RANKS OF FRINGE OPERATORS ON FINITE RUDIN TYPE INVARIANT SUBSPACES II

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Abstract. Let $\mathcal{M}$ be a finite Rudin type invariant subspace of the Hardy space over the bidisk with variables z, w. Let $\mathcal{F}_z$ be the fringe operator on $\mathcal{M} \ominus w\mathcal{M}$. In this paper we determine the rank of $\mathcal{F}_z^*$ on $\mathcal{M} \ominus w\mathcal{M}$.

1. Introduction

This paper is a continuation of [2]. In [2], we have determined the rank of the fringe operator $\mathcal{F}_z$ on $\mathcal{M} \ominus w\mathcal{M}$, where $\mathcal{M}$ is a fnite Rudin type invariant subspace of the Hardy space over the bidisk. In this paper we will determine the rank of $\mathcal{F}_z^*$ on $\mathcal{M} \ominus w\mathcal{M}$.

Let H be a separable Hilbert space and $T = (T_1, \ldots, T_n), n \geq 1$ a tuple of commuting bounded linear operators on H. A closed subspace M of H is called an invariant subspace for T if $T_i M \subset M, i = 1, \ldots, n$. If $E \subseteq H$, then we let $[E]_T = [E]_{\{T_1,\ldots,T_n\}}$ be the smallest invariant subspace for T containing E. A subset E of M is said to be a generating set of M for T if $[E]_T = M$. The minimum number of elements in the generating sets of M is called the rank of M for T, and we denote it by

$$\mathrm{rank}_T M.$$

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk with variables z, w, and T_z, T_w the multiplication operators with symbols z, w. If M is an invariant subspace for T_z, T_w, then we define the fringe operator $\mathcal{F}_z$ on $M \ominus wM$ by

$$\mathcal{F}_z = P_{M \ominus wM} T_z|_{M \ominus wM},$$

see [5].

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Let k be a fixed positive integer throughout this paper. Let $\varphi_1(z)$, $\varphi_2(z), \cdots, \varphi_k(z)$ and $\psi_1(w)$, $\psi_2(w)$, $\cdots, \psi_k(w)$ be non-constant one variable inner functions such that

$$\begin{cases} \varphi_k(z) \prec \varphi_{k-1}(z) \prec \cdots \prec \varphi_1(z), \\ \psi_1(w) \prec \psi_2(w) \prec \cdots \prec \psi_k(w), \end{cases} \tag{1.1}$$

where $\theta_2(z) \prec \theta_1(z)$ means $\theta_1(z)/\theta_2(z) \in H^2(z)$. Let

$$\mathcal{M} = \bigvee_{n=0}^{k} \varphi_{n+1}(z)\psi_n(w)H^2, \tag{1.2}$$

where $\varphi_{k+1}(z) = \psi_0(w) = 1$. Then $\mathcal{M}$ is an invariant subspace of H^2 for T_z, T_w. We call $\mathcal{M}$ a finite Rudin type invariant subspace. In the following when we use the notation $\mathcal{M}$, we always mean the finite Rudin type invariant subspace defined by (1.2).

Let

$$\zeta_n(z) = \frac{\varphi_n(z)}{\varphi_{n+1}(z)}, \quad \xi_n(w) = \frac{\psi_n(w)}{\psi_{n-1}(w)}, \quad 1 \le n \le k. \tag{1.3}$$

Then $\zeta_n(z)$ and $\xi_n(w)$ are inner functions, $\xi_1(w) = \psi_1(w)$, $\zeta_k(z) = \varphi_k(z)$, and

$$\varphi_\ell(z) = \prod_{n=\ell}^{k} \zeta_n(z) \quad \text{and} \quad \psi_\ell(w) = \prod_{n=1}^{\ell} \xi_n(w), \quad 1 \le \ell \le k.$$

Without loss of generality, we assume that

$$\zeta_1(z), \cdots, \zeta_k(z), \xi_1(w), \cdots, \xi_k(w) \text{ are non-constants.} \tag{1.4}$$

Note that

$$\mathcal{M} = \varphi_1(z)H^2 \oplus \bigoplus_{n=1}^{k} \varphi_{n+1}(z)K_{\zeta_n}(z) \otimes \psi_n(w)H^2(w),$$

where $K_{\zeta_n}(z) = H^2(z) \ominus \zeta_n(z)H^2(z)$, see e.g. [1,6]. Thus

$$\mathcal{M} \ominus w\mathcal{M} = \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z). \tag{1.5}$$

Let $\mathcal{I}$ be the set of non-constant one variable inner functions. The main result in this paper is the following.

Theorem 1.6. *Let $\mathcal{M}$ be the finite Rudin type invariant subspace defined by (1.2).*

(i) If $\psi_k(0) \neq 0$, then $\operatorname{rank}_{\mathcal{F}_z^*}(\mathcal{M} \ominus w\mathcal{M}) = 1$.

(ii) If $\psi_k(0)=0$*, and* $\{1\le n\le k:\xi_n(0)=0\}=\{n_1,n_2,\cdots,n_m\}$*,* $\theta_\ell(z)=\prod_{n=n_{\ell-1}}^{n_\ell-1}\zeta_n(z)$*,* $n_{m+1}=k+1$*, then*

$$\operatorname{rank}_{\mathcal{F}_z^*}(\mathcal{M}\ominus w\mathcal{M})=\max_{\sigma(z)\in\mathcal{I}}\#\big\{2\le\ell\le m+1:\sigma(z)\prec\theta_\ell(z)\big\}.$$

The proof of the above theorem is divided into two parts, see Theorems 2.5 and 3.9. In section 4, we discuss the Fredholm index of $\mathcal{F}_z$ and the index of $\mathcal{M}$.

2. The case $\psi_k(0)\neq 0$

Suppose $\varphi_1(z),\varphi_2(z),\cdots,\varphi_k(z)$ and $\psi_1(w),\ \psi_2(w),\ \cdots,\psi_k(w)$ are non-constant inner functions satisfying condition (1.1), and $\zeta_1(z),\zeta_2(z)$, $\ldots,\zeta_k(z),\ \xi_1(w),\cdots,\xi_k(w)$ are defined by (1.3) satisfying (1.4). Let $\varphi_0(z)$ be a zero function or a non-constant inner function such that $\varphi_1(z)\prec\varphi_0(z)$, and $\zeta_0(z)=\varphi_0(z)/\varphi_1(z)$. Let

$$\Gamma=\bigoplus_{n=1}^{k+1}\varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z), \tag{2.1}$$

and

$$\widetilde{\Gamma}=\bigoplus_{n=1}^{k+1}\varphi_n(z)K_{\zeta_{n-1}}(z),$$

where $\varphi_{k+1}(z)=\psi_0(w)=1$, $K_{\zeta_n}(z)=H^2(z)\ominus\zeta_n(z)H^2(z)$. Then $\widetilde{\Gamma}=K_{\varphi_0}(z)$, and by (1.5), $\Gamma\subseteq\mathcal{M}\ominus w\mathcal{M}$. Note that when $\varphi_0(z)=0$, we have $\Gamma=\mathcal{M}\ominus w\mathcal{M}$, $\widetilde{\Gamma}=H^2(z)$.

Suppose $\psi_k(0)\neq 0$. Since $\psi_n(w)\prec\psi_k(w)$, we have $a_n:=\psi_n(0)\neq 0, 1\le n\le k$. Note that $a_0=\psi_0(0)=1$. Let $\Phi:\Gamma\to\widetilde{\Gamma}$ be defined by

$$\Phi G=\bigoplus_{n=1}^{k+1}a_{n-1}\varphi_n(z)g_{n-1}(z)\in\widetilde{\Gamma}=K_{\varphi_0}(z),$$

where $G=\bigoplus_{n=1}^{k+1}\varphi_n(z)\psi_{n-1}(w)g_{n-1}(z)\in\Gamma$, and $\Psi:\widetilde{\Gamma}\to\Gamma$ be defined by

$$\Psi F=\bigoplus_{n=1}^{k+1}\overline{a}_{n-1}\varphi_n(z)\psi_{n-1}(w)f_{n-1}(z)\in\Gamma,$$

where $F=\bigoplus_{n=1}^{k+1}\varphi_n(z)f_{n-1}(z)\in\widetilde{\Gamma}=K_{\varphi_0}(z)$. Then Φ and Ψ are bounded invertible operators.

Let $\mathcal{F}_{z,\Gamma}f=P_\Gamma(zf), f\in\Gamma$, and

$$S_{z,\varphi_0}f(z)=P_{K_{\varphi_0}(z)}T_zf(z),\quad f(z)\in K_{\varphi_0}(z)$$

be the compression of T_z on $K_{\varphi_0}(z)$, where P_E is the orthogonal projection onto E. The following result is Theorem 2.1 in [2].

Theorem 2.2 ([2]). *Suppose that* $\psi_k(0) \neq 0$. *Then*

$$\langle \Psi F, \mathcal{F}_{z,\Gamma}^j G\rangle = \langle F, S_{z,\varphi_0}^j \Phi G\rangle, \quad G \in \Gamma, F \in \widetilde{\Gamma}, j \geq 0.$$

The following is a key observation.

Theorem 2.3. *Suppose that* $\psi_k(0) \neq 0$. *Then* $\Psi = \Phi^*$, *and*

$$\Phi \mathcal{F}_{z,\Gamma} = S_{z,\varphi_0}\Phi, \quad \mathcal{F}_{z,\Gamma}^* \Psi = \Psi S_{z,\varphi_0}^*.$$

Proof. Let $j = 0$ in Theorem 2.2, we have $\langle \Psi F, G\rangle = \langle F, \Phi G\rangle$, $G \in \Gamma, F \in \widetilde{\Gamma}$. So $\Psi = \Phi^*$. Now let $j = 1$, we obtain $\langle \Psi F, \mathcal{F}_{z,\Gamma} G\rangle = \langle F, S_{z,\varphi_0}\Phi G\rangle$. The conclusion then follows from this. □

Corollary 2.4. *Suppose that* $\psi_k(0) \neq 0$.

(i) *Let* $G \in \Gamma$. *Then* $[G]_{\mathcal{F}_{z,\Gamma}} = \Gamma$ *if and only if* $[\Phi G]_{S_{z,\varphi_0}} = K_{\varphi_0}(z)$.

(ii) *Let* $F \in \widetilde{\Gamma}$. *Then* $[\Psi F]_{\mathcal{F}_{z,\Gamma}^*} = \Gamma$ *if and only if* $[F]_{S_{z,\varphi_0}^*} = K_{\varphi_0}(z)$.

Recall that when $\varphi_0(z) = 0$, $\Gamma = \mathcal{M} \ominus w\mathcal{M}$, $\widetilde{\Gamma} = H^2(z)$. In this case, $\mathcal{F}_{z,\Gamma} = \mathcal{F}_z$ and $S_{z,\varphi_0} = T_z$ on $H^2(z)$.

Theorem 2.5. *Suppose that* $\psi_k(0) \neq 0$. *Then*

$$\text{rank}_{\mathcal{F}_z^*}(\mathcal{M} \ominus w\mathcal{M}) = \text{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) = 1.$$

Proof. Note that $1-\overline{\zeta_k(0)}\zeta_k(z)$ is an outer function contained in $K_{\zeta_k}(z)$. Let $G = \psi_k(w)(1 - \overline{\zeta_k(0)}\zeta_k(z))$. Then $G \in \Gamma$, and $\Phi G = a_k(1 - \overline{\zeta_k(0)}\zeta_k(z))$ is cyclic for T_z on $H^2(z)$. So by Corollary 2.4 (i), $\text{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) = 1$.

It is known that there exists $F \in K_{\varphi_0}(z)$ such that $[F]_{S_{z,\varphi_0}^*} = K_{\varphi_0}(z)$, where $\varphi_0(z) = 0$ or an inner function, see [4]. So by Corollary 2.4 (ii), $\text{rank}_{\mathcal{F}_z^*}(\mathcal{M} \ominus w\mathcal{M}) = 1$. □

3. The case $\psi_k(0) = 0$

Suppose $\psi_k(0) = 0$. Recall that $\xi_n(w) = \frac{\psi_n(w)}{\psi_{n-1}(w)}$ and $\psi_k(w) = \prod_{n=1}^k \xi_n(w)$. So there exits $1 \leq n \leq k$ such that $\xi_n(0) = 0$. Suppose

$$\{1 \leq n \leq k : \xi_n(0) = 0\} = \{n_1, n_2, \cdots, n_m\}, \tag{3.1}$$

where $1 \leq n_1 < n_2 < \cdots < n_m \leq k$. Set $n_0 = 0$ and $n_{m+1} = k+1$. For each $1 \leq \ell \leq m+1$, let

$$\Gamma_\ell = \bigoplus_{n=n_{\ell-1}+1}^{n_\ell} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z). \tag{3.2}$$

Then $\Gamma = \bigoplus_{\ell=1}^{m+1} \Gamma_\ell$, where Γ is defined by (2.1). It was shown in [2] that

$$\mathcal{F}_{z,\Gamma}\Gamma_\ell \subset \Gamma_\ell, \quad 1 \le \ell \le m+1. \tag{3.3}$$

In fact, if $n_{\ell-1}+1 \le n \le n_\ell$, $i \le n_{\ell-1}$ or $i > n_\ell$, then

$$\langle \psi_{n-1}(w), \psi_{i-1}(w)\rangle = 0.$$

It then follows from the definition of Γ_ℓ that $\mathcal{F}_{z,\Gamma}\Gamma_\ell \subset \Gamma_\ell, 1 \le \ell \le m+1$.

Let $\Theta = \big(\theta_1(z), \theta_2(z), \cdots, \theta_d(z)\big)$ be a d-tuple consisting of zeros or non-constant inner functions. Let

$$\mathbf{K}_\Theta(z) = K_{\theta_1}(z) \oplus K_{\theta_2}(z) \oplus \cdots \oplus K_{\theta_d}(z)$$

be the direct sum of $K_{\theta_j}(z)$. If $F = (f_1, f_2, \cdots, f_d) \in \mathbf{K}_\Theta(z)$, then we define

$$\mathbf{S}_{z,\Theta}F = \big(S_{z,\theta_1}f_1, S_{z,\theta_2}f_2, \cdots, S_{z,\theta_d}f_d\big) \in \mathbf{K}_\Theta(z).$$

Theorem 3.4. *Suppose that $\psi_k(0) = 0$. For $1 \le \ell \le m+1$, let*

$$\theta_\ell(z) = \prod_{n=n_{\ell-1}}^{n_\ell - 1} \zeta_n(z) = \frac{\varphi_{n_{\ell-1}}(z)}{\varphi_{n_\ell}(z)}, \tag{3.5}$$

and $\Theta = \big(\theta_1(z), \theta_2(z), \cdots, \theta_{m+1}(z)\big)$. Then there is a bounded invertible operator $T : \Gamma \to \boldsymbol{K}_\Theta(z)$ such that $T\mathcal{F}_{z,\Gamma} = \boldsymbol{S}_{z,\Theta}T$.

Proof. Let

$$\Gamma'_\ell = \bigoplus_{n=n_{\ell-1}+1}^{n_\ell} \frac{\varphi_n(z)}{\varphi_{n_\ell}(z)} \frac{\psi_{n-1}(w)}{\psi_{n_{\ell-1}}(w)} K_{\zeta_{n-1}}(z).$$

Note that

$$\frac{\frac{\varphi_n(z)}{\varphi_{n_\ell}(z)}}{\frac{\varphi_{n+1}(z)}{\varphi_{n_\ell}(z)}} = \frac{\varphi_n(z)}{\varphi_{n+1}(z)} = \zeta_n(z), \quad n_{\ell-1}+1 \le n \le n_\ell,$$

and by (3.1),

$$\Big(\frac{\psi_{n_\ell - 1}}{\psi_{n_{\ell-1}}}\Big)(0) = \prod_{n=n_{\ell-1}+1}^{n_\ell - 1} \xi_n(0) \ne 0.$$

Hence we can apply Theorem 2.3 for Γ'_ℓ and $\mathcal{F}_{z,\Gamma'_\ell}$. To be precise, let

$$\begin{aligned}\widetilde{\Gamma}'_\ell &= \bigoplus_{n=n_{\ell-1}+1}^{n_\ell} \frac{\varphi_n(z)}{\varphi_{n_\ell}(z)} K_{\zeta_{n-1}}(z) \\ &= \bigoplus_{n=n_{\ell-1}+1}^{n_\ell} \left[\frac{\varphi_n(z)}{\varphi_{n_\ell}(z)} H^2(z) \ominus \frac{\varphi_{n-1}(z)}{\varphi_{n_\ell}(z)} H^2(z)\right] \\ &= H^2(z) \ominus \frac{\varphi_{n_{\ell-1}}(z)}{\varphi_{n_\ell}(z)} H^2(z) = K_{\theta_\ell}(z).\end{aligned}$$

Then by Theorem 2.3, there are invertible operators $\Phi'_\ell : \Gamma'_\ell \to \widetilde{\Gamma}'_\ell = K_{\theta_\ell}(z)$ and $\Psi'_\ell : \widetilde{\Gamma}'_\ell \to \Gamma'_\ell$ such that $\Phi'_\ell \mathcal{F}_{z,\Gamma'_\ell} = S_{z,\theta_\ell} \Phi'_\ell$. Note that

$$\Gamma_\ell = \varphi_{n_\ell}(z) \psi_{n_{\ell-1}}(w) \Gamma'_\ell.$$

We define $\Phi_\ell : \Gamma_\ell \to K_{\theta_\ell}(z)$ by

$$\Phi_\ell\big(\varphi_{n_\ell}(z)\psi_{n_{\ell-1}}(w) f\big) = \Phi'_\ell f \in \widetilde{\Gamma}'_\ell = K_{\theta_\ell}(z), \quad f \in \Gamma'_\ell.$$

Then $\Phi_\ell : \Gamma_\ell \to K_{\theta_\ell}(z)$ is an invertible operator. We have

$$\begin{aligned}&\Phi_\ell \mathcal{F}_{z,\Gamma_\ell}[\varphi_{n_\ell}(z)\psi_{n_{\ell-1}}(w) f] \\ &= \Phi_\ell[\varphi_{n_\ell}(z)\psi_{n_{\ell-1}}(w) \mathcal{F}_{z,\Gamma'_\ell} f] \\ &= \Phi'_\ell \mathcal{F}_{z,\Gamma'_\ell} f = S_{z,\theta_\ell} \Phi'_\ell f \\ &= S_{z,\theta_\ell} \Phi_\ell[\varphi_{n_\ell}(z)\psi_{n_{\ell-1}}(w) f], \quad f \in \Gamma'_\ell.\end{aligned}$$

Hence

$$\Phi_\ell \mathcal{F}_{z,\Gamma_\ell} = S_{z,\theta_\ell} \Phi_\ell, \quad 1 \le \ell \le m+1. \tag{3.6}$$

Now we define $T : \Gamma \to \mathbf{K}_\Theta(z)$ by

$$Tf = \bigoplus_{\ell=1}^{m+1} \Phi_\ell f_\ell \in \bigoplus_{\ell=1}^{m+1} K_{\theta_\ell}(z) = \mathbf{K}_\Theta(z),$$

where $f = \bigoplus_{\ell=1}^{m+1} f_\ell \in \bigoplus_{\ell=1}^{m+1} \Gamma_\ell = \Gamma$. Then $T : \Gamma \to \mathbf{K}_\Theta(z)$ is an invertible operator. By (3.3),

$$\mathcal{F}_{z,\Gamma} = \bigoplus_{\ell=1}^{m+1} \mathcal{F}_{z,\Gamma_\ell} \quad \text{on} \quad \Gamma = \bigoplus_{\ell=1}^{m+1} \Gamma_\ell.$$

Thus

$$T\mathcal{F}_{z,\Gamma}f = T\bigoplus_{\ell=1}^{m+1}\mathcal{F}_{z,\Gamma_\ell}f_\ell = \bigoplus_{\ell=1}^{m+1}\Phi_\ell\mathcal{F}_{z,\Gamma_\ell}f_\ell$$
$$= \bigoplus_{\ell=1}^{m+1}S_{z,\theta_\ell}\Phi_\ell f_\ell = \mathbf{S}_{z,\Theta}Tf, \quad f = \bigoplus_{\ell=1}^{m+1}f_\ell \in \Gamma.$$

So $T\mathcal{F}_{z,\Gamma} = \mathbf{S}_{z,\Theta}T$. The proof is complete. □

Let $\mathcal{I}$ be the set of non-constant one variable inner functions.

Lemma 3.7. *Let $\theta_1(z), \theta_2(z), \cdots, \theta_d(z)$ be non-constant inner functions. Then*

$$\operatorname{rank}_{\boldsymbol{S}^*_{z,\Theta}}\boldsymbol{K}_\Theta(z) = \operatorname{rank}_{\boldsymbol{S}_{z,\Theta}}\boldsymbol{K}_\Theta(z) = \max_{\sigma(z)\in\mathcal{I}}\#\big\{1 \le j \le d : \sigma(z) \prec \theta_j(z)\big\},$$

where $\#A$ denotes the number of elements in A.

Proof. It is known that

$$\operatorname{rank}_{\mathbf{S}_{z,\Theta}}\mathbf{K}_\Theta(z) = \max_{\sigma(z)\in\mathcal{I}}\#\big\{1 \le j \le d : \sigma(z) \prec \theta_j(z)\big\},$$

see [4, p. 269].

Let $\tau_\theta : K_\theta(z) \to K_\theta(z)$ be defined by

$$\tau_\theta f(z) = \overline{z}\theta(z)\overline{f}(z), \quad f(z) \in K_\theta(z).$$

Then τ_θ is an antilinear onto isometry on $K_\theta(z)$, $\tau_\theta(\tau_\theta f(z)) = f(z)$ and $\tau_\theta S^*_{z,\theta} = S_{z,\theta}\tau_\theta$. Now we define $\boldsymbol{\tau}_\Theta$ on $\mathbf{K}_\Theta(z)$ by

$$\boldsymbol{\tau}_\Theta F = (\tau_{\theta_1}f_1, \tau_{\theta_2}f_2, \cdots, \tau_{\theta_d}f_d) \in \mathbf{K}_\Theta(z),$$

where $F = (f_1, f_2, \cdots, f_d) \in \mathbf{K}_\Theta(z)$. Then $\boldsymbol{\tau}_\Theta$ is an antilinear onto isometry and $\boldsymbol{\tau}_\Theta\mathbf{S}^*_{z,\Theta} = \mathbf{S}_{z,\Theta}\boldsymbol{\tau}_\Theta$. Thus it follows that $\operatorname{rank}_{\mathbf{S}^*_{z,\Theta}}\mathbf{K}_\Theta(z) = \operatorname{rank}_{\mathbf{S}_{z,\Theta}}\mathbf{K}_\Theta(z)$. □

Lemma 3.8. *Suppose that $\psi_k(0) = 0$. Let θ_ℓ be given by (3.5) and*

$$\Theta_1 = \big(\theta_2(z), \theta_3(z), \cdots, \theta_{m+1}(z)\big).$$

Then there is an invertible operator $T_0 : \mathcal{M} \ominus w\mathcal{M} \to H^2(z) \oplus \boldsymbol{K}_{\Theta_1}(z)$ such that $T_0\mathcal{F}_z = (T_z \oplus \boldsymbol{S}_{z,\Theta_1})T_0$.

Proof. Let $\varphi_0(z) = 0$. In this case $\Gamma = \mathcal{M} \ominus w\mathcal{M}$. Recall that Γ_ℓ are defined by (3.2), i.e.

$$\Gamma_\ell = \bigoplus_{n=n_{\ell-1}+1}^{n_\ell}\varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z), \quad 1 \le \ell \le m+1.$$

Set $\Lambda = \bigoplus_{\ell=2}^{m+1} \Gamma_\ell$. Then $\mathcal{M} \ominus w\mathcal{M} = \Gamma_1 \oplus \Lambda$, $\mathcal{F}_z\Gamma_1 \subset \Gamma_1$ and $\mathcal{F}_z\Lambda \subset \Lambda$. Note that when $\varphi_0(z) = 0$, $\zeta_0(z) = 0$. So $\theta_1(z) = \prod_{n=0}^{n_1-1} \zeta_n(z) = 0$, and $S_{z,\theta_1} = T_z$ on $H^2(z)$. Thus the conclusion follows from Theorem 3.4. □

Now we can prove the main result in this section. The rank of $\mathcal{F}_z$ on $\mathcal{M} \ominus w\mathcal{M}$ was obtained in [2], we include a slightly different proof in the following.

Theorem 3.9. *Suppose that $\psi_k(0) = 0$. Let θ_ℓ be given by (3.5) and*

$$\Theta_1 = \big(\theta_2(z), \theta_3(z), \cdots, \theta_{m+1}(z)\big).$$

Then

$$\operatorname{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) = 1 + \max_{\sigma(z)\in\mathcal{I}} \#\big\{2 \le \ell \le m+1 : \sigma(z) \prec \theta_\ell(z)\big\},$$

and

$$\operatorname{rank}_{\mathcal{F}_z^*}(\mathcal{M} \ominus w\mathcal{M}) = \max_{\sigma(z)\in\mathcal{I}} \#\big\{2 \le \ell \le m+1 : \sigma(z) \prec \theta_\ell(z)\big\}.$$

Proof. We first study the rank of $\mathcal{F}_z$. Let

$$s_1 = \max_{\sigma(z)\in\mathcal{I}} \#\big\{2 \le \ell \le m+1 : \sigma(z) \prec \theta_\ell(z)\big\}.$$

By Lemma 3.8, we have

$$\begin{aligned}
&\operatorname{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) \\
&= \operatorname{rank}_{\{T_z \oplus \mathbf{S}_{z,\Theta_1}\}}(H^2(z) \oplus \mathbf{K}_{\Theta_1}(z)) \\
&\le \operatorname{rank}_{T_z} H^2(z) + \operatorname{rank}_{\mathbf{S}_{z,\Theta_1}} \mathbf{K}_{\Theta_1}(z) \\
&= 1 + s_1.
\end{aligned}$$

Let $\theta(z) = \prod_{n=2}^{m+1} \theta_n(z_1)$ and $\widetilde{\Theta} = \big(\theta(z), \theta_2(z_1), \cdots, \theta_{m+1}(z_1)\big)$. Then

$$\begin{aligned}
&\operatorname{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) \\
&= \operatorname{rank}_{\{T_z \oplus \mathbf{S}_{z,\Theta_1}\}}(H^2(z) \oplus \mathbf{K}_{\Theta_1}(z)) \\
&\ge \operatorname{rank}_{\mathbf{S}_{z,\widetilde{\Theta}}} \mathbf{K}_{\widetilde{\Theta}}(z) \\
&= 1 + \max_{\sigma(z)\in\mathcal{I}} \#\big\{2 \le \ell \le m+1 : \sigma(z) \prec \theta_\ell(z)\big\} \\
&= 1 + s_1.
\end{aligned}$$

Thus $\operatorname{rank}_{\mathcal{F}_z}(\mathcal{M} \ominus w\mathcal{M}) = 1 + s_1$.

Now we study the rank of $\mathcal{F}_z^*$. By Lemma 3.8, we have

$$\mathcal{F}_z^* T_0^* = T_0^*(T_z^* \oplus \mathbf{S}_{z,\Theta_1}^*) \quad \text{on } H^2(z) \oplus \mathbf{K}_{\Theta_1}(z),$$

and

$$\mathrm{rank}_{\mathcal{F}_z^*}(\mathcal{M}\ominus w\mathcal{M}) = \mathrm{rank}_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}(H^2(z)\oplus \mathbf{K}_{\Theta_1}(z)).$$

Lemma 3.7 implies that

$$\mathrm{rank}_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}(H^2(z)\oplus \mathbf{K}_{\Theta_1}(z)) \geq \mathrm{rank}_{\mathbf{S}_{z,\Theta_1}^*}\mathbf{K}_{\Theta_1}(z) = s_1.$$

Thus it is left to show $\mathrm{rank}_{\mathcal{F}_z^*}(\mathcal{M}\ominus w\mathcal{M}) \leq s_1$. Let $F_1, F_2, \cdots, F_{s_1} \in \mathbf{K}_{\Theta_1}(z)$ be such that

$$[F_1, F_2, \cdots, F_{s_1}]_{\mathbf{S}_{z,\Theta_1}^*} = \mathbf{K}_{\Theta_1}(z),$$

and let $f_1(z)\in H^2(z)$ satisfy $[f_1(z)]_{T_z^*} = H^2(z)$. Set

$$\eta(z) = \prod_{\ell=2}^{m+1}\theta_\ell(z)$$

and

$$F_0 = \eta(z)f_1(z)\oplus F_1 \in H^2(z)\oplus \mathbf{K}_{\Theta_1}(z).$$

We show that

$$[F_0, F_2, \cdots, F_{s_1}]_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}} = H^2(z)\oplus \mathbf{K}_{\Theta_1}(z). \tag{3.10}$$

Let $S_{\eta,\theta}$ be defined by $S_{\eta,\theta}f(z) = P_{K_\theta(z)}(\eta(z)f(z)), f(z)\in K_\theta(z)$, and let

$$\mathbf{S}_{\eta,\Theta_1} = S_{\eta,\theta_2}\oplus S_{\eta,\theta_3}\oplus\cdots\oplus S_{\eta,\theta_{m+1}} \quad \text{on } \mathbf{K}_{\Theta_1}(z).$$

Then

$$(T_\eta^*\oplus \mathbf{S}_{\eta,\Theta_1}^*)F_0 \in [F_0, F_2, \cdots, F_{s_1}]_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}$$

Note that $\mathbf{S}_{\eta,\Theta_1}^*F_1 = 0$, thus

$$\begin{aligned}&(T_\eta^*\oplus \mathbf{S}_{\eta,\Theta_1}^*)F_0\\ &= T_\eta^*[\eta(z)f_1(z)] = f_1(z)\\ &\in [F_0, F_2, \cdots, F_{s_1}]_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}.\end{aligned}$$

It follows that

$$\begin{aligned}&[F_0, F_2, \cdots, F_{s_1}]_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}\\ &\supset [f_1(z), F_1, F_2, \cdots, F_{s_1}]_{\{T_z^*\oplus \mathbf{S}_{z,\Theta_1}^*\}}\\ &\supset [f_1(z)]_{T_z^*}\oplus [F_1, F_2, \cdots, F_{s_1}]_{\mathbf{S}_{z,\Theta_1}^*}\\ &= H^2(z)\oplus \mathbf{K}_{\Theta_1}(z).\end{aligned}$$

Therefore (3.10) is established, which finishes the proof. □

4. Related topics

A bounded linear operator T is called Fredholm if T has closed range, $\dim\ker T < \infty$ and $\dim\ker T^* < \infty$. In this case, the Fredholm index is defined by $\operatorname{ind} T = \dim\ker T - \dim\ker T^*$. The following result is well-known.

Lemma 4.1. *The following hold.*

(i) T_z *is a Fredholm operator on* $H^2(z)$, $\ker T_z = \{0\}$, $\ker T_z^* = \mathbb{C}$.

(ii) For a non-constant inner function $\theta(z)$, $S_{z,\theta}$ *is a Fredholm operator and*

$$\dim\ker S_{z,\theta} = \dim\ker S_{z,\theta}^* = \begin{cases} 0, & \theta(0) \neq 0 \\ 1, & \theta(0) = 0. \end{cases}$$

Theorem 4.2. $\mathcal{F}_z$ *is a Fredholm operator on* $\mathcal{M} \ominus w\mathcal{M}$ *and* $\operatorname{ind}\mathcal{F}_z = -1$. *Moreover we have the following.*

(i) If $\psi_k(0) \neq 0$, *then* $\ker\mathcal{F}_z = \{0\}$ *and* $\dim\ker\mathcal{F}_z^* = 1$.

(ii) If $\psi_k(0) = 0$, *then*

$$\dim\ker\mathcal{F}_z = \#\big\{2 \leq \ell \leq m+1 : \theta_\ell(0) = 0\big\}$$

and

$$\dim\ker\mathcal{F}_z^* = 1 + \#\big\{2 \leq \ell \leq m+1 : \theta_\ell(0) = 0\big\},$$

where θ_ℓ *are defined by (3.5).*

Proof. (i) Suppose that $\psi_k(0) \neq 0$. By Theorem 2.3, there is an invertible operator $\Phi : \mathcal{M} \ominus w\mathcal{M} \to H^2(z)$ such that $\Phi\mathcal{F}_z = T_z\Phi$. Lemma 4.1 (i) then ensures that $\ker\mathcal{F}_z = \{0\}$ and $\dim\ker\mathcal{F}_z^* = 1$.

(ii) Suppose that $\psi_k(0) = 0$. By Lemma 3.8, there is an invertible operator $T_0 : \mathcal{M} \ominus w\mathcal{M} \to H^2(z) \oplus \mathbf{K}_{\Theta_1}(z)$ such that

$$T_0\mathcal{F}_z = (T_z \oplus \mathbf{S}_{z,\Theta_1})T_0,$$

where $\Theta_1 = \big(\theta_2(z), \theta_3(z), \cdots, \theta_{m+1}(z)\big)$. It is clear that $T_z \oplus \mathbf{S}_{z,\Theta_1}$ has closed range. By Lemma 4.1 (ii), we have

$$\begin{aligned}
\dim\ker(T_z \oplus \mathbf{S}_{z,\Theta_1}) &= \dim\ker T_z + \dim\ker\mathbf{S}_{z,\Theta_1} \\
&= \sum_{\ell=2}^{m+1} \dim\ker S_{z,\theta_\ell} \\
&= \#\big\{2 \leq \ell \leq m+1 : \theta_\ell(0) = 0\big\},
\end{aligned}$$

and

$$\begin{aligned}
\dim\ker(T_z^* \oplus \mathbf{S}_{z,\Theta_1}^*) &= \dim\ker T_z^* + \dim\ker\mathbf{S}_{z,\Theta_1}^* \\
&= 1 + \#\big\{2 \leq \ell \leq m+1 : \theta_\ell(0) = 0\big\}.
\end{aligned}$$

Thus (ii) holds. □

We can also obtain the Fredholmness of $\mathcal{F}_z$ as follows. Note that $\mathcal{M}$ is a Hilbert-Schmidt submodule, so $\mathcal{F}_z$ is a Fredholm operator, see [3, Propositions 2.2 and 3.7].

For an invariant subspace N of H^2, let

$$\mathrm{ind} N = \mathrm{ind}_{(0,0)} N = \dim \big(N \ominus (zN + wN) \big).$$

$\mathrm{ind}_{(0,0)} N$ is called the index of N at $(0,0)$. Note that

$$N \ominus (zN + wN) = \ker \mathcal{F}_z^*,$$

see [3]. Hence by Theorems 4.2, we have the following.

Corollary 4.3. *The following hold.*

(i) If $\psi_k(0) \neq 0$*, then* $\mathrm{ind}\mathcal{M} = 1$.

(ii) If $\psi_k(0) = 0$*, then*

$$\mathrm{ind}\mathcal{M} = 1 + \# \big\{ 2 \le \ell \le m+1 : \theta_\ell(0) = 0 \big\}.$$

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References

[1] B.K. Das, J. Sarkar, Rudin's submodules of $H^2(\mathbb{D}^2)$. C. R. Math. Acad. Sci. Paris 353 (2015), no. 1, 51-55.

[2] K.H. Izuchi, S. Luo, Ranks of fringe operators on finite Rudin type invariant subspaces. J. Math. Anal. Appl. 520 (2023), no. 2, Paper No. 126910, 16 pp.

[3] S. Luo, K.J. Izuchi, R. Yang, Hilbert-Schmidtness of some finitely generated submodules in $H^2(\mathbb{D}^2)$. J. Math. Anal. Appl. 465 (2018), no. 1, 531-546

[4] N.K. Nikol'skiĭ, Treatise on the shift operator, Springer, New York, 1986.

[5] R. Yang, Operator theory in the Hardy space over the bidisk. III. J. Funct. Anal. 186 (2001), no. 2, 521-545.

[6] Y. Yang, Two inner sequences based invariant subspaces in $H^2(\mathbb{D}^2)$. Integral Equations Operator Theory 77 (2013), no. 2, 279-290.

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