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INVERSE PROBLEM FOR ABSTRACT DIFFERENTIAL EQUATION WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT. In this manuscript, we consider an inverse problem for first order abstract non-instantaneous impulsive differential equation in a Banach space and identifies the parameter using an overdetermined condition on a mild solution. A direct approach using Volterra integral equations for sufficiently regular data and an optimal control approach for less regular data are the main techniques to find the result. Under certain hypotheses, the characterization of the limit of the sequence of approximate solutions demonstrates that it is a solution to the original inverse problem. At last, an example is provided in the support of our results.

1. Introduction

In a dynamical mathematical model describing a natural process, the problem of parameter identification using the given set of observations is of particular interest in an extensive range of sciences, biology, medicine, engineering, economy and environmental sciences. Its mathematical approach may be challenging and the problem is not one of a particular estimation technique. Of course, the ideal solution is to obtain an identification formula for the parameter, but generally, this can be done under some strong hypotheses. These types of problems are called inverse problems. Because they provide information about parameters that are difficult to study directly, inverse problems are among the most important mathematical problems in science and mathematics. Communication theory, system identification, acoustics, optics, astronomy, radar, medical imaging, remote sensing, computer vision [1, 2], signal processing, oceanography, natural language processing, machine learning [3], geophysics, nondestructive testing and many more fields all use them. See [4–7] for the core theory of inverse problems.

Many evolutionary systems, such as shocks, harvesting, and natural disasters, are prone to rapid changes in their dynamics. These phenomena are caused by short-term deviations from continuous and smooth dynamics, insignificant to the evolution's lifetime. Non-instantaneous impulses are defined as abrupt time changes that remain over time intervals. The theory of non-instantaneous impulsive differential equations was firstly introduced Hernadez & O'Regan [8]. The study of non-instantaneous impulsive differential equations is crucial because it has many applications, including the theory of stage-by-stage rocket combustion, preserving hemodynamical equilibrium, etc. The entry of insulin into the bloodstream, an abrupt change and the subsequent absorption, which is a slow process since it remains active for a certain period of time, is a well-known use of noninstantaneous impulses. The theory of impulsive differential equations has found a wide range of practical applications in realistic

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mathematical modeling. Modeling of impulsive problems in population dynamics, physics, biological systems, ecology, biotechnology, and other fields has benefited. For more detail, one can see [9–15].

An area of mathematics involved discovering control for a dynamical system over a period of time to optimize an objective function [16]. The optimal control approach is essential in a wide range of mathematical applications. In a time-optimal control problem, we investigate a control function that steers or transfers the position from a primary state to a specific destination state in the shortest time achievable. Optimal control is applied in various fields, including science, operations research, and engineering. Electric bulk power systems, aeronautical engineering, chemical process control, crystal growth, quantum systems theory, reactor control and vascular surgery are among the specialties studied. [17–22] have more recent works on optimal control.

Abstract differential equations are an important branch of mathematics that deals with the study of differential equations in the abstract space. These equations play a critical role in many areas of science and engineering, including physics, economics, biology, and control theory, to name a few. Abstract differential equations are an essential tool for modeling complex systems, unifying different fields of study, developing control strategies, and developing numerical methods for solving differential equations. For more detail, we refer [23, 24] and refrences therein.

A large class of identification problems for linear evolution equations of first-order in the hyperbolic and parabolic cases and of second-order has been studied in papersby Angelo Favini and co-authors [25, 26]. Recently, a particular attention has been given to impulsive differential equations because differential equations with noninstantaneous impulsive moments are the most effective at describing the evolutionary process of several mathematical models whose motions depend on rapid changes in their states (noninstantaneous impulsive differential equations). Fadi Awawdeh has solved an abstract second order inverse problem [27], in which he has used the Perturbation method to find out the solution. Next, V. Barbu and G. Marinoschi [28] have discussed the possibility of solving the first-order linear inverse problem in two situations, one is when data is regular, and the other is for irregular data. S. Ruhil and M. Malik [30] have solved the inverse problem for the Atangana-Baleanu fractional differential equation by two approaches based on the regularity and irregularity of data.

As far as we know, the inverse problem for abstract differential equations with impulsive conditions has yet to be solved, and no solutions have been reported in the existing literature. This indicates the need for further research and exploration of new methodologies to address this challenging problem. Solving the inverse problem with non-instantaneous impulsive conditions is crucial for enabling accurate modeling, prediction, control, and diagnosis of complex systems in a wide range of applications. Therefore, motivated by the preceding information and to fill this gap, in this paper, we shall discuss two approaches to solve the inverse problem for the first order abstract noninstantaneous impulsive differential equation in a Banach space X. The First one is a direct approach in which we will use the $\mathscr{C}_0$ semigroup theory and Volterra integral equations of the second kind. Another is an optimal control approach in which the characterization of the limit of the sequence of approximate solutions demonstrates that it is a solution to the original inverse problem under some specific hypotheses.

Let $A : D(A) \subset X \to X$ be a densely defined linear operator with a nonempty resolvent set $\rho(A)$ and generates a $\mathscr{C}_0$-semigroup on a Banach space X, $u : [0,b] \to \mathbb{R}$ be a function and φ be assigned as piecewise continuous linear functional on X. Let $0 = \xi_0 < \theta_1 \leq \xi_1 < \theta_2 \leq \xi_2 < \cdots \leq \xi_{m-1} < \theta_m \leq \xi_m < \theta_{m+1} = b < \infty$, and $I_0 = [0, \theta_1], J_k = (\theta_k, \xi_k], I_k = (\xi_k, \theta_{k+1}]; k = 1, 2, ..., m$. We pose the inverse problem of finding a pair (u,v), where u is control and v is solution for following impulsive problem

$$\begin{cases} v'(\theta) = Av(\theta) + u(\theta)\zeta(\theta);\ \theta \in I_k,\ k = 0,1,2,...,m, \\ v(\theta) = g_k(\theta, v(\theta_k^-));\ \theta \in J_k,\ k = 1,2,...,m, \\ v(0) = v_0 \end{cases} \tag{1.1}$$

with overdetermined condition

$$\varphi(v(\theta)) = \mathfrak{C}(\theta) = \begin{cases} h_k(\theta);\ \theta \in I_k,\ k = 0,1,2....m, \\ 0;\ \theta \in J_k,\ k = 1,2......m, \end{cases} \tag{1.2}$$

where $v(\theta) \in X$ is the state variable, $\zeta \in C(\bigcup_{k=0}^{m} I_k; X)$, $g_k : J_k \times X \to X$; $k = 1,2,...m$ are given continuous functions and satisfy Lipschitz condition in second variable on $J_k \times X$ for $k = 1,2,...m$ and $h_k \in C(I_k;\mathbb{R})$; $k = 0,1,2,.....m$. The general assumptions, besides the fact that A is a generator of a $\mathscr{C}_0$-semigroup on the Banach space X, are:

(i) $v_0 \in X$,
(ii) $\zeta \in L^2(0,b;X)$,
(iii) $\mathfrak{C} \in L^2(0,b;\mathbb{R})$.

This paper addresses the reconstruction of u and consequently of v, as a solution to the inverse problem (1.1)-(1.2), using the observation (1.2). Compared with the aforementioned work, the novelties of this paper are listed as below:

- To the best of the author's knowledge, this is the first attempt to deal with the inverse problem for abstract differential equation with non-instantaneous impulsive conditions.
- Also, for the first time, an inverse problem for abstract differential equation with impulsive conditions has been solved using the optimal control approach.
- We examine the unique solution of considered inverse problem by using the $\mathscr{C}_0$ semigroup theory, optimal control theory and Volterra integral equation of second kind.

The rest of this paper is organized as follows. In Section 2, we give some important definitions and lemma related to our manuscript. In Section 3, the first result is to solve this problem in an exact form using Volterra integral equations of the second kind, under stronger conditions, is discussed. For more relaxed hypotheses, including also $\varphi(\zeta(\theta)) = 0$ on an interval $[0,b]$, we shall introduce an optimal control approach providing an sequence of the approximating solutions to (1.1)-(1.2). The sequence of these solutions will tend to the solution to (1.1)-(1.2) if this exists in this case. These will be detailed in Section 4. Finally, in Section 5, we have given an example of applying the inverse problem for the first order abstract noninstantaneous impulsive differential equation, to demonstrate the validity and accuracy of our results.

2. Preliminaries

In this section, we briefly describe some notations, fundamental definitions and important lemma which are useful to prove the main results. $C([0,b];X)$ denote the set of all continuous X valued functions on $[0,b]$ and $\mathscr{L}([0,b];X)$ denote the set of all Lebesgue X valued integrable functions on $[0,b]$. Also $\mathscr{PC}([0,b];X)$ is set of all piecewise continuous functions from $[0,b]$ to X which is defined as $\mathscr{PC}([0,b];X) = \{v : [0,b] \to X; v \in C([0,\theta_1] \cup (\theta_k,\xi_k] \cup (\xi_k,\theta_{k+1}];X) \ \exists\, v(\theta_k^-), v(\theta_k^+)\ v(\xi_k^-)$ and $v(\xi_k^+)$ with $v(\theta_k^-) = v(\theta_k)$ and $v(\xi_k^-) = v(\xi_k); k = 1,2,...,m\}$. Here $v(\theta_k^+), v(\xi_k^+)$ denotes the right hand limit of the function v at $\theta_k,\ \xi_k$, respectively, and $v(\theta_k^-),\ v(\xi_k^-)$ denotes the left hand limit of the function v at $\theta_k,\ \xi_k$, respectively. The space $\mathscr{PC}([0,b];X)$ forms a Banach space with the standard sup norm. Also, $\mathscr{PC}^1([0,b];X)$ is the set of all piecewise continuously differentiable functions $[0,b]$ to X which is defined as $\mathscr{PC}^1([0,b];X) = \{v : [0,b] \to X; v \in C^1([0,\theta_1] \cup (\theta_k,\xi_k] \cup (\xi_k,\theta_{k+1}];X) \ \exists\, v(\theta_k^-), v(\theta_k^+) v(\xi_k^-), v(\xi_k^+), v'(\xi_k^+), v'(\xi_k^-), v'(\theta_k^+)$and $v'(\theta_k^-)$ with $v(\theta_k^-) = v(\theta_k)$, $v'(\theta_k^-) = v'(\theta_k), v(\xi_k^-) = v(\xi_k)$ and$v'(\xi_k^-) = v'(\xi_k); k = 1,2,...,m\}$.

Definition 2.1. [29] Let A is the infinitesimal generator of a C_0-semigroup $\{e^{A\theta}\}_{\theta\geq 0}$, $v_0 \in X$, $\zeta \in C\left(\bigcup_{k=0}^m I_k;X\right)$, and $g_k \in C(J_k \times X;X)$, $k = 1,2,...,m$ and g_k satisfies Lipschitz condition in the second variable on $(J_k \times X)$ for $k = 1,2,\cdots m$. Then, the piecewise continuous function $v \in \mathscr{PC}([0,b];X)$ given by

$$v(\theta) = \begin{cases} e^{A\theta}v_0 + \int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)u(\xi)d\xi;\ \theta \in I_0, \\ g_k(\theta, v(\theta_k^-));\ \theta \in J_k,\ k = 1,2,...,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k, v(\xi_k^-)) + \int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)u(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,...,m \end{cases} \tag{2.1}$$

is called the mild solution of (1.1) on $[0,b]$.

Definition 2.2. Let $u \in \mathscr{PC}([0,b];\mathbb{R})$, $\zeta \in C^1\left(\bigcup_{k=0}^m I_k;X\right)$ and $g_k \in C^1(J_k \times X;X)$. A strong solution to (1.1) is a function $v \in \mathscr{PC}^1\left([0,b];X\right) \cap \mathscr{PC}\left([0,b];D(A)\right)$ which satisfies (1.1) almost everywhere, for all $\theta \in [0,b]$.

Let us define an admissible control set $\mathfrak{U}_{\mathfrak{a}} \subset L^2[0,b]$ which is nonempty, bounded, closed and convex:

$$\mathfrak{U}_{\mathfrak{a}} = \left\{u \in L^2[0,b] : u(\theta) \in [c_k,d_k] \subset \mathbb{R},\ a.e.\ \theta \in I_k;\ k = 0,1,2,\cdots m\right\}.$$

3. Direct approach

In this part, we use the Volterra integral equations of the second kind to discover the unique strong solution to the inverse problem (1.1)-(1.2).

Theorem 3.1. Let us assume that

$$v_0 \in D(A), \zeta \in C^1\Big(\bigcup_{k=0}^m I_k;X\Big),\ g_k \in C^1(J_k \times X;X);\ k = 1,2,\cdots m, \tag{3.1}$$

$$\varphi(\zeta(\theta)) \neq 0 \text{ for all } \theta \in [0,b], \tag{3.2}$$

$$\mathfrak{C}(\theta) \in \mathscr{PC}^1\left([0,b];\mathbb{R}\right) \text{and } \varphi\left(g_k(\theta, v(\theta_k^-))\right) = 0. \tag{3.3}$$

Then, the unique solution of problem (1.1)-(1.2) is given by

$$(u,v) \in \mathscr{PC}([0,b];\mathbb{R}) \times \mathscr{PC}^1\left([0,b];X\right) \cap \mathscr{PC}\left([0,b];D(A)\right).$$

Proof. Since A is a generator of a $\mathscr{C}_0$-semigroup on X, so by using corollary 2.2 (on page no. 114) and corollary 2.5 (on page no. 115) of [29], the problem (1.1) has a unique mild solution $v \in \mathscr{PC}([0,b];X)$ given by

$$v(\theta) = \begin{cases} e^{A\theta}v_0 + \int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)u(\xi)d\xi;\ \theta \in I_0, \\ g_k(\theta, v(\theta_k^-));\ \theta \in J_k,\ k = 1,2,...,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k, v(\xi_k^-)) + \int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)u(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,...,m. \end{cases}$$

By (3.1) (using corollary 2.10 on page no. 117 of [29]) this turns out to be a strong solution. Now, applying φ on both side we have

$$\varphi(v(\theta)) = \begin{cases} \varphi(e^{A\theta}v_0) + \int_0^\theta \varphi\left(e^{A(\theta-\xi)}\zeta(\xi)u(\xi)\right)d\xi;\ \theta \in I_0, \\ \varphi\left(g_k(\theta, v(\theta_k^-))\right);\ \theta \in J_k,\ k = 1,2,...,m, \\ \varphi\left(e^{A(\theta-\xi_k)}g_k(\xi_k, v(\xi_k^-))\right) + \int_{\xi_k}^\theta \varphi\left(e^{A(\theta-\xi)}\zeta(\xi)u(\xi)\right)d\xi;\ \theta \in I_k,\ k = 1,2,...,m. \end{cases}$$

Now, we know that $\varphi\left(g_k(\theta, v(\theta_k^-))\right) = 0$ for $\theta \in J_k; k = 1,2,....m$. Hence,

$$\varphi(v(\theta)) = \begin{cases} \varphi(e^{A\theta}v_0) + \int_0^\theta \varphi\left(e^{A(\theta-\xi)}\zeta(\xi)\right)u(\xi)d\xi;\ \theta \in I_0, \\ 0;\ \theta \in J_k,\ k = 1,2,...,m; \\ \varphi\left(e^{A(\theta-\xi_k)}g_k(\xi_k, v(\xi_k^-))\right) + \int_{\xi_k}^\theta \varphi\left(e^{A(\theta-\xi)}\zeta(\xi)\right)u(\xi)d\xi; \\ \qquad\qquad\qquad\qquad \theta \in I_k,\ k = 1,2,...,m. \end{cases} \tag{3.4}$$

Denoting

$$G(\theta,\xi) = \varphi\left(e^{A(\theta-\xi)}\zeta(\xi)\right)\ 0 \le \xi \le \theta \le \theta_k;\ k = 0,1,2,...m$$

and

$$\Gamma u(\theta) = \begin{cases} \int_0^\theta G(\theta,\xi)u(\xi)d\xi;\ \theta \in I_0, \\ 0;\ \theta \in J_k, k = 1,2,3,....m, \\ \int_{\xi_k}^\theta G(\theta,\xi)u(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,3,.....m. \end{cases}$$

We note that $(\theta,\xi) \to G(\theta,\xi)$ is differentiable on $(\bigcup_{k=0}^m I_k) \times (\bigcup_{k=0}^m I_k)$. Now, recalling (1.2), we can write (3.4) as

$$\Gamma u(\theta) = \begin{cases} h_0(\theta) - \varphi\left(e^{A\theta}v_0\right);\ \theta \in I_0, \\ 0;\ \theta \in J_k,\ k = 1,2,3,.....m, \\ h_k(\theta) - \varphi\left(e^{A(\theta-\xi_k)}g_k\left(\xi_k, v(\xi_k^-)\right)\right);\ \theta \in I_k,\ k = 1,2,3.....m. \end{cases} \tag{3.5}$$

We note that R.H.S. of (3.5) is differentiable on $\bigcup_{k=0}^{m} I_k$. These are the integral equations of the first kind which can be reduced to a linear Volterra equations of second kind by a standard way. Namely, by differentiating (3.5) with respect to θ we obtain

$$\begin{cases} G(\theta,\theta)u(\theta)+\int_0^\theta G_\theta(\theta,\xi)u(\xi)d\xi+\varphi\left(e^{A\theta}Av_0\right)=h_0'(\theta);\ \theta\in I_0,\\ G(\theta,\theta)u(\theta)+\int_{\xi_k}^\theta G_\theta(\theta,\xi)u(\xi)d\xi+\varphi\left(e^{A(\theta-\xi_k)}Ag_k(\xi_k,v(\xi_k^-))\right)=h_k'(\theta);\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta\in I_k,\ k=1,2,...m,\\ 0=0;\ \theta\in J_k,\ k=1,2,.....m. \end{cases}$$

Next writing

$$f(\theta)=\begin{cases} \left[h_0'(\theta)-\varphi\left(e^{A\theta}Av_0\right)\right](G(\theta,\theta))^{-1};\ \theta\in I_0,\\ \left[h_k'(\theta)-\varphi\left(e^{A(\theta-\xi_k)}Ag_k(\xi_k,v(\xi_k)^-)\right)\right](G(\theta,\theta))^{-1};\ \theta\in I_k,\ k=1,2,\cdots m,\\ 0;\ \theta\in J_k,\ k=1,2,3.....m, \end{cases}$$

and observing that $G(\theta,\theta)=\varphi(\zeta(\theta))\neq 0$ for all $\theta\in[0,b]$, we get

$$u(\theta)=\begin{cases} \int_0^\theta K(\theta,\xi)u(\xi)d\xi+f(\theta);\ \theta\in I_0,\\ \int_{\xi_k}^\theta K(\theta,\xi)u(\xi)d\xi+f(\theta);\ \theta\in I_k,\ k=1,2,\cdots m,\\ 0;\ \theta\in J_k,\ k=1,2,....m, \end{cases}\tag{3.6}$$

where

$$K(\theta,\xi)=\frac{G_\theta(\theta,\xi)}{G(\theta,\theta)},\ G_\theta(\theta,\xi)=\varphi\left(e^{A(\theta-\xi)}A\zeta(\xi)\right),\ 0\leq\xi\leq\theta\leq\theta_k;\ k=0,1,2,....m.$$

For piecewise continuity of u, we should choose $u(\theta)=0$ for all $\theta=\theta_k$ and $\theta=\xi_k; k=1,2,3...m$. Indeed, by (3.6) we have

$$\begin{cases} u(0)=0=f(0)\\ u(\xi_k)=0=f(\xi_k);\ k=1,2,3,....m. \end{cases}$$

It implies that

$$\begin{cases} u(0)=\left[h_0'(0)-\varphi\left(Av_0\right)\right](\varphi(\zeta(0)))^{-1};\\ u(\xi_k)=\left[h_k'(\xi_k)-\varphi\left(Ag_k(\xi_k,v(\xi_k)^-)\right)\right](\varphi(\zeta(\xi_k)))^{-1};\ k=1,2,....m. \end{cases}$$

Hence, we have

$$\begin{cases} h_0'(0) = \varphi(A v_0) \\ h_k'(\xi_k) = \varphi\left(A g_k(\xi_k, v(\xi_k^-))\right);\ k = 1,2,\ldots.m. \end{cases}$$

Under our hypothesis $f \in \mathscr{PC}([0,b];\mathbb{R})$ and so the system of volterra integral equations of second kind (3.6) has a unique solution $u \in \mathscr{PC}([0,b];\mathbb{R})$, given by

$$u(\theta) = \begin{cases} f(\theta) + \int_0^\theta \mathscr{K}(\theta,\xi)u(\xi)d\xi;\ \theta \in I_0, \\ 0;\ \theta \in J_k,\ k = 1,2,3,\ldots.m, \\ f(\theta) + \int_{\xi_k}^\theta \mathscr{K}(\theta,\xi)u(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,3\ldots.m \end{cases} \tag{3.7}$$

with resolvent kernel

$$\mathscr{K}(\theta,\xi) = \sum_{j=0}^{\infty} K_j(\theta,\xi) \text{ for all } \xi,\theta \in \bigcup_{k=0}^{m} I_k,$$

where

$$K_n(\theta,\xi) = \int_\xi^\theta K(\theta,\tau)K_{n-1}(\tau,\xi)d\tau,\ K_0(\theta,\xi) = K(\theta,\xi).$$

As also known, $u(\theta)$ can be iteratively obtained as

$$u_{n+1}(\theta) = \begin{cases} \int_0^\theta K(\theta,\xi)u_n(\xi)d\xi;\ \theta \in I_0, \\ 0;\ J_k,\ k = 1,2,3,\ldots\ldots m, \\ \int_{\xi_k}^\theta K(\theta,\xi)u_n(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,3\ldots.m \end{cases} \tag{3.8}$$

and

$$u_0(\theta) = u(0).$$

The sequence $u_n(\theta)_{n\geq 1}$ converges strongly to $u(\theta)$, the solution to (3.6), as $n \to \infty$. By (2.1) it follows that corrsponding solution $v(\theta)$ is unique strong solution (u,v), as claimed. □

4. An Optimal Control Approach

An alternative for solving problem inverse problem is to use an optimal control approach by considering a minimization problem. Let v be a solution of the system (1.1) corresponding to the control $u \in \mathfrak{U}_a$. We assume that $\mathfrak{U}_a \neq 0$. Now, let us consider the optimal control problem as follows:

Problem 4.1. Find a piecewise continuous function $\hat{u} \in \mathscr{PC}([0,b];\mathbb{R})$ such that

$$\mathfrak{J}(\hat{u}) = \min_{u\in\mathscr{PC}([0,b];\mathbb{R})}\left[\mathfrak{J}(u) = \begin{cases} \frac{1}{2}\int_0^{\theta_1}[\varphi(v(\theta)) - h_0(\theta)]^2 d\theta;\ \theta \in I_0, \\ \frac{1}{2}\int_{\theta_k}^{\xi_k}[\varphi(v(\theta))]^2 d\theta;\ \theta \in J_k,\ k = 1,2,....m, \\ \frac{1}{2}\int_{\xi_k}^{\theta_{k+1}}[\varphi(v(\theta)) - h_k(\theta)]^2 d\theta;\ \theta \in I_k, \\ \qquad\qquad k = 1,2,....m \end{cases}\right] \tag{4.1}$$

for all (u,v) satisfying system (1.1).

Definition 4.1. If $\hat{u}$ is the optimal control for the problem (4.1) then a pair $(\hat{u},\hat{v})$ is called optimal pair for equation (4.1), where $\hat{v}$ is the mild solution of the system (1.1) corresponding to $\hat{u}$.

It is obvious that if system (1.1) has a unique solution, this turns out to be the unique solution to the problem (4.1), but the converse assertion is not generally true. It is unclear if the problem (4.1) may have a solution, especially if we do not assume that $\varphi(\zeta(\theta)) \neq 0$ for all $\theta \in [0,b]$. To find the optimal pair for the problem (4.1), we use an approximating control problem which provides an approximating solution $(\hat{u_\lambda},\hat{v}_\lambda)$ and to check if this could tend to the solution to the problem (4.1) if the latter has one. In this approximating problem we shall require less regularity for the data than in the first case of section (3). The hypotheses used in this part are (i), (ii), (iii) and X is a reflexive Banach space. We stress that we do not require that $\varphi(\zeta(\theta))$ is nonzero.
Next, our approximating minimization problem can be stated as follows

Problem 4.2. Let λ be any non negative real number then find $\hat{u}_\lambda \in \mathfrak{U}_\mathfrak{a}$ such that

$$\mathfrak{J}(\hat{u}_\lambda) = \min_{u\in\mathfrak{U}_\mathfrak{a}}\left[\mathfrak{J}_\lambda(u) = \begin{cases} \frac{1}{2}\int_0^{\theta_1}[\varphi(v(\theta)) - h_0(\theta)]^2 d\theta + \frac{\lambda}{2}\|u\|^2_{L^2(I_0)};\ \theta \in I_0, \\ \frac{1}{2}\int_{\theta_k}^{\xi_k}[\varphi(v(\theta))]^2 d\theta + \frac{\lambda}{2}\|u\|^2_{L^2(J_k)};\ \theta \in J_k,\ k = 1,2,....m \\ \frac{1}{2}\int_{\xi_k}^{\theta_{k+1}}[\varphi(v(\theta)) - h_k(\theta)]^2 d\theta + \frac{\lambda}{2}\|u\|^2_{L^2(I_k)};\ \theta \in I_k,\ k = 1,2,....m \end{cases}\right]$$

subject to system (1.1).

Proposition 4.2. If $\hat{u_\lambda}$ minimizes the functional $\mathfrak{J}_\lambda(u)$ and $\hat{v}_\lambda$ solve the system (1.1), then problem (4.2) has a solution.

Proof. We assume that $\inf\{\mathfrak{J}_\lambda(u);\ u_\lambda \in \mathfrak{U}_\mathfrak{a}\} < \infty$. Now, we can see from the the definition of $\mathfrak{J}_\lambda(u)$ that $\inf\{\mathfrak{J}_\lambda(u);\ u_\lambda \in \mathfrak{U}_\mathfrak{a}\} \geq 0$. We know by definition of infimum that there exist a sequence $\{u_{\lambda_n}\} \subset \mathfrak{U}_\mathfrak{a}$, which will minimize our functional. Suppose v_{λ_n} is the solution of corrsponding to u_{λ_n}, such that

$$\mathfrak{J}_\lambda(u_{\lambda_n}) \to \inf\{\mathfrak{J}_\lambda(u) : u \in \mathfrak{U}_\mathfrak{a}\},\ \text{as } n \to \infty.$$

As $\mathfrak{U}_\mathfrak{a}$ is bounded it is clear that the sequence u_{λ_n} is bounded in $L^2([0,b];\mathbb{R})$. We know that $L^2([0,b];\mathbb{R})$ is reflexive Banach space, then there exist a subsequence, relabeled as u_{λ_n}, and $\hat{u}_\lambda \in L^2([0,b];\mathbb{R})$ such that u_{λ_n} converges weakly to $\hat{u}_\lambda$ in $L^2([0,b];\mathbb{R})$. Since $\mathfrak{U}_\mathfrak{a}$ is closed and convex, $\hat{u}_\lambda \in \mathfrak{U}_\mathfrak{a}$. By remark

(2.6) of [7], we find that v is uniformaly bounded. Next, recall the proof in Lemma (2.11) of [7], we know that v is piecewise equicontinuous.

Since v is uniformaly bounded and piecewise equicontinuous, we have $v_{\lambda_n} \to \hat{v}_\lambda$ in $\mathscr{PC}([0,b];X)$. Then we have $v_{\lambda_n}(\xi_k^-) \to \hat{v}_\lambda(\xi_k^-)$ and $v_{\lambda_n}(\theta_k^-) \to \hat{v}_\lambda(\theta_k^-)$. Next we check that $\hat{v}_\lambda$ is the solution of corrsponding $\hat{u}_\lambda$. Since $e^{A(\theta-\xi)}$ is bounded, we have

$$\int_{\xi_k}^{\theta} e^{A(\theta-\xi)}\zeta(\xi)\left(u_{\lambda_n}(\xi)-\hat{u}_\lambda(\xi)\right)d\xi \to 0, \text{in } \mathscr{PC}([0,b];X)$$

as u_{λ_n} converges weakly to $\hat{u}_\lambda$. Note that

$$v_{\lambda_n}(\theta) = \begin{cases} e^{A\theta}v_0 + \int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)u_{\lambda_n}(\xi)d\xi;\ \theta \in I_0, \\ g_k(\theta, v_{\lambda_n}(\theta_k^-));\ \theta \in J_k,\ k=1,2,\dots,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k, v_{\lambda_n}(\xi_k^-)) + \int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)u_{\lambda_n}(\xi)d\xi;\ \theta \in I_k,\ k=1,2,\dots,m. \end{cases}$$

Thus we obtain

$$\hat{v}_\lambda(\theta) = \begin{cases} e^{A\theta}v_0 + \int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)\hat{u}_\lambda(\xi)d\xi;\ \theta \in I_0, \\ g_k(\theta, \hat{v}_\lambda(\theta_k^-));\ \theta \in J_k,\ k=1,2,\dots,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k, \hat{v}_\lambda(\xi_k^-)) + \int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)\hat{u}_\lambda(\xi)d\xi;\ \theta \in I_k,\ k=1,2,\dots,m. \end{cases}$$

Thus, $\hat{v}_\lambda$ follows to be a solution of (1.1). Next, since φ be a piecewise continuous functional on X and $h_k \in C(\bigcup_{k=0}^m I_k;\mathbb{R})$. Hence, we have

$$\varphi(v_{\lambda_n}(\theta)) \to \varphi(\hat{v}_\lambda(\theta)), \text{ for all } \theta \in [0,b].$$

Thus $(\hat{u}_\lambda, \hat{v}_\lambda)$ is solution to the problem (1.1)-(1.2). Hence,

$$\mathfrak{J}_\lambda(u_{\lambda_n}) \to \inf\{\mathfrak{J}_\lambda(u) : u \in \mathfrak{U}_\mathfrak{a}\} \text{ as } n \to \infty.$$

So, probelm (4.2) has a solution. □

Let us denote by $\langle .,. \rangle$ the pairing between the set of all piecewise continuous linear functionals $\mathscr{PCL}(X;\mathbb{R})$ and X. And $P_{[c,d]}(f)$ the projection of f on $[c,d]$.

Proposition 4.3. Let $(\hat{u}_\lambda, \hat{v}_\lambda)$ be the solution to problem (4.2) and for $\sigma > 0$, let us denote

$$u_\lambda^\sigma = \hat{u}_\lambda + \sigma w, \text{ with } w = v - \hat{u}_\lambda \text{ and } v \in \mathfrak{U}_\mathfrak{a}.$$

Also, assume that $\lim_{\sigma\to 0}\frac{1}{\sigma}\left[g_k(\theta, v^{u^\sigma}(\theta_k^-)) - g_k(\theta, v^{\hat{u}}(\theta_k^-))\right] = 0$. Then the first order necessary conditions of optimality are

$$\hat{u}_\lambda(\theta) = \begin{cases} P_{[c_0,d_0]}\left(\frac{1}{\lambda}\langle p_\lambda(\theta), \zeta(\theta)\rangle\right);\ \text{a.e. } \theta \in I_0, \\ 0;\ \text{a.e. } \theta \in J_k,\ k=1,2\dots.m, \\ P_{[c_k,d_k]}\left(\frac{1}{\lambda}\langle p_\lambda(\theta), \zeta(\theta)\rangle\right);\ \text{a.e } \theta \in I_k,\ k=1,2,\dots..m, \end{cases}$$

where p_λ is the solution to the problem

$$-\frac{\mathrm{d}p_\lambda}{\mathrm{d}\theta}=\begin{cases}A^*p_\lambda(\theta)-\big(\varphi(v_\lambda^*(\theta))-h_0(\theta)\big)\,\varphi;\ \theta\in I_0,\\ 0;\ \theta\in J_k,\ k=1,2,3.....m,\\ A^*p_\lambda(\theta)-\big(\varphi(v_\lambda^*(\theta))-h_k(\theta)\big)\,\varphi;\ \theta\in I_k,\ k=1,2,\cdots m\end{cases}\tag{4.2}$$

with

$$P_\lambda(\theta_{k+1})=0;\ k=0,1,2,\cdots m$$

and A^* is the adjoint of A.

Proof. Let $(\hat{u}_\lambda,\hat{v}_\lambda)$ be the solution to problem (4.2) and for $\sigma>0$. Let us denote

$$u_\lambda^\sigma=\hat{u}_\lambda+\sigma w,\ \text{with } w=v-\hat{u}_\lambda \text{ and } v\in\mathfrak{U}_\mathfrak{a}.$$

Now, we consider an equation in variable Θ as follows:

$$\begin{cases}\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}=A\Theta(\theta)+w(\theta)\zeta(\theta);\ \theta\in I_k,\ k=0,1,2.....m,\\ \Theta(\theta)=0;\ \theta\in J_k,\ k=1,2,......m,\\ \Theta(0)=0\end{cases}\tag{4.3}$$

which has a unique mild solution $\Theta\in\mathscr{PC}([0,b];X)$. It is easily seen by (1.1) that the function

$$\Theta=\lim_{\sigma\to 0}\frac{v^{u^\sigma}-v^{\hat{u}}}{\sigma}$$

is the unique mild solution to the problem, where v^{u^σ} and $v^{\hat{u}}$ are the solution to (1.1) corrsponding to u^σ and $\hat{u}$, respectively. To prove it we take

$$\Theta=\begin{cases}\lim_{\sigma\to 0}\frac{1}{\sigma}\left[\int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)\left(u^\sigma(\xi)-\hat{u}(\xi)\right)d\xi\right];\ \theta\in I_0,\\ \lim_{\sigma\to 0}\frac{1}{\sigma}\left[g_k(\theta,v^{u^\sigma}(\theta_k^-))-g_k(\theta,v^{\hat{u}}(\theta_k^-))\right];\ \theta\in J_k,\ k=1,2,3,....m,\\ \lim_{\sigma\to 0}\frac{1}{\sigma}\left[\int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)\left(u^\sigma(\xi)-\hat{u}(\xi)\right)d\xi\right];\ \theta\in I_k,\ k=1,2,3,.....m.\end{cases}$$

By using $u_\lambda^\sigma=\hat{u}_\lambda+\sigma w$, we get

$$\Theta=\begin{cases}\lim_{\sigma\to 0}\frac{1}{\sigma}\left[\int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)\left(\sigma w(\xi)\right)d\xi\right];\ \theta\in I_0,\\ 0;\ \theta\in J_k,\ k=1,2,3,....m,\\ \lim_{\sigma\to 0}\frac{1}{\sigma}\left[\int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)\left(\sigma w(\xi)\right)d\xi\right];\ \theta\in I_k,\ k=1,2,3,.....m.\end{cases}$$

It implies

$$\Theta = \begin{cases} e^{A\theta}\Theta(0) + \int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)w(\xi)d\xi;\ \theta \in I_0, \\ 0;\ \theta \in J_k,\ k = 1,2,3,\ldots.m, \\ e^{A\theta}\Theta(0) + \int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)w(\xi)d\xi;\ \theta \in I_k,\ k = 1,2,3,\ldots..m. \end{cases}$$

Hence, it has been proved that

$$\Theta = \lim_{\sigma\to 0}\frac{v^{u^\sigma} - v^{\hat{u}}}{\sigma} \text{ strongly in } \mathscr{PC}([0,b];X)$$

is mild solution of the system (4.3). Now, since $(\hat{u}_\lambda, \hat{v}_\lambda)$ is optimal in problem (4.2), it satisfies

$$\mathfrak{J}_\lambda(\hat{u}_\lambda) \leq \mathfrak{J}_\lambda(u_\lambda^\sigma),$$

when by performing a short calculation, dividing by λ and passing to the limit as $\lambda \to 0$, we get the optimality relation

$$\begin{cases} \int_0^{\theta_1}(\varphi(\hat{v}_\lambda) - h_0(\theta))\,\varphi(\Theta(\theta))d\theta + \lambda\int_0^{\theta_1}\hat{u}_\lambda(\theta)w(\theta)d\theta \geq 0;\ \theta \in I_0, \\ \lambda\int_{\theta_k}^{\xi_k}\hat{u}_\lambda(\theta)w(\theta) \geq 0;\ \theta \in J_k,\ k = 1,2,\ldots..m, \\ \int_{\xi_k}^{\theta_{k+1}}(\varphi(\hat{v}_\lambda) - h_k(\theta))\,\varphi(\Theta(\theta))d\theta + \lambda\int_{\xi_k}^{\theta_{k+1}}\hat{u}_\lambda(\theta)w(\theta)d\theta \geq 0; \\ \qquad\qquad \theta \in I_k,\ k = 1,2,3\ldots..m. \end{cases} \tag{4.4}$$

We recall that, since X^* (dual space of X) is reflexive, A^* is generating a $\mathscr{C}_0$ semigroup on X^* (see [29], Pg. 41). Then, since φ be linear piecewise continuous functional on X, equation (4.2) has a unique mild solution $p_\lambda \in \mathscr{PC}([0,b];\mathscr{PCL}(X;\mathbb{R}))$ defined as

$$p_\lambda(\theta) = \begin{cases} \int_\theta^{\theta_1} e^{A^*(\xi-\theta)}(\varphi(\hat{v}_\lambda(\xi)) - h_0(\xi))\,\varphi\,d\xi;\ \theta \in I_0, \\ 0;\ \theta \in J_k,\ \theta \in J_k,\ k = 1,2,\ldots..m, \\ \int_\theta^{\theta_{k+1}} e^{A^*(\xi-\theta)}(\varphi(\hat{v}_\lambda(\xi)) - h_k(\xi))\,\varphi\,d\xi;\ \theta \in I_k,\ k = 1,2,\cdots m. \end{cases}$$

Now, if p_λ and Θ would be strong solutions, we could multiply (4.3) by p_λ and integrate it as

$$\begin{cases} \int_0^{\theta_1}\left\langle p_\lambda(\theta), \frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}t}\right\rangle d\theta = \int_0^{\theta_1}\langle p_\lambda(\theta), A\Theta(\theta)\rangle d\theta + \int_0^{\theta_1}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle d\theta;\ \theta \in I_0, \\ \int_{\theta_k}^{\xi_k}\langle p_\lambda(\theta), \Theta(\theta)\rangle d\theta = 0;\ \theta \in J_k,\ k = 1,2,3\ldots..m, \\ \int_{\xi_k}^{\theta_{k+1}}\left\langle p_\lambda(\theta), \frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}t}\right\rangle d\theta = \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), A\Theta(\theta)\rangle d\theta + \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle d\theta; \\ \qquad\qquad \theta \in I_k,\ k = 1,2,\cdots m. \end{cases}$$

Now, using integartion by parts formula we get

$$\begin{cases} \langle p_\lambda(\theta_1),\Theta(\theta_1)\rangle - \langle p_\lambda(0),\Theta(0)\rangle - \int_0^{\theta_1}\langle (p_\lambda)_\theta(\theta),\Theta(\theta)\rangle\, d\theta \\ \qquad\qquad = \int_0^{\theta_1}\langle p_\lambda(\theta), A\Theta(\theta)\rangle\, d\theta + \int_0^{\theta_1}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle\, d\theta;\ \theta\in I_0, \\ 0=0; \qquad\qquad \theta\in J_k,\ k=1,2,\cdots m, \\ \langle p_\lambda(\theta_{k+1}),\Theta(\theta_{k+1})\rangle - \langle p_\lambda(\xi_k),\Theta(\xi_k)\rangle - \int_{\xi_k}^{\theta_{k+1}}\langle (p_\lambda)_\theta(\theta),\Theta(\theta)\rangle\, d\theta \\ \qquad\qquad = \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), A\Theta(\theta)\rangle\, d\theta + \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle\, d\theta; \\ \qquad\qquad\qquad \theta\in I_k,\ k=1,2,\cdots m. \end{cases}$$

Hence, in our case the data is not regular, the same result of integration by parts can actually follow by passing to the limit in regularizing problems for θ and p_λ, corrsponding to regular data.

Next, by some calculations taking into account the initial and final conditions in (4.2) and (4.3) we get

$$\begin{cases} -\int_0^{\theta_1}\langle (p_\lambda)_\theta(\theta),\Theta(\theta)\rangle\, d\theta = \int_0^{\theta_1}\langle p_\lambda(\theta),\Theta(\theta)\rangle\, d\theta + \int_0^{\theta_1}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle\, d\theta;\ \theta\in I_0, \\ 0=0; \qquad\qquad \theta\in J_k,\ k=1,2,\cdots m, \\ -\int_{\xi_k}^{\theta_{k+1}}\langle (p_\lambda)_\theta(\theta),\Theta(\theta)\rangle\, d\theta = \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), A\Theta(\theta)\rangle\, d\theta + \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta), w(\theta)\zeta(\theta)\rangle\, d\theta; \\ \qquad\qquad\qquad \theta\in I_k,\ k=1,2,\cdots m. \end{cases}$$

It implies

$$\begin{cases} \int_0^{\theta_1}\langle -(p_\lambda)_\theta(\theta) - A^* p_\lambda(\theta),\Theta(\theta)\rangle\, d\theta = \int_0^{\theta_1}\langle p_\lambda(\theta),\zeta(\theta)\rangle\, w(\theta)d\theta;\ \theta\in I_0, \\ 0=0;\ \theta\in J_k,\ k=1,2,\cdots m, \\ \int_{\xi_k}^{\theta_{k+1}}\langle -(p_\lambda)_\theta(\theta) - A^* p_\lambda(\theta),\Theta(\theta)\rangle\, d\theta = \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta),\zeta(\theta)\rangle\, w(\theta)d\theta;\ \theta\in I_k,\ k=1,2,\cdots m. \end{cases}$$

Using (4.2) we get

$$\begin{cases} -\int_0^{\theta_1}\left(\varphi(v_\lambda^*(\theta)) - h_0(\theta)\right)\varphi(\Theta(\theta))d\theta = \int_0^{\theta_1}\langle p_\lambda(\theta),\zeta(\theta)\rangle\, w(\theta)d\theta;\ \theta\in I_0, \\ 0=0;\ \theta\in J_k,\ k=1,2,\cdots m, \\ -\int_{\xi_k}^{\theta_{k+1}}\left(\varphi(v_\lambda^*(\theta)) - h_0(\theta)\right)\varphi(\Theta(\theta))d\theta = \int_{\xi_k}^{\theta_{k+1}}\langle p_\lambda(\theta),\zeta(\theta)\rangle\, w(\theta)d\theta;\ \theta\in I_k,\ k=1,2,\cdots m. \end{cases}$$

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By comparision with (4.4), we find

$$\begin{cases} \int_0^{\theta_1} \left(-\langle p_\lambda(\theta), \zeta(\theta)\rangle + \lambda \hat{u}_\lambda(\theta)\right) w(\theta) d\theta \geq 0; \ \theta \in I_0, \\ 0 = 0; \ \theta \in J_k, \ k = 1, 2, \cdots m, \\ \int_{\xi_k}^{\theta_{k+1}} \left(-\langle p_\lambda(\theta), \zeta(\theta)\rangle + \lambda \hat{u}_\lambda(\theta)\right) w(\theta) d\theta \geq 0; \ \theta \in I_k, \ k = 1, 2, \cdots m. \end{cases}$$

When taking into account that $w = v - \hat{u}_\lambda$, we get

$$\begin{cases} \int_0^{\theta_1} \left(-\langle p_\lambda(\theta), \zeta(\theta)\rangle + \lambda \hat{u}_\lambda(\theta)\right) (\hat{u}_\lambda(\theta) - v(\theta)) d\theta \geq 0; \ \theta \in I_0, \\ 0 = 0; \ \theta \in J_k, \ k = 1, 2, \cdots m, \\ \int_{\xi_k}^{\theta_{k+1}} \left(-\langle p_\lambda(\theta), \zeta(\theta)\rangle + \lambda \hat{u}_\lambda(\theta)\right) (\hat{u}_\lambda(\theta) - v(\theta)) d\theta \geq 0; \ \theta \in I_k, \ k = 1, 2, \cdots m. \end{cases}$$

Thus we deduce that

$$(\langle p_\lambda(\theta), \zeta(\theta)\rangle - \lambda \hat{u}_\lambda(\theta)) \in N_{[c_k,d_k]}(\hat{u}_\lambda(\theta)) = \partial S_{[c_k,d_k]}(\hat{u}_\lambda(\theta)), \text{ a.e. for } \theta \in I_k; \ k = 0, 1, 2, ...m,$$

where $N_{[c_k,d_k]}(\hat{u}_\lambda(\theta))$ are normal cones to $[c_k, d_k]$ at $\hat{u}_\lambda(\theta)$ (for $k = 0, 1, 2, ...m$) and $\partial S_{[c_k,d_k]} : \mathbb{R} \to 2^{\mathbb{R}}$ is the subdifferential of the indicator function of $[c_k, d_k]$ for $k = 0, 1, 2, 3, ...m$ [31]. Then,

$$\frac{1}{\lambda} \langle (p_\lambda)(\theta), \zeta(\theta)\rangle \in \hat{u}_\lambda(\theta) + \frac{1}{\lambda} \partial S_{[c_k,d_k]}(\hat{u}_\lambda(\theta)) \text{ a.e. } \theta \in I_k; \ k = 0, 1, 2, \cdots m.$$

It implies that

$$\hat{u}_\lambda(\theta) = \begin{cases} P_{[c_0,d_0]}\left(\frac{1}{\lambda} \langle p_\lambda(\theta), \zeta(\theta)\rangle\right); \text{ a.e. } \theta \in I_0, \\ 0; \text{ a.e. } \theta \in J_k, \ k = 1, 2....m, \\ P_{[c_k,d_k]}\left(\frac{1}{\lambda} \langle p_\lambda(\theta), \zeta(\theta)\rangle\right); \text{ a.e } \theta \in I_k, \ k = 1, 2, ...m. \end{cases}$$

□

Let

$$Q = \left\{ u \in L^2(0,b); u(\theta) \in [c_k, d_k] \subset I_k; \ k = 0, 1, 2 \cdots m, \text{ a.e., } (u, v) \text{ solves (1.1)- (1.2)} \right\}.$$

We can easily see that Q is a close and convex set. Let us denote $P_Q(0)$ by the projection of 0 on Q.

Theorem 4.4. Let us assume that $(\hat{u}_\lambda, \hat{v}_\lambda)$ be a solution to the problem (4.2) and $\hat{v}_{\lambda_n}$ converges to v^* at the end points ξ_k of the impulsive intervals J_k i.e. $\hat{v}_{\lambda_n}(\xi_k^-) \to v^*(\xi_k^-)$ as $\lambda_n \to 0$, where λ_n is the decreasing sequence of positive real numbers. If Q is non-empty then for $\lambda \to 0$, we have $\hat{u}_\lambda$ and $\hat{v}_\lambda$ converges to u^* and v^* strongly in $L^2(0,b)$ and $\mathscr{PC}([0,b];X)$, respectively, where (u^*, v^*) is a solution to the inverse problem (1.1)- (1.2). Moreover, $u^* = P_Q(0)$.

Proof. Let $(\hat{u}_\lambda, \hat{v}_\lambda)$ be a solution to problem (4.2). By the optimality condition we have

$$\mathfrak{J}_\lambda(\hat{u}_\lambda) \leq \mathfrak{J}_\lambda(u).$$

It implies

$$\begin{cases} \frac{1}{2}\int_0^{\theta_1}[\varphi(\hat{v}_\lambda(\theta))-h_0(\theta)]^2+\frac{\lambda}{2}\|\hat{u}_\lambda\|^2_{L^2(I_0)}d\theta;\ \theta\in I_0, \\ \frac{1}{2}\int_{\theta_k}^{\xi_k}[\varphi(\hat{v}_\lambda(\theta))]^2+\frac{\lambda}{2}\|\hat{u}_\lambda\|^2_{L^2(J_k)}d\theta;\ \theta\in J_k,\ k=1,2,\dots.m, \\ \frac{1}{2}\int_{\xi_k}^{\theta_{k+1}}[\varphi(\hat{v}_\lambda(\theta))-h_k(\theta)]^2+\frac{\lambda}{2}\|\hat{u}_\lambda\|^2_{L^2(I_k)}d\theta;\ \theta\in I_k,\ k=1,2,\dots.m, \end{cases}$$

$$\leq \begin{cases} \frac{1}{2}\int_0^{\theta_1}[\varphi(v(\theta))-h_0(\theta)]^2+\frac{\lambda}{2}\|u\|^2_{L^2(I_0)}d\theta;\ \theta\in I_0, \\ \frac{1}{2}\int_{\theta_k}^{\xi_k}[\varphi(v(\theta))]^2+\frac{\lambda}{2}\|u\|^2_{L^2(J_k)}d\theta;\ \theta\in J_k,\ k=1,2,\dots.m \\ \frac{1}{2}\int_{\xi_k}^{\theta_{k+1}}[\varphi(v(\theta))-h_k(\theta)]^2+\frac{\lambda}{2}\|u\|^2_{L^2(I_k)}d\theta;\ \theta\in I_k,\ k=1,2,\dots.m \end{cases}$$

for all $u\in\mathfrak{U}_{\mathfrak{a}}$. In particular, let us $u=\hat{u}$ and $v=\hat{v}$, where $(\hat{u},\hat{v})$ is any solution to (1.1)-(1.2), if Q is non-empty. Hence, $\varphi(\hat{v}(\theta))=\mathfrak{C}(\theta)$, a.e. $\theta\in(0,b)$ and we have

$$\begin{cases} \|\hat{u}_{\lambda_n}\|^2_{L^2(I_k)}\leq\|\hat{u}\|^2_{L^2(I_k)};\ k=0,1,2,\dots\dots m \\ \|\hat{u}_{\lambda_n}\|^2_{L^2(J_k)}\leq\|\hat{u}\|^2_{L^2(J_k)};\ k=1,2,\dots\dots m. \end{cases} \tag{4.5}$$

Hence, on a subsequence $\lambda_n\to 0$, we get

$$\begin{cases} \hat{u}_{\lambda_n}\to u^*\ \text{weakly in } L^2(I_k);\ k=0,1,2,\dots\dots.m \\ \hat{u}_{\lambda_n}\to u^*\ \text{weakly in } L^2(J_k);\ k=1,2,\dots\dots.m. \end{cases} \tag{4.6}$$

Now we know that $(\hat{u}_{\lambda_n},\hat{v}_{\lambda_n})$ is the solution of (1.1)-(1.2) so,

$$\hat{v}_{\lambda_n}(\theta)=\begin{cases} e^{A\theta}v_0+\int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)\hat{u}_{\lambda_n}(\xi)d\xi;\ \theta\in I_0, \\ g_k(\theta,\hat{v}_{\lambda_n}(\theta_k^-));\ \theta\in J_k,\ k=1,2,\dots,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k,\hat{v}_{\lambda_n}(\xi_k^-))+\int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)\hat{u}_{\lambda_n}(\xi)d\xi;\ \theta\in I_k,\ k=1,2,\dots,m. \end{cases}$$

Let us define

$$v^*(\theta)=\begin{cases} e^{A\theta}v_0+\int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)u^*(\xi)d\xi;\ \theta\in I_0, \\ g_k(\theta,v^*(\theta_k^-));\ \theta\in J_k,\ k=1,2,\dots,m, \\ e^{A(\theta-\xi_k)}g_k(\xi_k,v^*(\xi_k^-))+\int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)u^*(\xi)d\xi;\ \theta\in I_k,\ k=1,2,\dots,m. \end{cases}$$

Now, we consider

$$\hat{v}_{\lambda_n}(\theta)-v^*(\theta)=\begin{cases}\int_0^\theta e^{A(\theta-\xi)}\zeta(\xi)(\hat{u}_{\lambda_n}(\xi)-u^*(\xi))d\xi;\ \theta\in I_0,\\ g_k(\theta,\hat{v}_{\lambda_n}(\theta_k^-))-g_k(\theta,v^*(\theta_k^-));\ \theta\in J_k,\ k=1,2,\ldots,m\\ e^{A(\theta-\xi_k)}(g_k(\xi_k,\hat{v}_{\lambda_n}(\xi_k^-))-g_k(\xi_k,v^*(\xi_k^-)))+\int_{\xi_k}^\theta e^{A(\theta-\xi)}\zeta(\xi)(\hat{u}_{\lambda_n}(\xi)-u^*(\xi))d\xi;\\ \theta\in I_k,\ k=1,2,\ldots,m.\end{cases}$$

We know that $(\hat{u}_{\lambda_n}(\xi)-u^*(\xi))$ and $(\hat{v}_{\lambda_n}(\xi_k^-)-v^*(\xi_k^-))$ converges weakly to 0. Hence, by using these facts and using g_k are continuous functions from $(J_k\times X)$ to X, for $k=1,2,\ldots m$, we can easily observe that $g_k(\xi_k,\hat{v}_{\lambda_n}(\xi_k^-))-g_k(\xi_k,v^*(\xi_k^-))$ converges weakly to 0. It implies that $(\hat{v}_{\lambda_n}(\theta)-v^*(\theta))$ converges weakly to 0 on I_k for $k=1,2,\cdots m$. So, from here we can easily observe that $g_k(\theta,\hat{v}_{\lambda_n}(\theta_k^-))-g_k(\theta,v^*(\theta_k^-))$ converges weakly to 0, because $I_k=(\xi_k,\theta_{k+1}]$. So, from here we can say that

$$\begin{cases}\hat{v}_{\lambda_n}\to v^*\text{ weakly in }L^2(I_k);\ k=0,1,2,\ldots\ldots.m,\\ \hat{v}_{\lambda_n}\to v^*\text{ weakly in }L^2(J_k);\ k=1,2,\ldots\ldots.m\end{cases}$$

and so $v^*(\theta)$ is a solution to (1.1). Moreover,

$$\varphi(\hat{v}_{\lambda_n})\to\varphi(v^*)\text{ as }\lambda\to 0$$

and so (u^*,v^*) is mild solution to (1.1)-(1.2), that is $u^*\in Q$. On the other hand, by (4.5) we have

$$\begin{cases}\|u^*\|_{L^2(I_k)}\le\liminf_{\lambda_n\to 0}\|\hat{u}_{\lambda_n}\|_{L^2(I_k)}\le\|\hat{u}\|_{L^2(I_k)};\ k=0,1,2\ldots\ldots.m,\\ \|u^*\|_{L^2(J_k)}\le\liminf_{\lambda_n\to 0}\|\hat{u}_{\lambda_n}\|_{L^2(J_k)}\le\|\hat{u}\|_{L^2(J_k)};\ k=1,2\ldots\ldots.m\end{cases}$$

for any $\hat{u}\in Q$ and so we deduce that the distance from u^* to 0 is the smallest, that is

$$u^*=P_Q(0). \tag{4.7}$$

Namely, $u^*=0$ if $0\in Q$ and $u^*\in\partial Q$ if 0 does not belongs to Q. On the other hand, it follows that

$$\begin{cases}\hat{u}_{\lambda_n}\to u^*\text{ strongly in }L^2(I_k);\ k=0,1,2,\ldots\ldots.m,\\ \hat{u}_{\lambda_n}\to u^*\text{ strongly in }L^2(J_k);\ k=1,2,\ldots\ldots.m\end{cases} \tag{4.8}$$

because by (4.5),

$$\begin{cases}\limsup_{\lambda_n\to 0}\|\hat{u}_{\lambda_n}\|_{L^2(I_k)}\le\|\hat{u}\|_{L^2(I_k)};\ k=0,1,2\ldots\ldots.m,\\ \limsup_{\lambda_n\to 0}\|\hat{u}_{\lambda_n}\|_{L^2(J_k)}\le\|\hat{u}\|_{L^2(J_k)};\ k=0,1,2\ldots\ldots.m.\end{cases}$$

Finally, by uniqueness of u^* is defined by (4.7) holds for all $\{\lambda_n\}\to 0$ so $\hat{u}_\lambda$ converges to u^* and it implies $\hat{v}_\lambda$ converges to v^* too. □

5. Example

Let $X = \mathscr{L}^2([0,\pi])$ and $I = [0,b]$ for $0 < b < \infty$. We consider the following inverse problem for first order linear evolution equation with non-instantaneous impulsive conditions:

$$(5.1)\qquad \begin{cases} \frac{\partial}{\partial\theta} w(\theta,\varpi) = \frac{\partial^2}{\partial\varpi^2} w(\theta,\varpi) + u(\theta)\rho(\theta,\varpi),\ (\theta,\varpi) \in I_k \times [0,\pi];\ k = 0,1,2,......N, \\ w(\theta,0) = w(\theta,\pi) = 0,\ \theta \in I, \\ w(0,\varpi) = z(\varpi),\ \varpi \in [0,\pi], \\ w(\theta,\varpi) = \frac{Sin(k\theta)}{e^{k\theta}} w(\theta_k^-,\varpi),\ \varpi \in [0,\pi],\ \theta \in J_k;\ k = 1,2,.....N \end{cases}$$

and the overdetermined condition is

$$(5.2)\qquad \int_0^{\pi/2} w(\theta,\varpi)\, d\varpi = \begin{cases} \alpha Sin(\theta),\ \theta \in I_k,\ k = 0,1,2,....N \\ 0,\ \theta \in J_k,\ k = 1,2,....N, \end{cases}$$

where $\alpha > 0$, and $0 = \theta_0 = \xi_0 < \theta_1 \le \xi_1 < .... < \theta_N < \xi_N < \theta_{N+1} = b$ are the fixed real numbers and $z \in X$, $\rho \in C(([0,\theta_1] \cup (\xi_k,\theta_{k+1}]) \times [0,\pi];\mathbb{R})$. We also assume that $w_{\lambda_n}(\xi_k^-,\varpi) \to w^*(\xi_k^-,\varpi)$ as $\lambda_n \to 0$, where λ_n is a decreasing sequence of positive real numbers. We denote $I_0 = [0,\theta_1]$, $I_k = (\xi_k,\theta_{k+1}]$ and $J_k = (\theta_k,\xi_k]$ for $k = 1,2,3,.....N$. We define $A : D(A) \subset X \to X$ is the operator given by $Av = v''$ on

$$D(A) = \left\{ v \in X : v', v'' \in X, v(0) = v(\pi) = 0 \right\},$$

where $v(\theta) = w(\theta,.)$, that is $v(\theta)(\varpi) = w(\theta,\varpi)$. By Lemma (2.1) of [29] (see pg. 234), it is well known that A is the infinitesimal generator of a C_0- semigroup. The inverse problem (5.1)-(5.2) can be reformulated as the following abstract differential equation in X :

$$(5.3)\qquad \begin{cases} v'(\theta) = Av(\theta) + u(\theta)\zeta(\theta);\ \theta \in I_k,\ k = 0,1,2,...,N, \\ v(\theta) = g_k(\theta, v(\theta_k^-));\ \theta \in J_k,\ k = 1,2,...,N, \\ v(0) = 0 \end{cases}$$

with overdetermined condition

$$(5.4)\qquad \varphi(v(\theta)) = \begin{cases} \alpha Sin(\theta);\ \theta \in I_k,\ k = 0,1,2....N, \\ 0;\ \theta \in J_k,\ k = 1,2......N, \end{cases}$$

where functions $g_k(\theta, v(\theta_k^-))(\varpi) = \frac{Sin(k\theta)}{e^{k\theta}} w(\theta_k^-,\varpi)$, $\zeta(\theta)(\varpi) = \rho(\theta,(\varpi))$ and

$$\varphi(v(\theta)) = \int_0^{\pi/2} v(\theta)(\varpi)\, d\varpi.$$

Next, we say that (u,w) is the mild solution of (5.1)-(5.2) if (u,v) is a mild solution of the associated abstract problem (5.3)-(5.4).

Now, from here we can see that $v_0 = 0 \in D(A)$, $\zeta(\theta) = \rho(\theta,.) \in C([0,b];\mathbb{R})$, $\mathfrak{C}(\theta) \in \mathscr{PC}([0,b],\mathbb{R})$ and $\varphi(\zeta(\theta)) \neq 0$ for all $\theta \in I_k$; $k = 0,1,2,....N$. Hence, by Theorem (3.1), the inverse problem with non-instantaneous impulsive conditions (5.1)-(5.2) has unique solution.

Now, if we do not assume that $\varphi(\zeta(\theta)) \neq 0$ for all $\theta \in I_k$; $k = 0, 1, 2, ...N$. Then we will solve our inverse problem (5.1)-(5.2) by an optimal control approach and for that our minimization problem is defined as follows:
Find a piecewise continuous function $\hat{u} \in \mathscr{PC}([0,b];\mathbb{R})$ such that

$$\mathfrak{J}(\hat{u}) = \min_{u \in \mathscr{PC}([0,b];\mathbb{R})} \left[\mathfrak{J}(u) = \begin{cases} \frac{1}{2}\int_0^{\theta_1} [\int_0^{\pi/2} w(\theta,\varpi)\, d\varpi - \alpha Sin(\theta)]^2 d\theta;\ \theta \in I_0, \\ \frac{1}{2}\int_{\theta_k}^{\xi_k} [\int_0^{\pi/2} w(\theta,\varpi)\, d\varpi]^2 d\theta;\ \theta \in J_k,\ k = 1, 2, ...m, \\ \frac{1}{2}\int_{\xi_k}^{\theta_{k+1}} [\int_0^{\pi/2} w(\theta,\varpi)\, d\varpi - \alpha Sin(\theta)]^2 d\theta;\ \theta \in I_k, \\ \qquad k = 1, 2, ...m \end{cases} \right]$$

for all (u, v) satisfying system (5.3) and applying the results of section (4) we will get our required results.

6. Conclusion

In this work, we have considered the inverse problem for the first order abstract non-instantaneous impulsive differential equation. The main aim of this work is the reconstruction of u and consequently of v, as a solution to the considered system. Here we have solved our problem in two situations, one is for regular data and another is for irregular data. For regular data, we have solved the problem using a direct approach in which we have used the $\mathscr{C}_0$ semigroup theory, Volterra integral equation for second-kind and duality in functional analysis. Here we found a unique strong solution to the inverse problem (1.1)-(1.2) under the condition that $\varphi(\zeta(\theta)) \neq 0$ for all $\theta \in [0,b]$. Next, for irregular data and more relaxed hypotheses, including also $\varphi(\zeta(\theta)) = 0$ for all $\theta \in [0,b]$, we shall introduce an optimal control approach. First, we have defined an optimal control problem corresponding to our inverse problem. Then by defining an approximate optimal control problem, we ensured that the solution of the original optimal control exists, and then we found the optimal pair. Next, we show that the characterization of the limit of the sequence of approximate solutions demonstrates that it is a solution to the original inverse problem (1.1)-(1.2) under specific hypotheses. Note that by a direct approach, we get the strong solution while, by the optimal control approach, we get the approximate solution to the considered inverse problem. At last, we have given an example to demonstrate the validity and accuracy of our results.

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