Homoclinic solutions for damped vibration systems with sublinear or superquadratic nonlinearities

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Abstract. In this paper we prove the existence of fast homoclinic solutions for the following class of damped vibration systems

$$\ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+b(t)\left|u(t)\right|^{p-2}u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R}$$

where $L(t)$ is a symmetric matrix-valued function only uniformly positive definite, $p>2$, $b\in C(\mathbb{R},\mathbb{R})$ and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is with sublinear nonlinearity or satisfying a new superquadratic condition generalizing the well-known Ambrosetti-Rabinowitz condition. To the best of our knowledge, our results are new and generalize some recent results in the literature.

Keywords. Damped vibration systems, fast homoclinic solutions, variational methods, critical point theory, sublinear nonlinearity, superquadratic condition.

Mathematical subject classification: 34J45, 35J61, 58E30.

1. Introduction.

In this paper, we are concerned with the following damped vibration system

$$(\mathcal{DV})\qquad \ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+b(t)\left|u(t)\right|^{p-2}u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R}$$

where $q,b:\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, $p>2$ is a constant, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function only uniformly positive definite and $W:\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a continuous function, differentiable with respect to the second variable with continuous derivative $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$.

When $q(t)=0$ and $b(t)=0$ for all $t\in\mathbb{R}$, system $(\mathcal{DV})$ reduces to the following second order Hamiltonian system

$$(\mathcal{HS})\qquad \ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R}.$$

As usual, a solution u of $(\mathcal{HS})$ is called homoclinic (to 0) if $u(t)\longrightarrow 0$ as $|t|\longrightarrow\infty$. If moreover $u\neq 0$, then u is called a nontrivial homoclinic solution. In the last three decades, the existence of homoclinic solutions for system $(\mathcal{HS})$ has been studied by many mathematicians via critical point theory and variational methods, see for example 2-4,6,8-14,17-21,35,38,39] and the references cited therein.

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During the last ten more years, some authors have been concerned with the fast homoclinic solutions (see Definition 2.1) for system $(\mathcal{DV})$, with $b(t) = 0$ for all $t \in \mathbb{R}$, see [1,5,15,16,28-34,36,37] and the references listed therein. In these last papers, the function L is required to satisfy different coercive conditions such as
(1.1) $L(t)$ is a positive definite matrix for all $t \in \mathbb{R}$ and the smallest eigenvalue of $L(t)$

$$l(t) = \inf_{|\xi|=1} L(t)\xi \cdot \xi \longrightarrow \infty \; as \; |t| \longrightarrow \infty;$$

(1.2) there exists a constant $\gamma < 0$ such that

$$l(t)\,|t|^{\gamma-1} \longrightarrow \infty \; as \; |t| \longrightarrow \infty;$$

(1.3) $l(t)$ is bounded from below and there exists a constant $r_0 > 0$ such that

$$\lim_{|s|\longrightarrow\infty} meas_Q(\{t \in (s-r_0, s+r_0)/L(t) < MI_N\}) = 0, \; \forall M > 0,$$

where $meas_Q$ denotes the Lebesgue's measure on $\mathbb{R}$ with density $e^{Q(t)}$ and $Q(t) = \int_0^t q(s)ds$;
(1.4) $l(t)$ is bounded from below and there exists a constant $\gamma > 1$ such that

$$meas_Q(\left\{t \in \mathbb{R}/\,|t|^{-\gamma} L(t) < MI_N\right\}) < +\infty, \; \forall M > 0.$$

Moreover in these papers, Q is such that $Q(t) \longrightarrow \infty$ as $|t| \longrightarrow \infty$ and the potential $W(t,x)$ is assumed to be subquadratic, superquadratic, asymptotically quadratic at infinity with respect to the second variable or a combination of a subquadratic and a superquadratic terms.

Furthermore, in [7], Cheng consider the second-order Hamiltonian system

$$\ddot{u}(t) + b(t)\,|u(t)|^{\mu-2}\,u(t) + \nabla H(t, u(t)) = 0 \tag{1.5}$$

where $\mu > 2$ is a constant, $b \in C(\mathbb{R}, \mathbb{R})$ and $H \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ are $T-$periodic in the first variable and obtained the existence of periodic solutions for system (1.5) under the following conditions

$$b \in C(\mathbb{R}, \mathbb{R}) \; and \; \int_0^T b(t)dt > 0; \tag{1.6}$$

$$\limsup_{|x|\longrightarrow 0} \frac{H(t,x)}{|x|^2} = 0, \; uniformly \; for \; all \; t \in \mathbb{R}; \tag{1.7}$$

(1.8) there exist two periodic functions $g, h \in L^1(0,T;\mathbb{R}^+)$ and a constant $0 \leq \nu < 1$ such that

$$|\nabla H(t,x)| \leq g(t)\,|x|^{\nu} + h(t), \; \forall (t,x) \in [0,T] \times \mathbb{R}^N;$$

$$H(t,-x) = H(t,x), \; \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.9}$$

Motivated by the previous works, in this paper we are interested to the existence of nontrivial fast homoclinic solutions for $(\mathcal{DV})$ when $L(t)$ is uniformly positive definite not necessary coercive and the potential $W(t,x)$ satisfies some new conditions. More precisely, Section 3 is devoted to the case when $b \neq 0$ and the nonlinearity $\nabla W(t,x)$ grows faster than $|x|^{\nu}$, $0<\nu<1$. In Section 4, $b=0$ and the potential $W(t,x)$ satisfies a new condition weaker than the well-known Ambrosetti-Rabinowitz superquadratic condition. To the best of our knowledge, our results are new and generalize some recent results in the literature.

2. Preliminaries.

In order to introduce the concept of fast homoclinic solutions for $(\mathcal{DV})$ conveniently, we firstly describe some properties of the weighted Sobolev space E on which the certain variational functional associated with $(\mathcal{DV})$ is defined and the fast homoclinic solutions of $(\mathcal{DV})$ are the critical points of such functional. We shall use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$< u,v >_{L^2_Q}= \int_{\mathbb{R}} e^{Q(t)} u(t)\cdot v(t)dt$$

and the induced norm

$$\|u\|_{L^2_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Similarly, $L^s_Q(\mathbb{R})$ $(1\leq s<\infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^s_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s dt\right)^{\frac{1}{s}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = esssup\left\{e^{\frac{Q(t)}{2}} |u(t)| /t\in\mathbb{R}\right\}.$$

In this paper, we assume that L satisfies the following condition
(L) $L(t)$ is uniformly positive definite.
Let

$$E=\left\{u\in H^1_Q(\mathbb{R})/\int_{\mathbb{R}} e^{Q(t)}L(t)u(t)\cdot u(t)dt<\infty\right\}$$

where

$$H^1_Q(\mathbb{R})=\left\{u\in L^2_Q(\mathbb{R})/\dot{u}\in L^2_Q(\mathbb{R})\right\}.$$

Then E equipped with the following inner product and norm is a Hilbert space

$$< u,v >= \int_{\mathbb{R}} e^{Q(t)}\left(\dot{u}(t)\cdot\dot{v}(t)+L(t)u(t)\cdot v(t)\right)dt,\ u,v\in E$$

$$\|u\| =< u,u >^{\frac{1}{2}},\ u\in E.$$

Evidently, under assumption (L), E is continuously embedded in $H_Q^1(\mathbb{R})$ and hence E is continuously embedded in $L_Q^s(\mathbb{R})$ for $2 \leq s \leq \infty$, that is for all $2 \leq s \leq \infty$, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L_Q^s} \leq \eta_s \|u\|, \ \forall u \in E. \tag{2.1}$$

Definition 2.1. A solution u of $(\mathcal{DV})$ is called a fast homoclinic orbit if $u \in E$.

To study the critical points of the variational functional associated with $(\mathcal{DV})$, we recall the following critical point theorem.

Lemma 2.1 (Mountain Pass Theorem) [22]. Let E be a Banach space and $I \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition and $I(0) = 0$. If I satisfies the following conditions
(i) there exist constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho} \geq \alpha$;
(ii) there exists $e \in E \setminus \overline{B}_\rho$ such that $I(e) \leq 0$.
Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s))$$

where B_ρ is the open ball in E of radius ρ about 0, and

$$\Gamma = \{g \in C([0,1], E)/g(0) = 0, \ g(1) = e\}.$$

3. Sublinear nonlinearity.

In this Section, we are concerned with the sublinear nonlinearity case. More precisely, we consider the following conditions
(B) $b \in C(\mathbb{R}, \mathbb{R})$ and $\int_{\mathbb{R}} e^{Q(t)} b(t)dt > 0$;
(W_1) there exists a constant $r > 0$ such that

$$W(t, x) \leq \frac{1}{4\eta_2^2} |x|^2, \ \forall t \in \mathbb{R}, \ |x| \leq r,$$

where η_2 is a sobolev constant defined in Section 2;
(W_2) there exist two functions $g, h \in L_Q^1(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla W(t, x)| \leq g(t)\gamma(|x|) + h(t), \ \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a nondecreasing function with the properties $\lim_{s \longrightarrow \infty} \frac{\gamma(s)}{s} = 0$ and $\lim_{s \longrightarrow \infty} \gamma(s) = \infty$.
Our main result in this Section reads as follows

Theorem 3.1. Assume that (L), (B), (W_1) and (W_2) are satisfied. Then $(\mathcal{DV})$ possesses a nontrivial fast homoclinic solution.

Example 3.1. Consider the map

$$W(t,x)=\frac{1}{2\eta_2^2(1+|t|^2)}\frac{|x|^2}{\ln(e+|x|^2)}\ \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N$$

and set $g(t)=\frac{1}{\eta_2^2(1+|t|^2)}$. We have for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$

$$\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)=g(t)x\frac{(e+|x|^2)\ln(e+|x|^2)-|x|^2}{(e+|x|^2)\ln^2(e+|x|^2)}.$$

Hence one gets

$$|\nabla W(t,x)|\le g(t)\frac{|x|}{\ln(e+|x|^2)}.$$

Set $\gamma(s)=\frac{s}{\ln(e+s^2)}$. It is clear that $\lim_{s\longrightarrow\infty}\gamma(s)=+\infty$ and $\lim_{s\longrightarrow\infty}\frac{\gamma(s)}{s}=0$. It remains to prove that γ is nondecreasing. For $s>0$, we have

$$\gamma'(s)=\frac{(e+s^2)\ln(e+s^2)-2s^2}{(e+s^2)\ln^2(e+s^2)}.$$

Let $\theta(u)=(e+u)\ln(e+u)-2u$, we have for $u>0$

$$\theta'(u)=\ln(e+u)-1>0.$$

Hence θ is nondecreasing and then $\theta(u)\ge\theta(0)=0$. Therefore γ is nondecreasing and the function $W(t,x)$ satisfies all the conditions of Theorem 3.1.

Proof of Theorem 3.1. Consider the variational functional associated to system $(\mathcal{DV})$, defined on the space E introduced in Section 2, by

$$\varphi(u)=\frac{1}{2}\int_{\mathbb{R}}e^{Q(t)}\left[|\dot u(t)|^2+L(t)u(t)\cdot u(t)\right]dt-\frac{1}{p}\int_{\mathbb{R}}e^{Q(t)}b(t)\,|u(t)|^p\,dt-\int_{\mathbb{R}}e^{Q(t)}W(t,u(t))dt.$$

It is well known that under assumption (W_2), the functional φ is continuously differentiable on E and

$$\begin{aligned}\varphi'(u)v=&\int_{\mathbb{R}}e^{Q(t)}\left[\dot u(t)\cdot\dot v(t)+L(t)u(t)\cdot v(t)\right]dt\\&-\int_{\mathbb{R}}e^{Q(t)}\,|u(t)|^{p-2}\,u(t)\cdot v(t)dt-\int_{\mathbb{R}}e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt\\&=<u,v>-\int_{\mathbb{R}}e^{Q(t)}\,|u(t)|^{p-2}\,u(t)\cdot v(t)dt-\int_{\mathbb{R}}e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt\end{aligned}$$

for all $u,v\in E$. Moreover, the nontrivial critical points of φ on E are fast homoclinic solutions of $(\mathcal{DV})$. In the following, we will proceed by successive lemmas.

Lemma 3.1. Suppose that (W_2) holds, then there exist two positive constants c_0, c_1 such that

$$\|\nabla W(t,u)\|_{L^1_Q} \leq c_0\gamma(\|u\|_{L^\infty}) + c_1,\ \forall u \in E.$$

Proof: Let $u \in E$. By (W_2) and the increasing property of γ, one has

$$\begin{aligned}\|\nabla W(t,u)\|_{L^1_Q} &= \int_{\mathbb{R}} e^{Q(t)}\,|\nabla W(t,u)|\,dt \leq \int_{\mathbb{R}} e^{Q(t)}\,[g(t)\gamma(|u(t)|) + h(t)]\,dt \\ &\leq c_0\gamma(\|u\|_{L^\infty}) + c_1,\end{aligned}$$

where $c_0 = 1 + \|g\|_{L^1_Q}$ and $c_1 = 1 + \|h\|_{L^1_Q}$. ■

Lemma 3.2. Suppose that (L), (B) and (W_1) are satisfied. Then there exist two positive constants ρ, α such that $\varphi_{|\partial B_\rho} \geq \alpha$.

Proof: Let $\rho_0 \in (0, \frac{r}{\eta_\infty})$ and $u \in B_\rho$. Then we have $|u(t)| \leq \|u\|_{L^\infty} \leq r$ for all $t \in \mathbb{R}$. Hence (B) implies

$$\begin{aligned}\varphi(u) &\geq \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\,|\dot{u}(t)|^2\,dt - \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}b(t)dt\,\|u\|^p_{L^\infty} - \frac{1}{2\eta_2^2}\int_{\mathbb{R}} e^{Q(t)}\,|u|^2\,dt \\ &\geq \frac{1}{4}\|u\|^2 - \frac{\eta_\infty^p}{p}\int_{\mathbb{R}} e^{Q(t)}b(t)dt\,\|u\|^p.\end{aligned}$$

Choosing $\rho \in (0, \min\left\{\rho_0, \left(\frac{1}{2\eta_\infty^p\int_{\mathbb{R}} e^{Q(t)}b(t)dt}\right)^{\frac{1}{p-2}}\right\})$ small enough such that

$$\alpha = \frac{1}{4}\rho^2 - \frac{\eta_\infty^p}{p}\int_{\mathbb{R}} e^{Q(t)}b(t)dt\rho^p > 0$$

then one has $\varphi_{|\partial B_\rho} \geq \alpha$. ■

Lemma 3.3. Assume that (L) and (W_1) hold. Then φ satisfies the Palais-Smale condition.

Proof: Let (u_n) be a Palais-Smale sequence in E, that is $(\varphi(u_n))$ is bounded and $\varphi'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, there exists a constant $M > 0$ such that

$$|\varphi(u_n)| \leq M \textit{ and } \|\varphi'(u_n)\| \leq M,\ \forall n \in \mathbb{N}.$$

By the Mean Value Theorem and Lemma 3.1, we have

$$\begin{aligned}
&\left|p\int_{\mathbb{R}} e^{Q(t)}W(t,u_n)dt - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n)\cdot u_n dt\right| \\
&= \left|p\int_{\mathbb{R}} e^{Q(t)}\nabla W(t,\theta_n u_n)\cdot u_n dt - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n)\cdot u_n dt\right| \\
&\leq (p+1)\left\|u_n\right\|_{L^\infty}\left[c_0\gamma(\|u\|_{L^\infty})+c_1\right] \\
&\leq (p+1)\eta_\infty\left\|u_n\right\|\left[c_0\gamma(\|u\|_{L^\infty})+c_1\right]
\end{aligned}$$

where $0<\theta_n<1$. Therefore, we have

$$\begin{aligned}
\frac{p-2}{2}\left\|u_n\right\|^2 &= p\varphi(u_n)-\varphi'(u_n)u_n + p\int_{\mathbb{R}} e^{Q(t)}W(t,u_n)dt - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n)\cdot u_n dt \\
&\leq pM + M\left\|u_n\right\| + (p+1)\eta_\infty\left\|u_n\right\|\left[c_0\gamma(\eta_\infty\left\|u_n\right\|)+c_1\right].
\end{aligned}$$

Since $\lim_{s\longrightarrow\infty}\frac{\gamma(s)}{s}=0$, then (u_n) is bounded. By a standard argument, we prove that (u_n) possesses a convergent subsequence. ∎

Lemma 3.4. Assume that (L), (B), (W_1) and (W_2) are satisfied. Then there exists $e\in E$ such that $\|e\|>\rho$ and $\varphi(e)\leq 0$.

Proof : Condition (B) implies that there exists $t_0\in\mathbb{R}$ such that $b(t_0)>0$. By the continuity of b, there exists a constant $\nu>0$ such that

$$b(t)>\frac{1}{2}b(t_0),\ \forall t\in(t_0-\nu,t_0+\nu)\subset\mathbb{R}.$$

Let $v_0\in E\setminus\{0\}$ with support included in $(t_0-\nu,t_0+\nu)$ and define

$$u_0(t)=\begin{cases} v_0(t)\ if\ t\in[t_0-\nu,t_0+\nu] \\ 0\ elsewhere.\end{cases}$$

Then $u_0\in E$. For $\xi\in\mathbb{R}\setminus\{0\}$, one has

$$\begin{aligned}
\varphi(\xi u_0) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\left[|\xi\dot{u}_0(t)|^2+L(t)\xi u_0(t)\cdot\xi u_0(t)\right]dt \\
&\quad -\frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}b(t)\left|\xi u_0(t)\right|^p dt - \int_{\mathbb{R}} e^{Q(t)}W(t,\xi u_0(t))dt \\
&\leq \frac{|\xi|^2}{2}\left\|u_0\right\|^2 - \frac{|\xi|^p}{p}\frac{b(t_0)}{2}\int_{t_0-\nu}^{t_0+\nu} e^{Q(t)}\left|u_0(t)\right|^p dt \\
&\quad + |\xi|\left\|u_0\right\|_{L^\infty}\left[c_0\gamma(|\xi|\left\|u_0\right\|_{L^\infty})+c_1\right].
\end{aligned}$$

Since $p>2$, the property $\lim_{s\longrightarrow\infty}\frac{\gamma(s)}{s}=0$ implies that $\lim_{|\xi|\longrightarrow\infty}\varphi(\xi u_0)=-\infty$. Take ξ_0 large enough such that $\varphi(\xi_0 u_0)\leq 0$, then $e=\xi_0 u_0$ satisfies condition (ii) of Lemma 2.1. ∎

Lemma 3.2-3.4 imply that all the conditions of Lemma 2.1 are satisfied. Therefore φ has a critical point u satisfying $\varphi(u) \geq \alpha > \varphi(0)$ and then system $(\mathcal{DV})$ possesses a nontrivial fast homoclinic solution.

4. Superquadratic growth.

In this Section we are concerned with the existence of fast homoclinic solutions for the following damped vibration system

$$(\mathcal{DV}) \qquad \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0,\ t \in \mathbb{R}$$

when the potential W satisfies a new condition generalizing the well-known Ambrosetti-Rabinowitz superquadratic condition. More precisely, we consider the following conditions

(W_3) there exist a bounded set $D \subset \mathbb{R}$ with $int(D) \neq \phi$, $\mu > 2$ and $\theta > \frac{\mu}{\mu-2}$ such that

$$(i) \qquad 0 < \mu W(t,x) \leq \nabla W(t,x) \cdot x,\ \forall t \in D,\ \forall x \in \mathbb{R}^N \setminus \{0\},$$

$$(ii) \qquad 0 \leq 2W(t,x) \leq \nabla W(t,x) \cdot x \leq \frac{1}{\theta} L(t)x \cdot x,\ \forall t \notin D,\ \forall x \in \mathbb{R}^N;$$

$$(W_4) \qquad |\nabla W(t,x)| = o(|x|)\ as\ |x| \longrightarrow 0\ uniformly\ in\ t \in \mathbb{R}.$$

We state our main result in this Section.

Theorem 4.1. Assume that (L), (W_3) and (W_4) hold. Then $(\mathcal{DV})$ possesses a nontrivial fast homoclinic solution.

Example 4.1. Let $a \in C(\mathbb{R}, \mathbb{R})$ be such that $a(t) > 0$ on $(-1, 1)$ and $a(t) = 0$ on $\mathbb{R} \setminus (-1, 1)$. Consider the potential

$$W(t,x) = a(t)\,|x|^3\,.$$

Choosing $D = (-1, 1)$, it is easy to show that $W(t,x)$ satisfies conditions (W_3) and (W_4) but $W(t,x)$ does not satisfy the Ambrosetti-Rabinowitz condition.

Proof of Theorem 4.1. Consider the continuously differentiable functional ψ associated to system $(\mathcal{DV})$

$$\psi(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^2 + L(t)u(t)\cdot u(t)\right] dt - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t))dt$$

defined on the space E introduced in Section 2. It is well known that under condition (W_4), ψ is continuously differentiable on E and we have

$$\begin{aligned}\psi'(u)v &= \int_{\mathbb{R}} e^{Q(t)} \left[\dot{u}(t)\cdot\dot{v}(t) + L(t)u(t)\cdot v(t)\right] dt - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t))\cdot v(t)dt \\ &= < u, v > - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t))\cdot v(t)dt\end{aligned}$$

for all $u, v \in E$. Moreover, the critical points of ψ on E are fast homoclinic solutions of $(\mathcal{DV})$. In the following, we will proceed by successive lemmas.

Lemma 4.1. Assume that (L) and (W_4) hold. Then there exist positive constants ρ, α such that $\psi_{|\partial B_\rho} \geq \alpha$.

Proof: By (W_4), for all $\epsilon > 0$ there exists a constant $r > 0$ such that

$$|\nabla W(t,x)| \leq \epsilon \, |x| \, , \ \forall t \in \mathbb{R}, \ \forall \, |x| \leq r.$$

Taking $\epsilon = \frac{1}{2\eta_2^2}$, $\rho = \frac{r}{\eta_\infty}$ and $\alpha = \frac{\rho^2}{4}$ yields for all $u \in \partial B_\rho$

$$\begin{aligned} \psi(u) &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\epsilon}{2} \int_{\mathbb{R}} e^{Q(t)} \, |u(t)|^2 \, dt \\ &\geq \frac{1}{4} \|u\|^2 = \alpha. \end{aligned}$$

■

Lemma 4.2. Suppose that (L), (W_3) and (W_4) are satisfied. Then there exists $e \in E$ such that $\|e\| > \rho$ and $\psi(e) \leq 0$.

Proof: By $(W_3)(i)$, there exists a constant $c_1 > 0$ such that

$$W(t,x) \geq c_1 \, |x|^\mu \, , \ \forall t \in D, \ |x| \geq 1. \tag{4.1}$$

Let $u_0 \in E \setminus \{0\}$ with support contained in D. By (W_4), Fatou's lemma and (4.1), one has

$$\begin{aligned} \limsup_{s \longrightarrow \infty} \frac{\psi(su_0)}{s^2} &= \frac{1}{2} \|u_0\|^2 - \liminf_{s \longrightarrow \infty} \int_{\mathbb{R}} e^{Q(t)} \frac{W(t, su_0)}{s^2} dt \\ &= \frac{1}{2} \|u_0\|^2 - \liminf_{s \longrightarrow \infty} \int_{D \setminus \{t/u_0(t)=0\}} e^{Q(t)} \frac{W(t, su_0)}{|su_0|^\mu} s^{\mu-2} \, |u_0|^\mu \, dt \\ &\leq \frac{1}{2} \|u_0\|^2 - \liminf_{s \longrightarrow \infty} \int_{D \setminus \{t/u_0(t)=0\}} e^{Q(t)} c_1 s^{\mu-2} \, |u_0|^\mu \, dt \\ &\leq \frac{1}{2} \|u_0\|^2 - \int_{D \setminus \{t/u_0(t)=0\}} e^{Q(t)} \liminf_{s \longrightarrow \infty} c_1 s^{\mu-2} \, |u_0|^\mu \, dt = -\infty. \end{aligned}$$

Hence there exists a constant s_0 large enough such that $\psi(s_0 u_0) < 0$ and $\|s_0 u_0\| > \rho$. Choosing $e = s_0 u_0$, then e satisfies $\|e\| > \rho$ and $\psi(e) < 0$. ■

Lemma 4.3. Under assumptions (L), (W_3) and (W_4), ψ satisfies the (PS) condition.

Proof : Let (u_n) be a Palais-Smale sequence, that is

$$(\psi(u_n)) \text{ is bounded and } \psi'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.2}$$

By $(W_3)(i)$ and (4.2) we have

$$\begin{aligned}
&\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right] dt \\
&= \int_{\mathbb{R}} e^{Q(t)} \nabla W(t, u_n(t)) \cdot v_n(t)dt + \psi'(u_n)u_n \\
&\geq \int_{D} e^{Q(t)} \nabla W(t, u_n(t)) \cdot u_n dt + o(\|u_n\|) \\
&\geq \mu \int_{D} e^{Q(t)} W(t, u_n(t))dt + o(\|u_n\|).
\end{aligned} \tag{4.3}$$

Combining (4.2), (4.3) and $(W_3)(ii)$, there exists a positive constant c_2 such that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right] dt \\
&= \int_{\mathbb{R}} e^{Q(t)} W(t, u_n(t))dt + \psi(u_n) \\
&\leq \int_{D} e^{Q(t)} W(t, u_n(t))dt + \int_{\mathbb{R}\setminus D} e^{Q(t)} W(t, u_n(t))dt + c_2 \\
&\leq \int_{D} e^{Q(t)} W(t, u_n(t))dt + \frac{1}{2\theta}\int_{\mathbb{R}\setminus D} e^{Q(t)} L(t)u_n(t)\cdot u_n(t)dt + c_2 \\
&\leq \frac{1}{\mu}\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right] dt + \frac{1}{2\theta}\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right] dt \\
&+ o(\|u_n\|) + c_2,
\end{aligned}$$

which implies

$$\left(\frac{\mu}{2} - 1 - \frac{\mu}{2\theta}\right) \|u_n\|^2 \leq \mu c_2 + o(\|u_n\|).$$

Since $\frac{\mu}{2} - 1 - \frac{\mu}{2\theta} > 0$ we deduce that (u_n) is bounded in E. It remains to prove that (u_n) is strongly convergent in E. Since E is reflexive, then up to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in E. Sinse D is bounded, there exists a positive constant r such that $D \subset B_r$. Let χ_r be a cut-off function satisfying

$$\chi_r = 0 \text{ on } B_r,\ \chi_r = 1 \text{ on } \mathbb{R}\setminus B_{2r},\ 0 \leq \chi_r \leq 1 \text{ and } \left|\chi_r'\right| \leq \frac{c_3}{r}$$

for a positive constant c_3. We have

$$\begin{aligned}\psi'(u_n)\chi_r u_n &= \int_{\mathbb{R}} e^{Q(t)}\left[\dot{u}_n(t)\cdot\dot{\overbrace{\chi_r u_n}}(t) + L(t)u_n(t)\cdot u_n(t)\chi_r\right]dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot u_n(t)\chi_r dt \\ &= \int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]\chi_r dt \\ &\quad + \int_{\mathbb{R}} e^{Q(t)}\dot{u}_n(t)\cdot u_n(t)\dot{\chi}_r dt - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot v_n(t)\chi_r(t)dt\end{aligned}$$

which with $(W_3)(ii)$ implies

$$\begin{aligned}&\int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]\chi_r dt \\ &= -\int_{\mathbb{R}} e^{Q(t)}\dot{u}_n(t)\cdot u_n(t)\dot{\chi}_r dt + \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot v_n(t)\chi_r(t)dt + \psi'(u_n)\chi_r u_n \\ &\leq -\int_{\mathbb{R}} e^{Q(t)}\dot{u}_n(t)\cdot u_n(t)\dot{\chi}_r dt + \frac{1}{\theta}\int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]\chi_r dt \\ &+ \psi'(u_n)\chi_r u_n.\end{aligned} \tag{4.4}$$

Combining (4.4) with Hölder's inequality, for positive constants c_4, c_5, c_6 yields

$$\begin{aligned}&\left(1-\frac{1}{\theta}\right)\int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]\chi_r dt \\ &\leq \frac{c_4}{r}\|\dot{u}_n\|_{L^2_Q}\|u_n\|_{L^2_Q} + \psi'(u_n)\chi_r u_n \\ &\leq \frac{c_4}{r}\eta_2\|u_n\|^2 + \|\psi'(u_n)\|\,\|\chi_r u_n\| \\ &\leq \frac{c_5}{r} + c_6\|\psi'(u_n)\|.\end{aligned}$$

For all $\epsilon > 0$, we can choose $r_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus B_{2r}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]dt \leq \int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}_n(t)|^2 + L(t)u_n(t)\cdot u_n(t)\right]\chi_r dt \leq \epsilon$$

for all $r \geq r_0$ and $n \geq n_0$. Hence, it is easy to check that (u_n) converges strongly to u in E. ■

Lemmas 4.1-4.3 imply that all the conditions of Lemma 2.1 are satisfied. Therefore ψ possesses a critical point u satisfying $\psi(u) \geq \alpha > 0$, and then $(\mathcal{DV})$ possesses a nontrivial fast homoclinic solution.

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References.

[1] R.P. Agarwal, P. Chen and X. Tang, Fast homoclinic solutions for a class of damped vibration problems, Applied Mathematics and Computation 219 (2013), 6053-6065.

[2] G. Chen, Homoclinic orbits for second order Hamiltonian systems with asymptotically linear terms at infinity, Advances in Difference systems 2014:114 (2014), 1-9.

[3] G.W. Chen, superquadratic or asymptotically quadratic Hamiltonian systems: ground state homoclinic orbits, Ann. Mat. Pura Appl. 194(3) (2013), 903-918.

[4] G. Chen, Z. He, Infinitely many homoclinic solutions for a class of second order Hamiltonian systems, Advances in Difference systems 2014:161 (2014), 1-15.

[5] P. Chen and X.H. Tang, Fast homoclinic solutions for a class of damped vibration problems with sub-quadratic potentials, Math. Nachr. 286, No.1 (2013), 4-16.

[6] G. Chen, L. Zhang, Infinitely many homoclinic solutions for perturbed second-order Hamiltonian systems with subquadratic potentials, Electr. J. Qual. Theo. Diff. Eq., No 9 (2020), 1-23.

[7] B. Cheng, Multiplicity of periodic orbits for second order Hamiltonian systems with superlinear or sublinear nonlinearity, Chaos, Solitons and Fractals 44 (2011), 811-816.

[8] L. Chu, Q. Zhang, Homoclinic solutions for a class of second order Hamiltonian systems with locally defined potentials, Nonlinear Analysis 75 (2012), 3188-3197.

[9] C. Deng, D.L. Wu, Multiple homoclinic solutions for a class of nonhomogeneous Hamiltonian systems, Boundary Value Problems (2018) 2018:56, 1-10.

[10] Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Analysis, 25, No.11 (1995), 1095-1113.

[11] S. Guo, Z. Liu, Z. Zhang, Homoclinic orbits for the second order Hamiltonian systems, Nonlinear Analysis (RWA) 36 (2017), 116-138.

[12] Z.Q. Han, M.H. Yang, Infinitely many solutions for second order Hamiltonian systems with odd nonlinearities, Nonlinear Analysis (TMA) 74 (7) (2011), 2633-2646.

[13] J. Jiang, S. Lu, X. Lv, Homoclinic solutions for a class of second order Hamiltonian systems, Nonlinear Analysis: Real word Applications, 13 (2012), 176-185.

[14] W. Jiang, Q. Zhang, Multiple homoclinic solutions for superquadratic Hamiltonian systems, Electr. J. Diff. Equ., Vol. 2016, No. 66 (2016), 1-12.

[15] F. Khelifi, M. Timoumi, Even homoclinic orbits for a class of damped vibration systems, Indigationes Mathematicae 28 (2017), 1111-1125.

[16] Y. Li, Q. Zhang, Existence and multiplicity of fast homoclinic solutions for a class of nonlinear second order nonautonomous systems in a weighted Sobolev space, Journal of Functional Spaces, Vol. 2015, Article ID 495040 (2015), 1-12.

[17] X. Lin and X.H. Tang, Homoclinic solutions for a class of second order Hamiltonian systems, J. Math. Anal. Appl. 354 (2009), 539-549.

[18] X. Lin and X.H. Tang, Infinitely many homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials, Nonlinear Analysis 74 (2011), 6314-6325.

[19] X. Lin and X.H. Tang, New conditions on homoclinic solutions for a subquadratic second order Hamiltonian system, Boundary Value Problems 2015:111 (2015), 1-17.

[20] C. Liu, Q. Zhang, Infinitely many homoclinic solutions for second order Hamiltonian systems, Nonlinear Analysis 72 (2010), 894-903.

[21] S. Lu, X. Lv and P. Yan, Existence of homoclinics for a class of Hamiltonian systems, Nonlinear Analysis 72 (2010), 390-398.

[22] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Reg. Conf. Ser. in Math., Vol. 65, American Mathematical Society, Providence, R.I, (1986).

[23] J. Sun, T-f. Wu, Homoclinic solutions for a second order Hamiltonian system with a positive semi-definite matrix, Chaos, Solitons and Fractals 76 (2015), 24-31.
[24] J. Sun, T-f. Wu, Multiplicity and concentration of homoclinic solutions for some second order Hamiltonian systems, Nonlinear Analysis 114 (2015), 105-115.
[25] X.H. Tang, Infinitely many homoclinic solutions for a second-order Hamiltonian system, Math. Nachr. 289, No. 1 (2016) 116-127.
[26] C.L. Tang, Li-Li Wan, Existence of homoclinic orbits for second order Hamiltonian systems without (AR) condition, Nonlinear Analysis 74 (2011), 5303-5313.
[27] X.H. Tang, L. Xiao, Homoclinic solutions for a class of second order Hamiltonian systems, Nonlinear Analysis 71 (2009), 1140-1152.
[28] M. Timoumi, Existence and multiplicity of fast homoclinic solutions for a class of damped vibration problems, J. Nonlinear Funct. Anal. 2016, Article ID 9 (2016), 1-22.
[29] M. Timoumi, Ground state homoclinic orbits of a class of superquadratic damped vibration problems, Communications in Optimisation Theory 2017, Article ID 29 (2017), 1-25.
[30] M. Timoumi, Infinitely many fast homoclinic solutions for a class of superquadratic damped vibration systems, Journal of Elliptic and Parabolic Equations, (2020), 1-21.
[31] M. Timoumi, Infinitely many fast homoclinic solutions for a class of superquadratic damped vibration systems, J. Nonlinear Funct. Anal. 2020, Article ID 8 (2020), 1-16.
[32] M. Timoumi, Infinitely many fast homoclinic solutions for damped vibration systems with superquadratic conditions or combined nonlinearities, J. Nonlinear Var. Anal. 5, No. 2 (2021), 299-318.
[33] M. Timoumi, Infinitely many fast homoclinic solutions for different classes of damped vibration systems, J. Nonlinear Funct. Anal. 2020, Article ID 46 (2020), 1-20.
[34] M. Timoumi, On ground state homoclinic orbits of a class of superquadratic damped vibration systems, Mediterr.J. Math. 2018 (2018), 1-20.
[35] J. Wei, J. Wang, Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials, J. Math. Anal. Appl., Vol. 366, Issue 2 (2010), 694-699.
[36] X. Wu, W. Zhang, Existence and multiplicity of homoclinic solutions for a class of damped vibration problems, Nonlinear Analysis 74 (2011), 4392-4398.
[37] R. Yuan and Z. Zhang, Fast homoclinic solutions for some second order non-autonomous systems, J. Math. Anal. Appl. 376 (2011), 51-63.
[38] R. Yuan, Z. Zhang: Homoclinic solutions for some second order nonautonomous hamiltonian systems without the globally superquadratic condition, Nonlinear Analysis 72 (2010), 1809-1819.
[39] Z. Zhang, Existence of homoclinic solutions for second order Hamiltonian systems with general potentials, J. Appl. Math. Comput. 44 (2014) 263-272.