PROOF PERTAINING TO SOME IMPORTANT PROPERTIES OF q-INTEGRAL

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Abstract. In the almost studies on quantum integral inequalities, the authors thought that the properties of the Riemann integral were also provided for the quantum integral. Due to misuse of these properties, many studies include some mistakes. In this paper, counterexamples are firstly given that the properties cause mistakes are not provided. Moreover, the correct versions of these properties are proved under some conditions. We believe that this study will be a important reference for authors who study quantum integral inequalities.

1. Introduction

The theory of integral inequalities is widely studied in recent decades. Hermite-Hadamard (trapezoid-midpoint), Fejér, Ostrowski, Simpson, Jansen, Opial, Hardy, Bullen, Mercer, Grüss, Wirtinger, etc. type inequalities, and some of their generalizations, investigated for Riemann integral.

The following elemental properties of the Riemann integral are very useful and necessary for investigating integral inequalities:

i) If $f, g : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_c^b f(x)\, dx \leq \int_c^b g(x)\, dx, \tag{1.1}$$

for all $c \in [a, b]$.

ii) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then

$$\left| \int_c^b f(x)\, dx \right| \leq \int_c^b |f(x)|\, dx, \tag{1.2}$$

for all $c \in [a, b]$.

iii) If $f(x) \leq 0$ on $[a, d]$, $f(x) \geq 0$ on $[d, b]$ and $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then

$$\int_a^b |f(x)|\, dx = \int_a^d -f(x)\, dx + \int_d^b f(x)\, dx. \tag{1.3}$$

Nowadays, the studies involving Riemann integral are commonly transporting to inequalities involving quantum integral. But quantum integral is working a bit different from the Riemann integral, and the properties i)-ii)-iii) are not generally valid in quantum integral. In many studies, these inequalities are assumed to be correct and they are used incorrectly by many authors. First of all, we will show why these problems occur with opposite examples. Then, we will solve these problems under favourable conditions.

2. Preliminaries and Definitions of q-Calculus

In this section, we present some required definitions about q-calculus. Throughout the paper, we consider that $0 < q < 1$.

In 2013-2014, when Tariboon et. al. [11,12] gave the definitions of q-derivative and q-integral, they added a continuity condition to these definitions. On the other hand, in 2003 Rackovic [10] did not give the continuity condition in the definition of q-integral. The definitions of q-integral and q-derivative

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have already been defined in discrete analysis. Therefore, there is no need to be continuous. More clearly, if we think for the q-derivative:

The arbitrary function defined on $[a,b]$ is quantum differentiable for each $x \in (a,b]$. Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If $x \in (a,b]$, for all $q \in (0,1)$ we know $qx + (1-q)a \in (a,b]$ and $f(x) - f(qx + (1-q)a) \in \mathbb{R}$. Also, $(1-q)(x-a) \in \mathbb{R}\backslash\{0\}$, so, for all $x \in (a,b]$ we have $\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)} \in \mathbb{R}$. Therefore, ${}_aD_qf(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}$ is defined for all $x \in (a,b]$. In order to be quantum differentiable at the point $x = a$, it is necessary and sufficient that ${}_aD_qf(a) = \lim_{x\to a} {}_aD_qf(x)$ limit exists.

Let's consider the quantum integral:

$f : [a,b] \to \mathbb{R}$ be an arbitrary function. In order for f to be q-integrable on $[a,b]$, it is sufficient for the $(1-q)(b-a)\sum\limits_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a)$ series to be convergent. More precisely, the continuity condition is not required for a function to be q-integrable. With all these, we will remove the continuity condition from the following definitions.

Definition 1. *[11, 12] For a function $f : [a,b] \to \mathbb{R}$, the q_a-derivative of f at $x \in [a,b]$ is characterized by the expression*

$$ {}_aD_qf(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{2.1}$$

If $x = a$, then we define ${}_aD_qf(a) = \lim_{x\to a} {}_aD_qf(x)$ if it exists and it is finite.

Definition 2. *[10–12] Let $f : [a,b] \to \mathbb{R}$ be a function. Then, the q_a-definite integral on $[a,b]$ is defined as:*

$$\begin{aligned} \int_a^b f(x)\ {}_ad_qx &= (1-q)(b-a)\sum_{n=0}^{\infty} q^n f[q^n b + (1-q^n)a] \\ &= (b-a)\int_0^1 f[(1-x)a + xb]\ d_qx \ . \end{aligned} \tag{2.2}$$

From (2.2) the following equation is written:

$$\int_c^b f(x)\ {}_ad_qx = \int_a^b f(x)\ {}_ad_qx - \int_a^c f(x)\ {}_ad_qx \ . \tag{2.3}$$

Also if $\int\limits_a^b f(x)\,dx$ is convergent, for $q \to 1^-$ we get Riemann integral as below

$$\lim_{q\to 1^-}\int_a^b f(x)\ {}_ad_qx = \int_a^b f(x)\,dx.$$

Moreover, we will need to use the following lemma in our main results:

Lemma 1. *[12] We have the equality*

$$\int_a^b (x-a)^{\alpha}\ {}_ad_qx = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q}$$

for $\alpha \in \mathbb{R}\backslash\{-1\}$.

We have to give the following notations which will be used many times in the next sections (see, [9]):

$$[n]_q = \frac{q^n - 1}{q-1}.$$

3. Counterexamples

In this section, we will give some counterexamples that the properties (1.1),(1.2) and (1.3) are not valid generally in quantum integral. In addition, some problems caused by these examples will be expressed.

Example 1. Let $f,g:[0,4]\to\mathbb{R}$, $f(x)=4-x$ and $g(x)=8-2x$. It's clear $f(x)\le g(x)$ on $[0,4]$, then let's calculate the following q-integrals:

$$\begin{aligned}\int_1^4 (4-x)\ {}_0d_qx &= \int_0^4 (4-x)\ {}_0d_qx - \int_0^1 (4-x)\ {}_0d_qx\\ &= \left[4x-\frac{x^2}{[2]_q}\right]_0^4 - \left[4x-\frac{x^2}{[2]_q}\right]_0^1\\ &= \frac{12q-3}{1+q},\end{aligned}$$

and

$$\begin{aligned}\int_1^4 (8-2x)\ {}_0d_qx &= \int_0^4 (8-2x)\ {}_0d_qx - \int_0^1 (8-2x)\ {}_0d_qx\\ &= \left[8x-\frac{2x^2}{[2]_q}\right]_0^4 - \left[8x-\frac{2x^2}{[2]_q}\right]_0^1\\ &= \frac{24q-6}{1+q}.\end{aligned}$$

Here, we have $\int_1^4 (4-x)\ {}_0d_qx \le \int_1^4 (8-2x)\ {}_0d_qx$ for $q\in\left[\frac{1}{4},1\right)$ and we have $\int_1^4 (4-x)\ {}_0d_qx \ge \int_1^4 (8-2x)\ {}_0d_qx$ for $q\in\left(0,\frac{1}{4}\right)$.

It means that, if $f(x)\le g(x)$ on $[0,b]$, then the following inequality is not provided generally

$$\int_c^b f(x)\ {}_0d_qx \le \int_c^b g(x)\ {}_0d_qx\ ,$$

for all $c\in[0,b]$. Similarly, if $f(x)\le g(x)$ on $[a,b]$, then the following inequality is not provided generally

$$\int_c^b f(x)\ {}_ad_qx \le \int_c^b g(x)\ {}_ad_qx\ ,$$

for all $c\in[a,b]$.

Problem 1. Assume f,g are continuous functions and $c\in(a,b)$. When the following inequality is satisfied?

$$\int_c^b f(x)\ {}_ad_qx \le \int_c^b g(x)\ {}_ad_qx\ .$$

Example 2. In here we refer [2, Page 12 , (ii)], because the authors proved that the following inequality is not provided always (means for all functions) for Jackson q-integral

$$\left|\int_c^b f(x)\ {}_0d_qx\right| \le \int_c^b |f(x)|\ {}_0d_qx$$

unless $c = 0$. Similarly, the following inequality is not provided always

$$\left| \int_c^b f(x) \ {}_a d_q x \right| \leq \int_c^b |f(x)| \ {}_a d_q x \ ,$$

unless $c = a$.

Problem 2. Assume f is continuous function and $c \in (a, b)$. When the following inequality is satisfied?

$$\left| \int_c^b f(x) \ {}_a d_q x \right| \leq \int_c^b |f(x)| \ {}_a d_q x \ . \tag{3.1}$$

We will investigate these two problems in the following section. Let we discuss the following problem.

Problem 3. *Assume $f(x) \leq 0$ on $[a, d]$, $f(x) \geq 0$ on $[d, b]$. Does the following equality hold?*

$$\int_a^b |f(x)| \ {}_a d_q x \ = \int_a^d (-f(x)) \ {}_a d_q x \ + \int_d^b f(x) \ {}_a d_q x \ . \tag{3.2}$$

Now let's show that (3.2) is not correct with the example below.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2 - x$. In almost all papers on quantum integral inequalities, the integral $\int_0^3 |2 - x| \ {}_0 d_q x$ is calculated by the equality (3.2) as follows:

$$\begin{aligned} & \int_0^2 (2 - x) \ {}_0 d_q x \ + \int_2^3 (x - 2) \ {}_0 d_q x \\ = \ & (1 - q) \sum_{n=0}^{\infty} q^n \left[2(2 - 2.q^n) + 3(3.q^n - 2) - 2(2.q^n - 2)\right] \\ = \ & (1 - q) \sum_{n=0}^{\infty} q^n \left[4(2 - 2.q^n) + 3(3.q^n - 2)\right] \\ = \ & (1 - q) \sum_{n=0}^{\infty} q^n (2 + q^n) \\ = \ & (1 - q) \sum_{n=0}^{\infty} \left(2q^n + q^{2n}\right) \\ = \ & (1 - q) \left(\frac{2}{1 - q} + \frac{1}{1 - q^2}\right) \\ = \ & 2 + \frac{1}{1 + q}, \end{aligned} \tag{3.3}$$

where $q \in (0, 1)$. If we choose $q = \frac{1}{3}$ in (3.3), then we have

$$\int_0^2 (2 - x) \ {}_0 d_q x \ + \int_2^3 (x - 2) \ {}_0 d_q x \ = \frac{11}{4}. \tag{3.4}$$

On the other hand, let calculate the integral $\int_0^3 |2-x| \ {}_0d_qx$ for $q=\frac{1}{3}$ by using the definition of q-integral directly, as follows:

$$\int_0^3 |x-2| \ {}_0d_{\frac{1}{3}}x = \left(1-\frac{1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\left[3\left|3\left(\frac{1}{3}\right)^n-2\right|\right] \tag{3.5}$$

$$= \frac{2}{3}\left\{3+\sum_{n=1}^{\infty}\frac{1}{3^n}\left[3\left|3\left(\frac{1}{3}\right)^n-2\right|\right]\right\}$$

$$= \frac{2}{3}\left\{3+\sum_{n=1}^{\infty}\frac{1}{3^n}.3\left[2-3\left(\frac{1}{3}\right)^n\right]\right\}$$

$$= 2+\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{6}{3^n}-\frac{9}{3^{2n}}\right)$$

$$= 2+\frac{2}{3}\left(3-\frac{9}{8}\right)$$

$$= \frac{13}{4}.$$

It turns out clearly that (3.4) and (3.5) are not equal. Therefore, we say that

$$\int_a^b |f(x)| \ {}_ad_qx \neq \int_a^d (-f(x)) \ {}_ad_qx + \int_d^b f(x) \ {}_ad_qx .$$

Now let's show how to compute the above integral for all q. From definition of q-integral we calculate as follows:

$$\int_0^3 |x-2| \ {}_0d_qx = (1-q)\sum_{n=0}^{\infty}q^n\left(3\left|3.q^n-2\right|\right)$$

$$= (1-q)\sum_{n=0}^{\infty}q^n\left(3\left|3.q^n-2\right|\right).$$

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There is a $m \in \mathbb{N}^{+}$ and $q_m = \sqrt[m]{\frac{2}{3}} \in (0,1)$ such that we get $(3q_m^n - 2) > 0$ for $\forall n < m$ and we have $(3q_m^n - 2) < 0$ for $\forall n > m$. Hence, for $\sqrt[m]{\frac{2}{3}} \leq q < \sqrt[m+1]{\frac{2}{3}}$, we obtain

$$\begin{aligned}
\int_0^3 |x-2| \ {}_0d_q x &= 3(1-q)\sum_{n=0}^{\infty} q^n |3.q^n - 2| \\
&= 3(1-q)\sum_{n=0}^{m} q^n (3.q^n - 2) + 3(1-q)\sum_{n=m+1}^{\infty} q^n (2-3q^n) \\
&= 3(1-q)\left(3\frac{1-q^{2m+2}}{1-q^2} - 2\frac{1-q^{m+1}}{1-q}\right) + 3(1-q)\sum_{n=0}^{\infty} q^{n+m+1}\left(2-3q^{n+m+1}\right) \\
&= 9\frac{1-q^{2m+2}}{1+q} - 6\left(1-q^{m+1}\right) + 3(1-q)\left(\frac{2q^{m+1}}{1-q} - \frac{3q^{2m+2}}{1-q^2}\right) \\
&= 9\frac{1-q^{2m+2}}{[2]_q} - 6 + 6q^{m+1} + 6q^{m+1} - \frac{9q^{2m+2}}{[2]_q} \\
&= \frac{9 - 18q^{2m+2} - 6 + 12q^{m+1} - 6q + 12q^{m+2}}{[2]_q} \\
&= \frac{3 - 6q - 18q^{2m+2} + 12q^{m+1} + 12q^{m+2}}{[2]_q}.
\end{aligned}$$

If $0 < q < \frac{2}{3}$, then $(3q^n - 2) < 0$ for $\forall n \geq 1$. Therefore, we get

$$\begin{aligned}
\int_0^3 |x-2| \ {}_0d_q x &= 3(1-q)\sum_{n=0}^{\infty} q^n |3.q^n - 2| \\
&= 3(1-q)\left(1 + \sum_{n=1}^{\infty} q^n |3.q^n - 2|\right) \\
&= 3(1-q)\left(1 + \sum_{n=1}^{\infty} q^n (2 - 3.q^n)\right) \\
&= 3(1-q)\left(2 + \sum_{n=0}^{\infty} q^n (2 - 3.q^n)\right) \\
&= 3(1-q)\left(2 + \frac{2}{1-q} - \frac{3}{1-q^2}\right) \\
&= \frac{3 + 6q - 6q^2}{1+q}.
\end{aligned}$$

As a result, we have

$$\int_0^3 |x-2| \ {}_0d_q x = \begin{cases} \frac{3-6q-18q^{2m+2}+12q^{m+1}+12q^{m+2}}{[2]_q}, & \sqrt[m]{\frac{2}{3}} \leq q < \sqrt[m+1]{\frac{2}{3}}, m \in \mathbb{N}^{+}, \\ \frac{3+6q-6q^2}{[2]_q}, & q \in \left(0, \frac{2}{3}\right). \end{cases}$$

On the other hand, $\sqrt[m]{\frac{2}{3}} \leq q < \sqrt[m+1]{\frac{2}{3}} \Rightarrow \frac{2}{3} \leq q^m < \left(\frac{2}{3}\right)^{\frac{m}{m+1}}$, then for the limit $m \to \infty$, we have $q \to 1^{-}$. Then,

$$\lim_{m\to\infty} \frac{2}{3} \leq \lim_{m\to\infty} q^m < \lim_{m\to\infty} \left(\frac{2}{3}\right)^{\frac{m}{m+1}} \Rightarrow q^m \to \frac{2}{3}.$$

This yields classical result for Riemann integral as below:

$$\int_0^3 |x-2| \, dx = \frac{5}{2}.$$

In Section 5, we show how to calculate the quantum integral of absolute value of a linear function.

Remark 1. *In Examples 1-3, it is proven that in quantum integral the similar properties as* (1.1)-(1.3) *of Riemann integral are not provided. This means that you need to be careful when using these properties in quantum integral.*

4. Proofs of Problem 1 and Problem 2

In this section we prove two theorems to solve the Problem 1 and Problem 2.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ are two functions and f, g are q-integrable functions on $[a, b]$. Assume that $f \le g$ on $[a, b]$ then, according to the given conditions, the following inequalities hold:*

1) If $g(x) - f(x)$ is nondecreasing, then the following inequality holds

$$\int_c^b f(x) \ {}_a d_q x \ \le \int_c^b g(x) \ {}_a d_q x \ ,$$

for all $q \in (0, 1)$,
2) If $\frac{g(b)-f(b)}{c-a} \ge \frac{g(c)-f(c)}{b-a}$ and $g(x) - f(x)$ is nonincreasing, then the following inequality holds

$$\int_c^b f(x) \ {}_a d_q x \ \le \int_c^b g(x) \ {}_a d_q x \ ,$$

for all $q \in (0, 1)$,
3) If $\frac{g(b)-f(b)}{c-a} < \frac{g(c)-f(c)}{b-a}$, then we have

$$\begin{cases} \int_c^b f(x) \ {}_a d_q x \ \ge \int_c^b g(x) \ {}_a d_q x \ , & \text{for } q \in (0, \min_{i \in \mathbb{N}} \{q_i\}], \\ \int_c^b f(x) \ {}_a d_q x \ \le \int_c^b g(x) \ {}_a d_q x \ , & \text{for } q \in [\max_{i \in \mathbb{N}} \{q_i\}, 1), \end{cases}$$

where $c \in (a, b)$ and $\left\{ q_i : \int_c^b (g(x) - f(x)) \ {}_a d_{q_i} x \ = 0, \ i \in \mathbb{N} \right\}$.There is absolutely at least one q_i for every f and q_i are specific to the given function.

Proof. Let us choose the function $h(q) = \int_c^b (g(x) - f(x)) \ {}_a d_q x$ for $f \le g$ and $c \in [a, b]$ with $q \in (0, 1)$. Firstly, since f, g are q-integrable $(1-q) \sum_{n=0}^{\infty} q^n \{(b-a)(g-f)(q^n b + (1-q^n) a) - (c-a)(g-f)(q^n c + (1-q^n) a$ is convergent for all $q \in (0, 1)$. Therefore $h(q)$ is continuous for $\forall q \in (0, 1)$.

Case 1: If $g(x) - f(x)$ is nondecreasing, then we take $q \to 1^-$ and we obtain Riemann integrals as below from q-integrals

$$\begin{aligned} \lim_{q \to 1^-} h(q) &= h(1^-) = \lim_{q \to 1^-} \left\{ \int_c^b [g(x) - f(x)] \ {}_a d_q x \ \right\} \\ &= \int_c^b (g(x) - f(x)) \, dx \ge 0, \end{aligned} \tag{4.1}$$

since $f \le g$. Moreover we have

$$\begin{aligned} \lim_{q \to 0^+} h(q) &= h(0^+) = \lim_{q \to 0^+} \left\{ \int_c^b [g(x) - f(x)] \ {}_a d_q x \ \right\} \\ &= (b-a)[g(b) - f(b)] - (c-a)[g(c) - f(c)] \ge 0. \end{aligned}$$

Since $g(x) - f(x)$ is nondecreasing,

$$(b-a)[g(b)-f(b)] - (c-a)[g(c)-f(c)] \geq \int_c^b [g(x)-f(x)]\,dx,$$

such that

$$h\left(0^+\right) \geq h(q) \geq h\left(1^-\right) \geq 0.$$

For this for $\forall q \in (0,1)$

$$\int_c^b f(x) \quad {}_ad_qx \quad \leq \int_c^b g(x) \quad {}_ad_qx \quad ,$$

holds.

Case 2: If $g(x) - f(x)$ is nonincreasing and $(b-a)[g(b)-f(b)] - (c-a)[g(c)-f(c)] \geq 0$, then, for $\forall q \in (0,1)$we get

$$\begin{aligned} \lim_{q\to 0^+} h(q) &= h\left(0^+\right) \\ &= (b-a)[g(b)-f(b)] - (c-a)[g(c)-f(c)] \geq 0. \end{aligned} \tag{4.2}$$

Since $g(x) - f(x)$ is nonincreasing, (4.1)-(4.2) yield

$$0 \leq h\left(0^+\right) \leq h(q) \leq h\left(1^-\right),$$

such that

$$\int_c^b f(x) \quad {}_ad_qx \quad \leq \int_c^b g(x) \quad {}_ad_qx \quad ,$$

holds.

Case 3: If $(b-a)[g(b)-f(b)] - (c-a)[g(c)-f(c)] < 0$, then we get $h(0^+) < 0$. In (4.1) without breaking the generality assume $h(1^-) > 0$. So, since $h(0^+)h(1^-) < 0$, by Bolzano Theorem there is at least one $q_i \in (0,1)$ value such that

$$h(q_i) = 0,$$

where $q_i \in \{q_1, \cdots q_k\}$. Consequently, we have

$$h(q) = \int_c^b [g(x)-f(x)] \quad {}_ad_qx \quad \to \begin{cases} \int_c^b [g(x)-f(x)] \quad {}_ad_qx \quad \leq 0, & \text{for } q \in (0, \min_{i\in\mathbb{N}}\{q_i\}], \\ \int_c^b [g(x)-f(x)] \quad {}_ad_qx \quad \geq 0 & \text{for } q \in [\max_{i\in\mathbb{N}}\{q_i\}, 1), \end{cases}$$

i.e.

$$\begin{cases} \int_c^b f(x) \quad {}_ad_qx \quad \geq \int_c^b g(x) \quad {}_ad_qx \quad , & \text{for } q \in (0, \min_{i\in\mathbb{N}}\{q_i\}], \\ \int_c^b f(x) \quad {}_ad_qx \quad \leq \int_c^b g(x) \quad {}_ad_qx & \text{for } q \in [\max_{i\in\mathbb{N}}\{q_i\}, 1), \end{cases}$$

and the proof is completed. $\square$

Theorem 2. *Assume $f : [a,b] \to \mathbb{R}$ is a function and f is q-integrable functions on $[a,b]$. Then,*

1) If $c = a$, for all $q \in (0,1)$, then the followin inequality holds

$$\left| \int_a^b f(x) \quad {}_ad_qx \quad \right| \leq \int_a^b |f(x)| \quad {}_ad_qx \quad .$$

2) If $c > a$, then there is at least one $q_i \in (0,1)$ number such that the following inequalities hold:

$$\begin{cases} \left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right| \geq \int\limits_c^b |f(x)| \ \ {}_a d_q x \ , & \text{for } q \in (0, \min_{i\in\mathbb{N}} \{q_i\}], \\ \left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right| \leq \int\limits_c^b |f(x)| \ \ {}_a d_q x & \text{for } q \in [\max_{i\in\mathbb{N}} \{q_i\}, 1). \end{cases}$$

where $\left\{ q_i : \left| \int\limits_c^b f(x) \ \ {}_a d_{q_i} x \ \right| = \int\limits_c^b |f(x)| \ \ {}_a d_{q_i} x \ , \ i \in \mathbb{N} \right\}$. There is absolutely at least one q_i number for every f and q_i are specific to the given function.

Proof. **Case 1**: If $c = a$, for $q \in (0,1)$, then we have

$$\begin{aligned} \left| \int\limits_a^b f(x) \ \ {}_a d_q x \ \right| &= \left| (1-q) \sum_{n=0}^{\infty} q^n \{(b-a) f[q^n b + (1-q^n) a]\} \right| \\ &\leq (1-q)(b-a) \sum_{n=0}^{\infty} q^n |f[q^n b + (1-q^n) a]| \\ &= \int\limits_a^b |f(x)| \ \ {}_a d_q x \ . \end{aligned}$$

Case 2: If $c > a$.
Then let's first define following $H(q)$ funtion.

$$H(q) = \int\limits_c^b |f(x)| \ \ {}_a d_q x \ - \left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right|.$$

$H(q)$ is continuous for $\forall q \in (0,1)$ since f is q-integrable, $\int\limits_c^b |f(x)| \ \ {}_a d_q x$ and $\left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right|$ are convergent for all $q \in (0,1)$. If $q \to 1^-$, then we obtain Riemann integrals as below from q-integrals

$$\begin{aligned} \lim_{q\to 1^-} H(q) &= H(1^-) = \lim_{q\to 1^-} \left(\int\limits_c^b |f(x)| \ \ {}_a d_q x \ - \left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right| \right) \\ &= \int\limits_c^b |f(x)| \, dx - \left| \int\limits_c^b f(x) \, dx \right| \\ &\geq 0. \end{aligned} \tag{4.3}$$

On the other hand, if $q \to 0^+$, then we get

$$\begin{aligned} \lim_{q\to 0^+} H(q) &= H(0^+) = \lim_{q\to 0^+} \left(\int\limits_c^b |f(x)| \ \ {}_a d_q x \ - \left| \int\limits_c^b f(x) \ \ {}_a d_q x \ \right| \right) \\ &= \lim_{q\to 0^+} \Bigg((1-q) \sum_{n=0}^{\infty} q^n \{(b-a) |f[q^n b + (1-q^n) a]| - (c-a) |f[q^n c + (1-q^n) a]|\} \\ &\quad - \left| (1-q) \sum_{n=0}^{\infty} q^n \{(b-a) f[q^n b + (1-q^n) a] - (c-a) f[q^n c + (1-q^n) a]\} \right| \Bigg) \\ &= (b-a)|f(b)| - (c-a)|f(c)| - |(b-a) f(b) - (c-a) f(c)|. \end{aligned}$$

From the triangle inequality the following inequality is clear

$$(b-a)|f(b)| - (c-a)|f(c)| - |(b-a) f(b) - (c-a) f(c)| \leq 0. \tag{4.4}$$

So, we have $H(0^+) \leq 0$. That is when q is very close to 0, the following inequality is valid

$$\left| \int_c^b f(x) \ {}_a d_q x \right| \geq \int_c^b |f(x)| \ {}_a d_q x \ . \tag{4.5}$$

Without breaking the generality assume $H(0^+) < 0$ and $H(1^-) > 0$. So, since $H(0^+) H(1^-) < 0$ from Bolzano Theorem there is at least one $q_i \in (0,1)$ value such that

$$H(q_i) = 0$$

where $q_i \in \{q_1, \cdots q_k\}$. Consequently, we have

$$\left| \int_c^b f(x) \ {}_a d_q x \right| \lesseqgtr \int_c^b |f(x)| \ {}_a d_q x \ \to \begin{cases} \left| \int_c^b f(x) \ {}_a d_q x \right| \geq \int_c^b |f(x)| \ {}_a d_q x \ , & \text{for } q \in (0, \min_{i\in\mathbb{N}} \{q_i\}], \\ \left| \int_c^b f(x) \ {}_a d_q x \right| \leq \int_c^b |f(x)| \ {}_a d_q x \ , & \text{for } q \in [\max_{i\in\mathbb{N}} \{q_i\}, 1), \end{cases}$$

where q_i are specific to the given function, varies from function to function. So the proof is completed. □

5. q-integral of Absolute Value of Linear Functions

In this Section, we present the quantum integral of absolute value of a linear function.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ and $f(x) = mx + n$ is a linear function. The q-integral of absolute valued functions has been calculated as below:*

1) If $-\frac{n}{m} \leq a \leq b$ or $a \leq b \leq -\frac{n}{m}$, for $\forall\ q \in (0,1)$, then the following equation holds:

$$\int_a^b |mx+n| \ {}_a d_q x \ = \left| \frac{m(b-a)^2}{[2]_q} + (ma+n)(b-a) \right|. \tag{5.1}$$

2) If $a \leq -\frac{n}{m} \leq b$, then we have

$$\begin{aligned} &\int_a^b |mx+n| \ {}_a d_q x \\ = &\begin{cases} \frac{1-2q^{2k+2}}{[2]_q} m(b-a)^2 + \left\{1 - 2q^{k+1}\right\}(ma+n)(b-a), & \sqrt[k]{\frac{n+ma}{m(a-b)}} \leq q < \sqrt[k+1]{\frac{n+ma}{m(a-b)}}, k \in \mathbb{N}^+, \\ (b-a)\left[(1-q)(|mb+n| + mb+n) \right. \\ \left. -\frac{m(b-a)}{[2]_q} - (ma+n)\right], & q \in \left(0, \frac{n+ma}{m(a-b)}\right). \end{cases} \end{aligned}$$

Proof. Assume that $f : [a,b] \to \mathbb{R}$ and $f(x) = mx + n$ is a linear function.

Case 1: If $-\frac{n}{m} \leq a \leq b$ or $a \leq b \leq -\frac{n}{m}$.

For $-\frac{n}{m} \leq a \leq b$, from definition of q-integral we calculate as follows:

$$\int_a^b |mx+n| \ {}_a d_q x \ = (1-q)(b-a) \sum_{i=0}^{\infty} q^i \left| m\left[q^i b + \left(1-q^i\right)a\right] + n \right|.$$

Since $-\frac{n}{m} \leq a \leq b$, for all $i \in \mathbb{N}$, we know $m\left[q^i b + \left(1-q^i\right)a\right] + n \leq 0$ or $m\left[q^i b + \left(1-q^i\right)a\right] + n \geq 0$. If $m\left[q^i b + \left(1-q^i\right)a\right] + n \geq 0$, then we get

$$\begin{aligned} \int_a^b |mx+n| \ {}_a d_q x &= (1-q)(b-a)\sum_{i=0}^{\infty}\left[q^{2i} m (b-a) + q^i (ma+n)\right] \\ &= (1-q)(b-a)\left[\frac{m(b-a)}{1-q^2} + \frac{(ma+n)}{1-q}\right] \\ &= \frac{m(b-a)^2}{[2]_q} + (ma+n)(b-a). \end{aligned} \tag{5.2}$$

Similarly, if $\left(m\left(q^i b + \left(1-q^i\right)a\right) + n\right) \leq 0$, then we have

$$\int_a^b |mx+n| \ {}_a d_q x = \frac{-m(b-a)^2}{[2]_q} - (ma+n)(b-a). \tag{5.3}$$

Same way, for $a \leq b \leq -\frac{n}{m}$, we obtain (5.2) and (5.3). So, (5.1) holds.

Case 2: If $a \leq -\frac{n}{m} \leq b$, then from definition of q-integral we calculate as follows:

$$\int_a^b |mx+n| \ {}_a d_q x = (1-q)(b-a)\sum_{i=0}^{\infty} q^i \left|m\left[q^i b + \left(1-q^i\right)a\right] + n\right|.$$

There is a $k\in\mathbb{N}^{+}$ and $q_k=\sqrt[k]{\frac{n+ma}{m(a-b)}}\in(0,1)$ such that for $\forall n<k$, $m\left[q_k^ib+\left(1-q_k^i\right)a\right]+n>0$ and for $\forall n>m$, $m\left[q_k^ib+\left(1-q_k^i\right)a\right]+n<0$, then, for $\sqrt[k]{\frac{n+ma}{m(a-b)}}\leq q<\sqrt[k+1]{\frac{n+ma}{m(a-b)}}$ we have

$$\begin{aligned}
\int_a^b |mx+n| \ {}_ad_qx &= (1-q)(b-a)\sum_{i=0}^{\infty}q^i\left|m\left[q^ib+\left(1-q^i\right)a\right]+n\right| \\
&= (1-q)(b-a)\sum_{i=0}^{k}q^i\left\{m\left[q^ib+\left(1-q^i\right)a\right]+n\right\} \\
&\quad -(1-q)(b-a)\sum_{i=k+1}^{\infty}q^i\left\{m\left[q^ib+\left(1-q^i\right)a\right]+n\right\} \\
&= (1-q)(b-a)\left[m(b-a)\frac{1-q^{2k+2}}{1-q^2}+(ma+n)\frac{1-q^{k+1}}{1-q}\right] \\
&\quad -(1-q)(b-a)\sum_{i=0}^{\infty}q^{i+k+1}\left\{m\left[q^{i+k+1}b+\left(1-q^{i+k+1}\right)a\right]+n\right\} \\
&= m(b-a)^2\frac{1-q^{2k+2}}{[2]_q}+(ma+n)\left(1-q^{k+1}\right)(b-a) \\
&\quad -(1-q)(b-a)\sum_{i=0}^{\infty}\left[(b-a)mq^{2i+2k+2}+(ma+n)q^{i+k+1}\right] \\
&= m(b-a)^2\frac{1-q^{2k+2}}{[2]_q}+(ma+n)\left(1-q^{k+1}\right)(b-a) \\
&\quad -(1-q)(b-a)\left[(b-a)mq^{2k+2}\frac{1}{1-q^2}+(ma+n)q^{k+1}\frac{1}{1-q}\right] \\
&= m(b-a)^2\frac{1-q^{2k+2}}{[2]_q}+(ma+n)\left(1-q^{k+1}\right)(b-a) \\
&\quad -(b-a)^2mq^{2k+2}\frac{1}{[2]_q}-(ma+n)q^{k+1}(b-a) \\
&= \frac{1-2q^{2k+2}}{[2]_q}m(b-a)^2+\left(1-2q^{k+1}\right)(ma+n)(b-a).
\end{aligned}$$

Also, if for $k=1$, $q<\frac{n+ma}{m(a-b)}$, $\forall i\in\mathbb{N}$, then we get $\left|m\left(q^{i}b+\left(1-q^{i}\right)a\right)+n\right|\leq 0$ then we have

$$\begin{aligned}
\int_a^b |mx+n| \ {}_a d_q x &= (1-q)(b-a)\sum_{i=0}^{\infty} q^i \left|m\left(q^i b+\left(1-q^i\right)a\right)+n\right| \\
&= (1-q)(b-a)\left\{|mb+n|+\sum_{i=1}^{\infty} q^i \left|m\left[q^i b+\left(1-q^i\right)a\right]+n\right|\right\} \\
&= (1-q)(b-a)\left(|mb+n|-\sum_{i=1}^{\infty} q^i \left\{m\left[q^i b+\left(1-q^i\right)a\right]+n\right\}\right) \\
&= (1-q)(b-a)\left(|mb+n|+\{mb+n\}-\sum_{i=0}^{\infty} q^i \left\{m\left[q^i b+\left(1-q^i\right)a\right]+n\right\}\right) \\
&= (1-q)(b-a)(|mb+n|+mb+n) \\
&\quad -(1-q)(b-a)\sum_{i=0}^{\infty} q^i \left\{m\left[q^i b+\left(1-q^i\right)a\right]+n\right\} \\
&= (1-q)(b-a)(|mb+n|+mb+n) \\
&\quad -(1-q)(b-a)\sum_{i=0}^{\infty}\left[q^{2i} m(b-a)+(ma+n)q^i\right] \\
&= (1-q)(b-a)(|mb+n|+mb+n) \\
&\quad -(1-q)(b-a)\left[\frac{m(b-a)}{1-q^2}+\frac{(ma+n)}{1-q}\right] \\
&= (1-q)(b-a)(|mb+n|+mb+n)-\frac{m(b-a)^2}{1+q}-(ma+n)(b-a).
\end{aligned}$$

So,

$$\begin{aligned}
&\int_a^b |mx+n| \ {}_a d_q x \\
= &\begin{cases} \frac{1-2q^{2k+2}}{[2]_q} m(b-a)^2+\left(1-2q^{k+1}\right)(ma+n)(b-a), & \sqrt[k]{\frac{n+ma}{m(a-b)}}\leq q<\sqrt[k+1]{\frac{n+ma}{m(a-b)}}, k\in\mathbb{N}^+, \\ (b-a)\left\{(1-q)(|mb+n|+mb+n)-\frac{m(b-a)}{[2]_q}-(ma+n)\right\}, & q\in\left(0,\frac{n+ma}{m(a-b)}\right). \end{cases}
\end{aligned}$$

So the proof is completed. ☐

Corollary 1. *In Theorem 3,* $\sqrt[k]{\frac{n+ma}{m(a-b)}}\leq q<\sqrt[k+1]{\frac{n+ma}{m(a-b)}}\Rightarrow\frac{n+ma}{m(a-b)}\leq q^k<\left(\frac{n+ma}{m(a-b)}\right)^{\frac{k}{k+1}}$*, then for* $k\to\infty\Rightarrow q\to 1^-$*, then*

$$\lim_{k\to\infty}\frac{n+ma}{m(a-b)}\leq\lim_{k\to\infty} q^k<\lim_{k\to\infty}\left(\frac{n+ma}{m(a-b)}\right)^{\frac{k}{k+1}}\Rightarrow q^k\to\frac{n+ma}{m(a-b)}$$

this yields classical result for Riemann integral as below:

$$\int_a^b |mx+n|\,dx=\frac{m^2\left(b^2+a^2\right)+2nm(a+b)+2n^2}{2m}.$$

Example 2. Let applying Theorem 3 to $f(x) = |2x-3|$ on $[0,4]$, then we have the following q-integral:

$$\int_0^3 |2x-3| \ {}_0d_qx \tag{5.4}$$

$$= \begin{cases} 18\frac{1-2q^{2k+2}}{[2]_q} - 9\left\{1-2q^{k+1}\right\}, & \sqrt[k]{\frac{1}{2}} \leq q < \sqrt[k+1]{\frac{1}{2}}, k \in \mathbb{N}^+, \\ \frac{-9-9q-18q^2}{[2]_q}, & q \in \left(0, \frac{1}{2}\right). \end{cases}$$

For example, in (5.4) if we choose $q = 0,75$, then we must take $k = 2$ since

$$\sqrt{\frac{1}{2}} = 0.7071067811 \leq q = 0,75 < \sqrt[3]{\frac{1}{2}} = 0.7937005259.$$

This yields

$$\int_0^3 |2x-3| \ {}_0d_{(0,75)}x \ = 5,5157645095.$$

6. Conclusion

The conditions in this study can be added to the previous studies that contain these inequalities and the mistakes made can be eliminated. In addition, if there are results including these inequalities in future studies, this study should definitely be taken into consideration.

Availability of data and material

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

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This work does not have any conflicts of interest.

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Consent to Participate (Ethics)

All participants have been informed about the purposes of the study and declare that there is no unethical behavior. It has not been used elsewhere and has not been posted. Compliance with the publishing ethics rules of the universities has been approved.

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