is complete. □

4 Partial functional differential equations with finite delay

In this part, We focus to the existence and uniqueness of $\vartheta-$Pseudo S-asymptotically $(\omega,\mathcal{E})$ periodic solution for neutral partial differential equation with finite delay. Now we need the following assumptions.

(A1) The operator $\mathcal{J}:\mathcal{D}(\mathcal{J})\subset\mathbb{L}\longmapsto\mathbb{L}$ is the infinitesimal generator of an exponentially stable C_0 Semi Group $(\Phi(w))_{w\geq 0}$, that is there exist constants $K_0,\varpi>0$, such that

$$\|\Phi(w)\|\leq K_0e^{-\varpi w}, w\geq 0.$$

(A2) $\mathcal{F}\in\mathfrak{BC}(\mathbb{R}\times\mathcal{C},\mathbb{W})$ and there exists $L_{\mathcal{F}}>0$ such that

$$\|\mathcal{F}(w,\xi_1)-\mathcal{F}(w,\xi_2)\|\leq L_{\mathcal{F}}\|\xi_1-\xi_2\|,$$

for every $w\in\mathbb{R}$, and $\xi_1,\xi_2\in\mathcal{C}$. Moreover $\forall(\eta,z)\in\mathbb{R}\times\mathbb{L},\mathcal{F}(\eta+\omega,z)=\mathcal{E}(\omega)\mathcal{F}(\eta,\mathcal{E}(\omega)^{-1}z)$.

(A3) $\mathcal{G}\in\mathfrak{BC}(\mathbb{R}\times\mathcal{C},\mathbb{W})$ and there exists $L_{\mathcal{G}}>0$ such that

$$\|\mathcal{G}(w,\xi_1)-\mathcal{G}(w,\xi_2)\|\leq L_{\mathcal{G}}\|\xi_1-\xi_2\|,$$

for every $w\in\mathbb{R}$, and $\xi_1,\xi_2\in\mathcal{C}$. Moreover $\forall(\eta,z)\in\mathbb{R}\times\mathbb{L},\mathcal{G}(\eta+\omega,z)=\mathcal{E}(\omega)\mathcal{G}(\eta,\mathcal{E}(\omega)^{-1}z)$.

Definition 4.1. A continuous function $y:\mathbb{R}\longmapsto\mathbb{L}$ is said to be a mild solution of Equation (1.1) if

$$\begin{cases} y(t)=\mathcal{F}(t,y_t)+\Phi(t)[\varphi(0)-\mathcal{F}(0,\varphi)]+\displaystyle\int_0^t\Phi(t-s)\mathcal{G}(s,y_s)ds\ for\ all\ t\geq 0,\\ y_0=\varphi.\end{cases}\tag{4.1}$$

Lemma 4.1. *Let $\vartheta \in \mathfrak{M}$ satisfies $(M1)$ and $u \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$. If Ψ is the function defined by*

$$\Psi(\alpha) = \int_{-\infty}^{\alpha} \Phi(\alpha - s)u(s)ds; \; for \; all \; \alpha \in \mathbb{R},$$

then $\Psi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$.

Proof. We have

$$\begin{aligned}
&\frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \|\Psi(\lambda + \omega) - \mathcal{E}(\omega)\Psi(\lambda)\| d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \| \int_{-\infty}^{\lambda+\omega} \Phi(\lambda + \omega - s)u(s)ds - \mathcal{E}(\omega) \int_{-\infty}^{\lambda} \Phi(\lambda - s)u(s)ds \| d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \| \int_{-\infty}^{\lambda} \Phi(\lambda - s)u(s + \omega)ds - \mathcal{E}(\omega) \int_{-\infty}^{\lambda} \Phi(\lambda - s)u(s)ds \| d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \int_{-\infty}^{\lambda} \|\Phi(\lambda - s)(u(s + \omega) - \mathcal{E}(\omega)u(s))\| ds d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \int_{0}^{+\infty} \|\Phi(n)(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| dn d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \sup_{\lambda \in [\alpha - q, \alpha]} \int_{0}^{+\infty} \|\Phi(n)\| \|(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| dn d\vartheta(\alpha) \\
&\leq \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} \int_{0}^{+\infty} \sup_{\lambda \in [\alpha - q, \alpha]} K_0 e^{-\varpi n} \|(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| dn d\vartheta(\alpha) \\
&\leq \int_{0}^{+\infty} \frac{K_0}{\vartheta([-l,l])} \int_{[-l,l]} e^{-\varpi n} \sup_{\lambda \in [\alpha - q, \alpha]} \|(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| d\vartheta(\alpha) dn.
\end{aligned}$$

Since $\lim_{l \to +\infty} \frac{1}{\vartheta([-l,l])} \int_{[-l,l]} e^{-\varpi n} \sup_{\lambda \in [\alpha - q, \alpha]} \|(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| d\vartheta(\alpha) = 0$, then

$\frac{K_0}{\vartheta([-l,l])} \int_{[-l,l]} e^{-\varpi n} \sup_{\lambda \in [\alpha - q, \alpha]} \|(u(\lambda + \omega - n) - \mathcal{E}(\omega)u(\lambda - n))\| d\vartheta(\alpha) \leq 2K_0.cst.\|u\|_\infty e^{-\varpi n} := h(n)$.

Since $h \in L^1(\mathbb{R}_+)$, then, by the Lebesgue's Dominated Convergence, we obtain $\Psi \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, q)$. □

Theorem 4.2. *Suppose that $(M1)$,(A1), (A2) and (A3) hold. If $L_{\mathcal{F}} + \frac{K_0}{\varpi} L_{\mathcal{G}} < 1$, then our equation has a unique mild solution $u \in PSABP_{\omega,\mathcal{E}}(\mathbb{R}, \mathbb{L}, \vartheta, r)$ that given by*

$$u(t) = \mathcal{F}(t, u_t) + \int_{-\infty}^{t} \Phi(t - s)\mathcal{G}(s, u_s)ds. \tag{4.2}$$

Proof. Suppose u satisfies (4.2), then

$$\begin{aligned} u(t) &= \mathcal{F}(t,u_t) + \int_{-\infty}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds \\ &= \mathcal{F}(t,u_t) + \int_{-\infty}^{0} \Phi(t-s)\mathcal{G}(s,u_s)ds + \int_{0}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds \\ &= \mathcal{F}(t,u_t) + \Phi(t)\int_{-\infty}^{0} \Phi(-s)\mathcal{G}(s,u_s)ds + \int_{0}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds \\ &= \mathcal{F}(t,u_t) + \Phi(t)[u(0)-\mathcal{F}(0,u_0)] + \int_{0}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds \\ &= \mathcal{F}(t,u_t) + \Phi(t)[\varphi(0)-\mathcal{F}(0,\varphi)] + \int_{0}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds, \end{aligned}$$

then $u(t) = \mathcal{F}(t,u_t) + \int_{-\infty}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds$ verified equation (4.1).
Now on $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$, we define the operator Λ from $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$ to $C(\mathbb{R},\mathbb{L})$ by

$$\Lambda u(t) = \mathcal{F}(t,u_t) + \int_{-\infty}^{t} \Phi(t-s)\mathcal{G}(s,u_s)ds.$$

Λu is well defined and continuous : moreover from Lemma 4.2, Proposition 3.9 and Theorem 3.10 $\Lambda u \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$ that is $\Lambda : PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r) \longmapsto PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$.

For $y,z \in PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$, we have

$$\begin{aligned} &\|\Lambda y(t) - \Lambda z(t)\| = \|\mathcal{F}(t,y_t) - \mathcal{F}(t,z_t) + \int_{-\infty}^{t} \Phi(t-s)(\mathcal{G}(s,y_s) - \mathcal{G}(s,z_s))ds\| \\ &\leq L_{\mathcal{F}}\|y-z\|_\infty + \int_{-\infty}^{t} L_{\mathcal{G}}\|\Phi(t-s)\|\|y-z\|_\infty ds \\ &\leq L_{\mathcal{F}}\|y-z\|_\infty + \int_{-\infty}^{t} L_{\mathcal{G}}K_0 e^{-\varpi(t-s)}\|y-z\|_\infty ds \\ &\leq L_{\mathcal{F}}\|y-z\|_\infty + \int_{0}^{+\infty} L_{\mathcal{G}}K_0 e^{-\varpi\xi}\|y-z\|_\infty d\xi \\ &\leq (L_{\mathcal{F}} + \frac{K_0}{\varpi}L_{\mathcal{G}})\|y-z\|_\infty \end{aligned}$$

From the Fixed point Theorem of Picard, we infer that our equation admits a unique solution in $PSABP_{\omega,\mathcal{E}}(\mathbb{R},\mathbb{L},\vartheta,r)$. □

5 Application

In this Section, we discuss the existence and uniqueness of $\vartheta-$pseudo S-asymptotically $(\omega,\mathcal{E})$ periodic solution of the scalar reaction-diffusion equation with delay given by

$$\begin{cases} \frac{\partial}{\partial t}[u(t,x) - \mathrm{h}_1(t,u(t-p,x))] = \frac{\partial}{\partial x^2}[u(t,x) - \mathrm{h}_1(t,u(t-p,x))] \\ \quad +\mathrm{h}_2(t,u(t-p,x)) for\ \ t \in \mathbb{R}\ and\ x \in [0,\pi] \\ u(t,0) = u(t,\pi) = 0 for\ t \in \mathbb{R} \\ u(\theta,x) = u_0(\theta,x)\ for\ \ -p \le \theta \le 0\ and\ x \in [0,\pi] \end{cases} \tag{5.1}$$

where $p > 0$, $u_0 \in C([-p,0] \times [0,\pi],\mathbb{R})$, $\mathrm{h}_1, \mathrm{h}_2 : \mathbb{R}^2 \longmapsto \mathbb{R}$ are continuous functions.

We pick the space $\mathbb{L} = L^2([0,\pi],\mathbb{R})$ equipped with the L^2 norm $\|.\|_2$ and $\mathcal{E}(x) = e^{ixk}$ where $k \in \mathbb{R}$, we take the operator $\mathcal{J} : \mathcal{D}(\mathcal{J}) \subset \mathbb{L} \longmapsto \mathbb{L}$ is defined by

$$\begin{cases} \mathcal{D}(\mathcal{J}) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ \mathcal{J}\chi = \chi^{''}\ for\ \chi \in\ \mathcal{D}(\mathcal{J}). \end{cases} \tag{5.2}$$

Then the spectrum $\sigma(\mathcal{J})$ of $\mathcal{J}$ is equal to the point spectrum $\sigma_p(\mathcal{J})$ and is given by

$\sigma(\mathcal{J}) = \sigma_p(\mathcal{J}) = \{-d^2;\ d \in \mathbb{N}\}$ and the associated eigenfunctions $(v_n)_{n\ge1}$ are given by

$v_n(k) = \sqrt{\frac{2}{\pi}}\sin(nk)$ for $k \in [0,\pi]$ so $\mathcal{J}y = -\sum_{n=1}^{+\infty} n^2(y,v_n)v_n$ for $y \in \mathcal{D}(\mathcal{J})$.

We knew that $\mathcal{J}$ is the infinitesimal of an analytic semi group uniformly exponentially stable $(\Phi(t))_{t\ge0}$ on $\mathbb{L}$ which is given by

$$\Phi(t)x = \sum_{n=1}^{+\infty} e^{-n^2t}(x,v_n)v_n, \text{ for } x \in \mathbb{L}.$$

It follows that $0 \in \rho(A)$ and $\|\Phi(\varrho)\| \le e^{-\varrho}$ for any $\varrho \ge 0$, which give us the hypothesis (A1) is verified . Now we select the measure ϑ where its Radon- Nikodym derivative of

$$\varrho(w) = \begin{cases} e^w\ if\ \ w \le 0, \\ 1\ \ if\ \ w > 0. \end{cases} \tag{5.3}$$

Then from [9], $\vartheta \in \mathfrak{M}$ satisfies $(M1)$. Let $\mathcal{F}$ and $\mathcal{G}$ two functions from $\mathbb{R} \times \mathcal{C}$ to $\mathbb{L}$ defined by

$\mathcal{G}(t,\zeta)(x) = h_2(t,\zeta(-p)(x))$ for $\zeta \in \mathcal{C}$ and $x \in [0,\pi]$;

$\mathcal{F}(t,\zeta)(x) = h_1(t,\zeta(-p)(x))$ for $\zeta \in \mathcal{C}$ and $x \in [0,\pi]$.

Let $x(t) = u(t,.)$ for $t \ge 0$ and $\zeta(\kappa) = u_0(\kappa,.)$ for $\kappa \in [-p,0]$.

Now, we pick $\mathcal{F}(t,\zeta)(x) = \mathcal{G}(t,\zeta)(x) = \Upsilon(t)\zeta(x)$, where ψ is a continuous ω periodic function, moreover there exist $\mathfrak{Y} \in \mathbb{R}$ such that $\|\psi\| \le \mathfrak{Y}$. The functions $\mathcal{F}$ and $\mathcal{G}$ verified hypotheses (A1) and (A2),for more precision, we can see Example 4.4 in [19].

7 Aug 2023 12:58:12 PDT
230409-Miraoui Version 2 - Submitted to Rocky Mountain J. Math.

If $\mathfrak{Y} < \frac{1}{2}$,then from theorem 4.2, we find a unique ϑ pseudo S-asymptotically $(\omega, \mathcal{E})$ periodic solution to our equation.

6 Conclusion

We have investigated the existence and uniqueness of $\vartheta-$pseudo-S-asymptotically $(\omega, \mathcal{E})$ type periodic mild solutions for some differential equations with finite delay. In this case, we used the fixed point theorem of Picard and several Lemmas of compositions to demonstrate the existence and uniqueness of mild solutions. Last but not least, an example is provided to illustrate the obtained theoretical results. Further research into stochastic differential equations with delay is planned in the future, and the results obtained in this paper can be used to investigate the existence of optimal control for some stochastic partial differential equations with variable delay.

7 Conflicts of Interest Statement

We have no conflicts of interest to disclose.

8 Data Availability Statement

My manuscript has no associate data.

References

[1] E. Ait Dads, S. Khelifi and M. Miraoui. On the Differential Equations with Piecewise Constant Argument. Journal of Dynamical and Control Systems. DOI: 10.1007/s10883-022-09633-x. (2022).

[2] H. Assel, M.A. Hammami, M. Miraoui. Dynamics and oscillations for some difference and differential equations with piecewise constant arguments. Asian Journal of Control (2021), 1-9.

[3] M. Ben Salah, Y. Khemili and M. Miraoui. Pseudo asymptotically Bloch periodic solutions with measures for some differential Equations. Rocky Mountain Journal of Mathematics. (Accepted 2023).

[4] R. Bellman and K. Cooke. Differential-Difference Equations, Academic Press,1963.

[5] Y. K. Chang and Y. Wei. Pseudo S-asymptotically Bloch type periodicity with applications to some evolution equations, Z. Anal. Anwend, vol. 40, pp. 3350, 2021.

[6] S. Chen, Y-K. Chang and Y. Wei. Pseudo S-asymptotically Bloch type periodic solutions to a damped evolution equation. Evolution Equations and Control Theory, 2022, 11 (3).

[7] O. Diekman, S.A.Van Gils, S.M. Verdayn Lunel and H.O. Wather. Delay Equation : Functional, complex, and Nonlinear Analysis, Springer-Verlag, New York (1995).

[8] L-E. ELsgolts and S-B. Norkin, Introduction to the Theory and Application of Diffential Equations with deviating Argument, Mathematcs in science and Engineering, Vol. 105. Academic Press, 1973).

[9] K. Ezzinbi, M. Miraoui and A Rebey. Measure Pseudo-Almost Periodic Solutions in the -Norm to Some Neutral Partial Differential Equations with Delay. Mediterr. J. Math. 13, 34173431 (2016).

[10] H. Gao, K. Wang and F. Wei. Massera-type theorem and asymptotically periodic logisitc equations. Nonlinear Anal RWA, 2006, 7: 1268-1283 .

[11] M-F. Hasler and G-M. N'Gurrkata. Bloch-periodic functions and some applications. Nonlinear Studied, 2014, 21: 21-30.

[12] H-R. Herniquez, M. Pierri and P. Táboas. On S-asymptotically ω-periodic functions on Banach spaces and applications. J Math Anal Appl, 2008, 343: 1119-1130.

[13] Miraoui M. (2017). *Existence of $\mu-$pseudo almost periodic solutions to some evolution equations.* Mathematical Methods in the Applied Sciences. 40(13), 4716–4726.

[14] Miraoui M. (2017). *Pseudo almost automorphic solutions for some differential equations with reflection of the argument.* Numerical Functional Analysis and Optimization. 38(3), 376–394.

[15] Miraoui M. and D.D. Repovš, Dynamics and oscillations of models for differential equations with delays. Boundary Value Problems, doi: 10.1186/s13661-020-01348-x (2020).

[16] E-R. Oueama-Guengai and G-M. N'Gurrkata. On S-asymptotically ω-periodic and Bloch periodic mild solutions to some fractional differential equations in abstract spaces. Math Meth Appl Sci, 2018, 41: 9116-9122.

[17] V. Volterra, Sur la Thorie Mathmatiques des phnomnes hrditaires, J.de Mathmatiques 7 (1928), 249-298.

[18] Z. Xia, D. Wang, C-F. Wen and J-C. Yao. Pseudo asymptotically periodic mild solutions of semilinear functional integro-differential equations in banach spaces. Math Meth Appl Sci .2017;123.

[19] Y-K.Chang and Y.Wei .Pseudo-S-Asymptotically Bloch Type Periodicity with applications to some evolution Equations ,Z. Anal .Anwend . , 40 (2021),33-50.