CHARACTERIZATIONS OF MATRIX VALUED ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS

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Abstract. Matrix valued asymmetric truncated Toeplitz operators are compressions of multiplication operators acting between two possibly different model spaces. In this paper, we characterize matrix valued asymmetric truncated Toeplitz operators by using compressed shifts, modified compressed shifts and shift invariance.

1. Introduction

Let H^2 be the classical Hardy space in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Truncated Toeplitz operators (TTO's) and asymmetric truncated Toeplitz operators (ATTO's) are compressions of multiplication operator to the backward shift invariant subspaces of H^2 (with two possibly different underlying subspaces in the asymmetric case). Each of these subspaces is of the form $K_\theta = (\theta H^2)^\perp = H^2 \ominus \theta H^2$, where θ a complex-valued inner function: $\theta \in H^\infty$ and $|\theta(z)| = 1$ a.e. on the unit circle $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Since D. Sarason's paper [27] TTO's, and later on ATTO's [3,6,7], have been intensely studied (see [1,9,11,13,15,28] and [5,16–19,23,24]).

It is natural to consider TTO's and ATTO's defined on subspaces of vector valued Hardy space $H^2(\mathcal{H})$ with $\mathcal{H}$ a separable finite dimensional complex Hilbert space (see Sections 2 and 3 for detailed definitions). A vector valued model space $K_\Theta \subset H^2(\mathcal{H})$ is the orthogonal complement of $\Theta H^2(\mathcal{H})$, that is, $K_\Theta = H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$. Here Θ is an operator valued inner function: a function with values in $\mathcal{L}(\mathcal{H})$ (the algebra of all bounded linear operators on $\mathcal{H}$), analytic in $\mathbb{D}$, bounded and such that the boundary values $\Theta(z)$ are unitary operators a.e. on $\mathbb{T}$. These spaces appear in connection with model theory of Hilbert space contractions (see [25]). Let P_Θ be the orthogonal projection from $L^2(\mathcal{H})$ onto K_Θ.

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For two operator valued inner functions $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ and $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$ (again, see Sections 2 and 3 for definitions) let

$$A_\Phi^{\Theta_1,\Theta_2} f = P_{\Theta_2}(\Phi f), \quad f \in K_{\Theta_1} \cap H^\infty(\mathcal{H}). \tag{1.1}$$

The operator $A_\Phi^{\Theta_1,\Theta_2}$ is called a matrix valued asymmetric truncated Toeplitz operator (MATTO), while $A_\Phi^{\Theta_1} = A_\Phi^{\Theta_1,\Theta_1}$ is called a matrix valued truncated Toeplitz operator (MTTO, see [22]). Both are densely defined. Let $\mathcal{MT}(\Theta_1, \Theta_2)$ be the set of all MATTO's of the form (1.1) which can be extended boundedly to the whole space K_{Θ_1} and for $\Theta_1 = \Theta_2 = \Theta$ let $\mathcal{MT}(\Theta) = \mathcal{MT}(\Theta, \Theta)$.

Two important examples of operators from $\mathcal{MT}(\Theta)$ are the model operators

$$S_\Theta = A_z^\Theta = A_{zI_{\mathcal{H}}}^\Theta \quad \text{and} \quad S_\Theta^* = A_{\bar{z}}^\Theta = A_{\bar{z}I_{\mathcal{H}}}^\Theta. \tag{1.2}$$

It is known that each C_0 contraction with finite defect indices is unitarily equivalent to S_Θ for some operator valued inner function Θ (see [25, Chapter IV]). On the other hand, operators from $\mathcal{MT}(\Theta_1, \Theta_2)$ with certain bounded analytic symbols appear as the operators intertwining S_{Θ_1} and S_{Θ_2} (see [2, p. 238]).

Some algebraic properties of MTTO's were studied in [22], while the asymmetric case was investigated in [21]. Here we continue the investigation started in [21].

Sections 2 and 3 contain preliminary material on spaces of vector valued functions (Section 2), model spaces and MATTO's (Section 3). In Section 4 we consider some model space operators and their action on $\mathcal{MT}(\Theta_1, \Theta_2)$. Section 5 is devoted to characterizations of MATTO's in terms of S_{Θ_1}, S_{Θ_2} and their adjoints. In Section 6 we consider the notion of shift invariance of operators from $\mathcal{MT}(\Theta_1, \Theta_2)$. In section 7 we use modified compressed shift to characterize MATTO's.

2. Spaces of vector valued functions and their operators

Let $\mathcal{H}$ be a complex separable Hilbert space. In what follows $\|\cdot\|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ will denote the norm and the inner product in $\mathcal{H}$, respectively. Moreover, we will assume that $\dim \mathcal{H} < \infty$. The space $L^2(\mathcal{H})$ can be defined as

$$L^2(\mathcal{H}) = \{f \colon \mathbb{T} \to \mathcal{H} : f \text{ is measurable and } \int_{\mathbb{T}} \|f(z)\|_{\mathcal{H}}^2 \, dm(z) < \infty\}$$

(m being the normalized Lebesgue measure on $\mathbb{T}$). As usual, each $f \in L^2(\mathcal{H})$ is interpreted as a class of functions equal to the representing

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f a.e. on $\mathbb{T}$ with respect to m. The space $L^2(\mathcal{H})$ is a (separable) Hilbert space with the inner product given by

$$\langle f, g\rangle_{L^2(\mathcal{H})} = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_{\mathcal{H}}\, dm(z), \quad f, g \in L^2(\mathcal{H}).$$

Equivalently, $L^2(\mathcal{H})$ consists of elements $f : \mathbb{T} \to \mathcal{H}$ of the form

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{(a.e. on } \mathbb{T}) \\ &\quad \text{with} \quad \{a_n\} \subset \mathcal{H} \text{ such that } \sum_{n=-\infty}^{\infty} \|a_n\|_{\mathcal{H}}^2 < \infty. \end{aligned} \tag{2.1}$$

The n-th Fourier coefficient a_n of $f \in L^2(\mathcal{H})$ is determined by

$$\langle a_n, x\rangle_{\mathcal{H}} = \int_{\mathbb{T}} \overline{z}^n \langle f(z), x\rangle_{\mathcal{H}} dm(z) \quad \text{for all } x \in \mathcal{H}. \tag{2.2}$$

If $f \in L^2(\mathcal{H})$ is given by (2.1), then its Fourier series converges in the $L^2(\mathcal{H})$ norm and

$$\|f\|_{L^2(\mathcal{H})}^2 = \int_{\mathbb{T}} \|f(z)\|_{\mathcal{H}}^2\, dm(z) = \sum_{n=-\infty}^{\infty} \|a_n\|_{\mathcal{H}}^2.$$

Moreover, for $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^2(\mathcal{H})$ we have

$$\langle f, g\rangle_{L^2(\mathcal{H})} = \sum_{n=-\infty}^{\infty} \langle a_n, b_n\rangle_{\mathcal{H}}.$$

For $\mathcal{H} = \mathbb{C}$ we denote $L^2 = L^2(\mathbb{C})$.

The vector valued Hardy space $H^2(\mathcal{H})$ is defined as the set of all the elements of $L^2(\mathcal{H})$ whose Fourier coefficients with negative indices vanish, that is,

$$H^2(\mathcal{H}) = \left\{ f \in L^2(\mathcal{H}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \right\}.$$

Each $f \in H^2(\mathcal{H})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, can also be identified with a function

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad \lambda \in \mathbb{D},$$

analytic in the unit disk $\mathbb{D}$ (the boundary values $f(z)$ can be obtained from the radial limits, which converge to the boundary function in the

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$L^2(\mathcal{H})$ norm). Denote by P_+ the orthogonal projection $P_+ : L^2(\mathcal{H}) \to H^2(\mathcal{H})$,

$$P_+\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n,$$

and let $H^2 = H^2(\mathbb{C})$.

We can also consider the spaces

$$L^\infty(\mathcal{H}) = \left\{ f\colon \mathbb{T} \to \mathcal{H} : \begin{array}{l} f \text{ is measurable and} \\ \|f\|_\infty = \text{ess sup}_{z\in\mathbb{T}} \|f(z)\|_{\mathcal{H}} < \infty \end{array} \right\}$$

(clearly, $L^\infty(\mathcal{H}) \subset L^2(\mathcal{H})$) and

$$H^\infty(\mathcal{H}) = L^\infty(\mathcal{H}) \cap H^2(\mathcal{H}),$$

the latter seen also as the space of all bounded $\mathcal{H}$–valued functions which are analytic in $\mathbb{D}$.

Now let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ equipped with the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$. In the case $\dim \mathcal{H} = d < \infty$ each element of $\mathcal{L}(\mathcal{H})$ can be identified with a $d \times d$ matrix. Denote

$$L^\infty(\mathcal{L}(\mathcal{H})) = \left\{ \mathbf{F}\colon \mathbb{T} \to \mathcal{L}(\mathcal{H}) : \begin{array}{l} \mathbf{F} \text{ is measurable and} \\ \|\mathbf{F}\|_\infty = \text{ess sup}_{z\in\mathbb{T}} \|\mathbf{F}(z)\|_{\mathcal{L}(\mathcal{H})} < \infty \end{array} \right\}$$

(a function $\mathbf{F}\colon \mathbb{T} \to \mathcal{L}(\mathcal{H})$ is measurable if $\mathbf{F}(\cdot)x\colon \mathbb{T} \to \mathcal{H}$ is measurable for every $x \in \mathcal{H}$). Each $\mathbf{F} \in L^\infty(\mathcal{L}(\mathcal{H}))$ admits a formal Fourier expansion (a.e. on $\mathbb{T}$)

$$\mathbf{F}(z) = \sum_{n=-\infty}^{\infty} F_n z^n \quad \text{with } \{F_n\} \subset \mathcal{L}(\mathcal{H}) \tag{2.3}$$

defined by

$$F_n x = \int_{\mathbb{T}} \overline{z}^n \mathbf{F}(z) x \, dm(z) \quad \text{for } x \in \mathcal{H} \tag{2.4}$$

(integrated in the strong sense). Let

$$H^\infty(\mathcal{L}(\mathcal{H})) = \left\{ \mathbf{F} \in L^2(\mathcal{L}(\mathcal{H})) : \mathbf{F}(z) = \sum_{n=0}^{\infty} F_n z^n \right\}.$$

The space $H^\infty(\mathcal{L}(\mathcal{H}))$ can equivalently be defined as the space of all analytic functions $\mathbf{F} : \mathbb{D} \to \mathcal{L}(\mathcal{H})$ such that

$$\|\mathbf{F}\|_\infty = \sup_{\lambda\in\mathbb{D}} \|\mathbf{F}(\lambda)\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

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Each such bounded analytic $\mathbf{F}$ is of the form

$$\mathbf{F}(\lambda) = \sum_{n=0}^{\infty} F_n \lambda^n, \quad \lambda \in \mathbb{D}, \tag{2.5}$$

and can be identified with the boundary function

$$\mathbf{F}(z) = \sum_{n=0}^{\infty} F_n z^n \in L^\infty(\mathcal{L}(\mathcal{H})). \tag{2.6}$$

Conversely, each $\mathbf{F} \in L^\infty(\mathcal{L}(\mathcal{H}))$ given by (2.6) can be extended by (2.5) to a function bounded and analytic in $\mathbb{D}$. In each case the coefficients $\{F_n\}$ can be obtained by (2.4) and the norms $\|\cdot\|_\infty$ of the boundary function and its extension coincide (see [2, p. 232]).

Note that for each $\lambda \in \mathbb{D}$ the function $\mathbf{k}_\lambda(z) = (1 - \bar{\lambda} z)^{-1} I_{\mathcal{H}}$ belongs to $H^\infty(\mathcal{L}(\mathcal{H}))$. Moreover, for every $x \in \mathcal{H}$ the function $\mathbf{k}_\lambda x : z \mapsto \mathbf{k}_\lambda(z)x$ belongs to $H^\infty(\mathcal{H})$ and has the following reproducing property

$$\langle f, \mathbf{k}_\lambda x \rangle_{L^2(\mathcal{H})} = \langle f(\lambda), x \rangle_{\mathcal{H}}, \quad f \in H^2(\mathcal{H}).$$

To each $\mathbf{F} \in L^\infty(\mathcal{L}(\mathcal{H}))$ there corresponds a multiplication operator $M_{\mathbf{F}} : L^2(\mathcal{H}) \to L^2(\mathcal{H})$: for $f \in L^2(\mathcal{H})$,

$$(M_{\mathbf{F}} f)(z) = \mathbf{F}(z) f(z) \quad \text{a.e. on } \mathbb{T}.$$

We will write $\mathbf{F}f$ instead of $M_{\mathbf{F}}f$. For a constant $\mathbf{F}$, that is, $\mathbf{F}(z) = F$ a.e. on $\mathbb{T}$ for some $F \in \mathcal{L}(\mathcal{H})$, we will also write Ff instead of $\mathbf{F}f$. By $T_{\mathbf{F}}$ we will denote the compression of $M_{\mathbf{F}}$ to the Hardy space: $T_{\mathbf{F}} : H^2(\mathcal{H}) \to H^2(\mathcal{H})$,

$$T_{\mathbf{F}} f = P_+ M_{\mathbf{F}} f \quad \text{for } f \in H^2(\mathcal{H}).$$

It is clear that $(M_{\mathbf{F}})^* = M_{\mathbf{F}^*}$ and $(T_{\mathbf{F}})^* = T_{\mathbf{F}^*}$, where $\mathbf{F}^*(z) = \mathbf{F}(z)^*$ a.e. on $\mathbb{T}$. It is also not difficult to verify that for $\mathbf{F} \in L^\infty(\mathcal{L}(\mathcal{H}))$ we have that $\mathbf{F} \in H^\infty(\mathcal{L}(\mathcal{H}))$ if and only if $M_{\mathbf{F}}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. In particular, for $M_z = M_{zI_{\mathcal{H}}}$ we have $M_z^* = M_{\bar{z}} = M_{\bar{z}I_{\mathcal{H}}}$ and $M_z(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. The operator $S = T_z = M_{z|H^2(\mathcal{H})}$ is called the (forward) shift operator. Its adjoint, the backward shift operator $S^* = T_{\bar{z}}$, is given by the formula

$$S^* f(z) = \bar{z}\big(f(z) - f(0)\big).$$

Here we assume that $\dim \mathcal{H} < \infty$ so we can consider $\mathcal{L}(\mathcal{H})$ as a Hilbert space with the Hilbert–Schmidt norm and we may also define as above the spaces $L^2(\mathcal{L}(\mathcal{H}))$ and $H^2(\mathcal{L}(\mathcal{H}))$. Recall that the norm and the corresponding inner product are defined as follows: for Hilbert-Schmidt operators $A, B \in \mathcal{L}(\mathcal{H})$ we have

$$\|A\|_2^2 = tr(A^*A) = \sum_{e \in \varepsilon} \langle Ae, Ae \rangle_{\mathcal{H}}$$

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and

$$\langle A, B\rangle = tr(B^*A) = \sum_{e\in\varepsilon}\langle Ae, Be\rangle_{\mathcal{H}}$$

ε being any orthonormal basis for $\mathcal{H}$ (see [13, Chapter 3]). Hence for $\mathbf{F}, \mathbf{G} \in L^2(\mathcal{L}(\mathcal{H}))$

$$\begin{aligned}\langle \mathbf{F}, \mathbf{G}\rangle_{L^2(\mathcal{L}(\mathcal{H}))} &= \int_{\mathbb{T}} \langle \mathbf{F}(z), \mathbf{G}(z)\rangle_2 dm(z) \\ &= \int_{\mathbb{T}} \sum_{e\in\varepsilon}\langle \mathbf{F}(z)e, \mathbf{G}(z)e\rangle_{\mathcal{H}} dm(z).\end{aligned}$$

Since $\langle A, B\rangle_2 = \langle B^*, A^*\rangle_2$, it follows that $\langle \mathbf{F}, \mathbf{G}\rangle_{L^2(\mathcal{L}(\mathcal{H}))} = \langle \mathbf{G}^*, \mathbf{F}^*\rangle_{L^2(\mathcal{L}(\mathcal{H}))}$.

Since here the Hilbert–Schmidt norm and the operator norm are equivalent, we have

$$L^\infty(\mathcal{L}(\mathcal{H})) \subset L^2(\mathcal{L}(\mathcal{H})), \quad H^\infty(\mathcal{L}(\mathcal{H})) \subset H^2(\mathcal{L}(\mathcal{H})).$$

Moreover, it is not difficult to verify that if $\mathbf{F} \in L^2(\mathcal{L}(\mathcal{H}))$ is given by

$$\mathbf{F}(z) = \sum_{n=-\infty}^{\infty} F_n z^n, \quad F_n \in \mathcal{L}(\mathcal{H}),$$

where the series is convergent in the $L^2(\mathcal{L}(\mathcal{H}))$-norm, then

$$\mathbf{F}^*(z) = \mathbf{F}(z)^* = \sum_{n=-\infty}^{\infty} (F_{-n})^* z^n.$$

We thus have

$$L^2(\mathcal{L}(\mathcal{H})) = \left[zH^2(\mathcal{L}(\mathcal{H}))\right]^* \oplus H^2(\mathcal{L}(\mathcal{H})).$$

For $\mathbf{F} \in L^2(\mathcal{L}(\mathcal{H}))$ the operators $M_{\mathbf{F}}$ and $T_{\mathbf{F}}$ can be densely defined, on $L^\infty(\mathcal{H})$ and $H^\infty(\mathcal{H})$, respectively. For more details on spaces of vector valued functions we refer the reader to [2,25,26].

3. Model spaces and MATTO's

An element $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ is called an (operator valued) inner function if its boundary values $\Theta(z)$ are unitary operators a.e. on $\mathbb{T}$ (in general it is assumed that the boundary values are isometries, see e.g. [2, p. 113], but here $\dim \mathcal{H} < \infty$). We will consider only pure inner functions, that is Θ such that $\|\Theta(0)\|_{\mathcal{L}(\mathcal{H})} < 1$.

The model space

$$K_\Theta = H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$$

corresponding to an inner function Θ is invariant under the backward shift S^*. Moreover, by the vector valued version of Beurling's invariant subspace theorem, each closed (nontrivial) S^*–invariant subspace of $H^2(\mathcal{H})$ is a model space ([2, Chapter 5, Theorem 1.10]). Let P_Θ be the orthogonal projection from $L^2(\mathcal{H})$ onto K_Θ. Then

$$P_\Theta = P_+ - M_\Theta P_+ M_{\Theta^*}.$$

Note that $M_\Theta P_+ M_{\Theta^*}$ is the orthogonal projection from $L^2(\mathcal{H})$ onto $\Theta H^2(\mathcal{H})$.

For each $\lambda \in \mathbb{D}$ we can consider

$$\mathbf{k}_\lambda^\Theta(z) = \tfrac{1}{1-\bar{\lambda}z}(I_{\mathcal{H}} - \Theta(z)\Theta(\lambda)^*) \in H^\infty(\mathcal{L}(\mathcal{H})).$$

For $x \in \mathcal{H}$ we will denote the function $z \mapsto \mathbf{k}_\lambda^\Theta(z)x$ simply by $\mathbf{k}_\lambda^\Theta x$. Then, for each $x \in \mathcal{H}$ and $\lambda \in \mathbb{D}$, the function $\mathbf{k}_\lambda^\Theta x = P_\Theta(\mathbf{k}_\lambda x)$ belongs to $K_\Theta^\infty = K_\Theta \cap H^\infty(\mathcal{H})$ and has the following reproducing property

$$\langle f, \mathbf{k}_\lambda^\Theta x\rangle_{L^2(\mathcal{H})} = \langle f(\lambda), x\rangle_{\mathcal{H}} \quad \text{for every } f \in K_\Theta.$$

It follows in particular that $K_\Theta^\infty = K_\Theta \cap H^\infty(\mathcal{H})$ is a dense subset of K_Θ.

Now let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two inner functions. For any $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$ define

$$A_\Phi^{\Theta_1,\Theta_2} f = P_{\Theta_2} M_\Phi f = P_{\Theta_2}(\Phi f), \quad f \in K_{\Theta_1}^\infty.$$

The operator $A_\Phi^{\Theta_1,\Theta_2}$ is called a matrix valued asymmetric truncated Toeplitz operator (MATTO) with symbol $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$. It is densely defined and if bounded, it can be extended to a bounded linear operator $A_\Phi^{\Theta_1,\Theta_2} : K_{\Theta_1} \to K_{\Theta_2}$ (in which case we simply say that $A_\Phi^{\Theta_1,\Theta_2}$ is bounded). Let us denote

$$\mathcal{MT}(\Theta_1,\Theta_2) = \{A_\Phi^{\Theta_1,\Theta_2} : \Phi \in L^2(\mathcal{L}(\mathcal{H})) \text{ and } A_\Phi^{\Theta_1,\Theta_2} \text{ is bounded}\}.$$

For $\Theta_1 = \Theta_2 = \Theta$ we put $A_\Phi^\Theta = A_\Phi^{\Theta,\Theta}$ (a matrix valued truncated Toeplitz operator, MTTO) and $\mathcal{MT}(\Theta) = \mathcal{MT}(\Theta,\Theta)$. Observe that $\left(A_\Phi^{\Theta_1,\Theta_2}\right)^* = A_{\Phi^*}^{\Theta_2,\Theta_1}$, so $A \in \mathcal{MT}(\Theta_1,\Theta_2)$ if and only if $A^* \in \mathcal{MT}(\Theta_2,\Theta_1)$.

Two important examples of MTTO's are the model operators: the compressed shift S_Θ and its adjoint S_Θ^*, defined by (1.2). Clearly, $S_\Theta f = P_\Theta S f = P_\Theta(M_z f)$ and since K_Θ is S^*–invariant, we have $S_\Theta^* f = S^* f = P_+(M_{\bar{z}} f)$.

These operators are models for a class of Hilbert space contractions. For example, each C_0 contraction with finite defect indices is unitarily equivalent to S_Θ for some operator valued inner function Θ (see [25, Chapter IV]).

Let

$$\mathcal{D}_\Theta = \{(I_{\mathcal{H}} - \Theta\Theta(0)^*)x : \ x \in \mathcal{H}\} = \{\mathbf{k}_0^\Theta x : \ x \in \mathcal{H}\} \subset K_\Theta.$$

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Then for $f \in K_\Theta$ we have $f \perp \mathcal{D}_\Theta$ if and only if $f(0) = 0$. Indeed, $f \perp \mathcal{D}_\Theta$ if and only if

$$0 = \langle f, \mathbf{k}_0^\Theta x\rangle_{L^2(\mathcal{H})} = \langle f(0), x\rangle_{\mathcal{H}} \text{ for every } x \in \mathcal{H}.$$

It follows that

$$(S_\Theta^* f)(z) = \begin{cases} \bar{z} f(z) & \text{for } f \perp \mathcal{D}_\Theta, \\ -\bar{z}\big(\Theta(z) - \Theta(0)\big)\Theta(0)^* x & \text{for } f = \mathbf{k}_0^\Theta x \in \mathcal{D}_\Theta. \end{cases}$$

Now denote (the defect operator)

$$D_\Theta = I_{K_\Theta} - S_\Theta S_\Theta^*.$$

Since for each $f \in H^2(\mathcal{H})$ we have $(I_{H^2(\mathcal{H})} - SS^*)f = f(0)$ (a constant function in $H^2(\mathcal{H})$), it follows that for $f \in K_\Theta$,

$$\begin{aligned} D_\Theta f &= (I_{K_\Theta} - S_\Theta S_\Theta^*)f = P_\Theta(I_{H^2(\mathcal{H})} - SS^*)f \\ &= (I_{\mathcal{H}} - \Theta\Theta(0)^*)f(0) = \mathbf{k}_0^\Theta f(0) \in \mathcal{D}_\Theta. \end{aligned} \tag{3.1}$$

More precisely,

$$D_\Theta f = \begin{cases} 0 & \text{for } f \perp \mathcal{D}_\Theta, \\ \mathbf{k}_0^\Theta (I_{\mathcal{H}} - \Theta(0)\Theta(0)^*)x & \text{for } f = \mathbf{k}_0^\Theta x \in \mathcal{D}_\Theta. \end{cases}$$

Since $\mathbf{k}_0^\Theta$ is invertible in $H^\infty(\mathcal{L}(\mathcal{H}))$, the formula

$$\Omega_\Theta(\mathbf{k}_0^\Theta x) = x, \quad x \in \mathcal{H},$$

gives a well defined operator $\Omega_\Theta : \mathcal{D}_\Theta \to \mathcal{H}$. Clearly, Ω_Θ is bounded (here for example as an operator acting between two finite dimensional Hilbert spaces). Since $\mathcal{H}$ can be identified with a subspace of $H^2(\mathcal{H})$ (the space of all constant $\mathcal{H}$–valued functions), Ω_Θ can be seen as an operator from $\mathcal{D}_\Theta$ into $H^2(\mathcal{H})$. For each $f \in K_\Theta$ we then have

$$\Omega_\Theta D_\Theta f = \Omega_\Theta(\mathbf{k}_0^\Theta f(0)) = f(0) = (I_{H^2(\mathcal{H})} - SS^*)f. \tag{3.2}$$

4. MATTO's and some model space operators

In this section we consider the action of some model space operators on $\mathcal{MT}(\Theta_1, \Theta_2)$.

In [20] the author considers the generalized Crofoot transform. A bounded linear operator $W \in \mathcal{L}(\mathcal{H})$ is called a contraction if $\|W\|_{\mathcal{L}(\mathcal{H})} \leq 1$ and a strict contraction if $\|W\|_{\mathcal{L}(\mathcal{H})} < 1$. The operators $D_W = (I - W^*W)^{\frac{1}{2}}$ and $D_{W^*} = (I - WW^*)^{\frac{1}{2}}$ are called the defect operators of W. For a pure inner function $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ and $W \in \mathcal{L}(\mathcal{H})$ such that $\|W\|_{\mathcal{L}(\mathcal{H})} < 1$ define the generalized Crofoot transform $J_W^\Theta : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ by

$$J_W^\Theta f = D_{W^*}(I_{L^2(\mathcal{H})} - \Theta W^*)^{-1} f, \quad f \in L^2(\mathcal{H}).$$

Then J_W^Θ is unitary and maps K_Θ onto K_{Θ^W}, where

$$\Theta^W(z) = -W + D_{W^*}(I_{L^2(\mathcal{H})} - \Theta(z)W^*)^{-1}\Theta(z)D_W.$$

The following theorem describes the action of the Crofoot transform on $\mathcal{MT}(\Theta_1, \Theta_2)$ (see [21] for asymmetric matrix valued truncated Toeplitz operators and [17] for the scalar case):

Theorem 4.1. [21] *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $W_1, W_2 \in \mathcal{L}(\mathcal{H})$ be such that $\|W_1\|_{\mathcal{L}(\mathcal{H})} < 1$ and $\|W_2\|_{\mathcal{L}(\mathcal{H})} < 1$. A bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $J_{W_2}^{\Theta_2} A (J_{W_1}^{\Theta_1})^*$ belongs to $\mathcal{MT}(\Theta_1^{W_1}, \Theta_2^{W_2})$. More precisely, $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $J_{W_2}^{\Theta_2} A (J_{W_1}^{\Theta_1})^* = A_\Psi^{\Theta_1^{W_1},\Theta_2^{W_2}} \in \mathcal{MT}(\Theta_1^{W_1}, \Theta_2^{W_2})$ with*

$$\Psi = D_{W_2^*}(I_{\mathcal{L}(\mathcal{H})} - \Theta_2 W_2^*)^{-1}\Phi D_{W_1^*}(I_{\mathcal{L}(\mathcal{H})} + \Theta_1^{W_1} W_1^*)^{-1}.$$

Recall that if $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ is an inner function, then so is

$$\widetilde{\Theta}(z) = \Theta(\bar{z})^*.$$

Let us now consider the map $\tau_\Theta : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ defined for $f \in L^2(\mathcal{H})$ by

$$(\tau_\Theta f)(z) = \bar{z}\Theta(\bar{z})^* f(\bar{z}) = \bar{z}\widetilde{\Theta}(z) f(\bar{z}) \quad \text{a.e. on } \mathbb{T}. \tag{4.1}$$

The map τ_Θ is an isometry and its adjoint $\tau_\Theta^* = \tau_{\widetilde{\Theta}}$ is also its inverse. Hence τ_Θ is unitary. Moreover, it is easy to verify that

$$\tau_\Theta(\Theta H^2(\mathcal{H})) \subset H^2(\mathcal{H})^\perp \quad \text{and} \quad \tau_\Theta(H^2(\mathcal{H})^\perp) \subset \widetilde{\Theta} H^2(\mathcal{H}),$$

which implies that

$$\tau_\Theta(K_\Theta) = K_{\widetilde{\Theta}}.$$

Theorem 4.2. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions. A bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\tau_{\Theta_2} A \tau_{\Theta_1}^*$ belongs to $\mathcal{MT}(\widetilde{\Theta}_1, \widetilde{\Theta}_2)$. More precisely, $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\tau_{\Theta_2} A \tau_{\Theta_1}^* = A_\Psi^{\widetilde{\Theta}_1,\widetilde{\Theta}_2} \in \mathcal{MT}(\widetilde{\Theta}_1, \widetilde{\Theta}_2)$ with*

$$\begin{aligned} \Psi(z) &= \Theta_2(\bar{z})^* \Phi(\bar{z}) \Theta_1(\bar{z}) \\ &= \widetilde{\Theta}_2(z) \Phi(\bar{z}) \widetilde{\Theta}_1(z)^* \quad \textit{a.e. on } \mathbb{T}. \end{aligned} \tag{4.2}$$

Proof. Let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. Assume that $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with some $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$, and take

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$f \in K^\infty_{\widetilde{\Theta}_1}$ and $g \in K^\infty_{\widetilde{\Theta}_2}$. Note that $\tau_{\widetilde{\Theta}_1} f \in K^\infty_{\Theta_1}$ and $\tau_{\widetilde{\Theta}_2} g \in K^\infty_{\Theta_2}$. Therefore

$$\begin{aligned}\langle \tau_{\Theta_2} A\, \tau^*_{\widetilde{\Theta}_1} f, g\rangle_{L^2(\mathcal{H})} &= \langle A^{\Theta_1,\Theta_2}_{\Phi} \tau_{\widetilde{\Theta}_1} f, \tau^*_{\Theta_2} g\rangle_{L^2(\mathcal{H})}\\ &= \langle \Phi\, \tau_{\widetilde{\Theta}_1} f, \tau^*_{\Theta_2} g\rangle_{L^2(\mathcal{H})} = \langle \tau_{\Theta_2}(\Phi\, \tau_{\widetilde{\Theta}_1} f), g\rangle_{L^2(\mathcal{H})}\\ &= \int_{\mathbb{T}} \langle \bar z \widetilde{\Theta}_2(z)(\Phi\, \tau_{\widetilde{\Theta}_1} f)(\bar z), g(z)\rangle_{\mathcal{H}}\, dm(z)\\ &= \int_{\mathbb{T}} \langle \bar z \widetilde{\Theta}_2(z)\Phi(\bar z) z \Theta_1(\bar z) f(z), g(z)\rangle_{\mathcal{H}}\, dm(z)\\ &= \int_{\mathbb{T}} \langle \Psi(z) f(z), g(z)\rangle_{\mathcal{H}}\, dm(z) = \langle A^{\widetilde{\Theta}_1,\widetilde{\Theta}_2}_{\Psi} f, g\rangle_{L^2(\mathcal{H})}\end{aligned}$$

with $\Psi \in L^2(\mathcal{L}(\mathcal{H}))$ given by (4.2).

Now, if $\tau_{\Theta_2} A\, \tau^*_{\widetilde{\Theta}_1} = A^{\widetilde{\Theta}_1,\widetilde{\Theta}_2}_{\Psi} \in \mathcal{MT}(\widetilde{\Theta}_1, \widetilde{\Theta}_2)$ for some $\Psi \in L^2(\mathcal{L}(\mathcal{H}))$, then $A = \tau_{\widetilde{\Theta}_2} A^{\widetilde{\Theta}_1,\widetilde{\Theta}_2}_{\Psi} \tau^*_{\widetilde{\Theta}_1}$ and by the first part of the proof $A = A^{\Theta_1,\Theta_2}_{\Phi} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with

$$\Phi(z) = \widetilde{\Theta}_2(\bar z)^* \Psi(\bar z) \widetilde{\Theta}_1(\bar z) = \Theta_2(z) \Psi(\bar z) \Theta_1(z)^* \quad \text{a.e. on } \mathbb{T}. \tag{4.3}$$

Hence $\Psi(z) = \Theta_2(\bar z)^* \Phi(\bar z) \Theta_1(\bar z)$ and (4.2) is satisfied. □

Denote

$$\widetilde{D}_\Theta = I - S^*_\Theta S_\Theta.$$

Applying Theorem 4.2 to the model operator S_Θ ($\Theta_1 = \Theta_2 = \Theta$) we obtain

$$\tau_\Theta S_\Theta \tau^*_\Theta = \tau_\Theta S_\Theta \tau_{\widetilde{\Theta}} = S^*_{\widetilde{\Theta}} \tag{4.4}$$

(see [22, p. 1001]). It follows that

$$\widetilde{D}_\Theta = \tau_{\widetilde{\Theta}} D_{\widetilde{\Theta}} \tau_\Theta = \tau_{\widetilde{\Theta}} D_{\widetilde{\Theta}} \tau^*_{\widetilde{\Theta}} \tag{4.5}$$

and by (3.1),

$$\widetilde{D}_\Theta f = \tau_{\widetilde{\Theta}}\big(\mathbf{k}^{\widetilde{\Theta}}_0 (\tau_\Theta f)(0)\big) \quad \text{for all } f \in K_\Theta.$$

For $\lambda \in \mathbb{D}$ let $\widetilde{\mathbf{k}}^\Theta_\lambda x = \tau_{\widetilde{\Theta}}(\mathbf{k}^{\widetilde{\Theta}}_{\bar\lambda} x)$, $x \in \mathcal{H}$. Then (a.e. on $\mathbb{T}$)

$$\begin{aligned}\widetilde{\mathbf{k}}^\Theta_\lambda(z) x &= \tau_{\widetilde{\Theta}}(\mathbf{k}^{\widetilde{\Theta}}_{\bar\lambda}(z) x) = \bar z \Theta(z) \mathbf{k}^{\widetilde{\Theta}}_{\bar\lambda}(\bar z) x\\ &= \tfrac{\bar z}{1-\lambda \bar z} \Theta(z)(I_{\mathcal{H}} - \widetilde{\Theta}(\bar z)\widetilde{\Theta}(\bar\lambda)^*) x\\ &= \tfrac{1}{z-\lambda} \Theta(z)(I_{\mathcal{H}} - \Theta(z)^*\Theta(\lambda)) x\\ &= \tfrac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda)) x \in K_\Theta \quad \text{for each } x \in \mathcal{H}.\end{aligned}$$

In particular,

$$\widetilde{\mathbf{k}}^\Theta_0(z) x = \bar z(\Theta(z) - \Theta(0)) x$$

and

$$\widetilde{D}_\Theta f = \widetilde{\mathbf{k}}_0^\Theta(\tau_\Theta f)(0) \in \widetilde{\mathcal{D}}_\Theta,$$

where

$$\widetilde{\mathcal{D}}_\Theta = \tau_{\widetilde{\Theta}}\mathcal{D}_{\widetilde{\Theta}} = \{\widetilde{\mathbf{k}}_0^\Theta x : x \in \mathcal{H}\} = \{\bar{z}(\Theta(z) - \Theta(0))x : x \in \mathcal{H}\}.$$

Observe that for $f \in K_\Theta$, $x \in \mathcal{H}$,

$$\begin{aligned}\langle f, \widetilde{\mathbf{k}}_\lambda^\Theta x\rangle_{L^2(\mathcal{H})} &= \langle f, \tau_{\widetilde{\Theta}}(\mathbf{k}_{\bar\lambda}^{\widetilde{\Theta}} x)\rangle_{L^2(\mathcal{H})} \\ &= \langle \tau_\Theta f, \mathbf{k}_{\bar\lambda}^{\widetilde{\Theta}} x\rangle_{L^2(\mathcal{H})} = \langle (\tau_\Theta f)(\bar\lambda), x\rangle_{\mathcal{H}}.\end{aligned}$$

It follows that for $f \in K_\Theta$ we have $M_z f \in K_\Theta$ if and only if $f \perp \widetilde{\mathcal{D}}_\Theta$. Indeed, $M_z f \in K_\Theta$ if and only if $\Theta P_+(\Theta^* M_z f)) = 0$. Since

$$(\Theta^* M_z f)(z) = \Theta(z)^* z f(z) = (\tau_\Theta f)(\bar{z}),$$

we have $P_+(\Theta^* M_z f)) = (\tau_\Theta f)(0)$ and so $M_z f \in K_\Theta$ if and only if

$$0 = \langle (\tau_\Theta f)(0), x\rangle = \langle f, \widetilde{\mathbf{k}}_0^\Theta x\rangle_{L^2(\mathcal{H})} \quad \text{for every} \quad x \in \mathcal{H},$$

i.e, $f \perp \widetilde{\mathcal{D}}_\Theta$. Therefore

$$(S_\Theta f)(z) = \begin{cases} z f(z) & \text{for } f \perp \widetilde{\mathcal{D}}_\Theta, \\ -\big(I_{\mathcal{H}} - \Theta(z)\Theta(0)^*\big)\Theta(0)x & \text{for } f = \widetilde{\mathbf{k}}_0^\Theta x \in \widetilde{\mathcal{D}}_\Theta. \end{cases}$$

Hence

$$\widetilde{D}_\Theta f = \begin{cases} 0 & \text{for } f \perp \widetilde{\mathcal{D}}_\Theta, \\ \widetilde{\mathbf{k}}_0^\Theta (I_{\mathcal{H}} - \Theta(0)\Theta(0)^*)x & \text{for } f = \widetilde{\mathbf{k}}_0^\Theta x \in \widetilde{\mathcal{D}}_\Theta. \end{cases}$$

Let us now consider conjugations. A conjugation J in a Hilbert space $\mathcal{H}$ is an antilinear map $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = I_{\mathcal{H}}$ and

$$\langle Jf, Jg\rangle = \langle g, f\rangle \quad \text{for all} \quad f, g \in \mathcal{H}$$

The importance of conjugations comes, for example, from their connection with complex symmetric operators. Recall that a bounded linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be J-symmetric (J being a conjugation on $\mathcal{H}$) if $JTJ = T^*$. We say that T is complex symmetric if it is J-symmetric with respect to some conjugation J (see, e.g., [14] for more details on conjugations and complex symmetric operators).

In [4] the authors consider certain classes of conjugations in $L^2(\mathcal{H})$. One such conjugation is $\mathbf{J}^* : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ defined for a fixed conjugation J in $\mathcal{H}$ by

$$(\mathbf{J}^* f)(z) = J(f(\bar{z})) \quad \text{a.e. on } \mathbb{T}. \tag{4.6}$$

It is not difficult to verify that for $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n \in L^2(\mathcal{L}(\mathcal{H}))$ we have

$$(\mathbf{J}^* f)(z)=\sum_{n=-\infty}^{\infty} J(a_n)z^n.$$

Hence, $\mathbf{J}^*$ is an M_z-commuting conjugation, i.e, $\mathbf{J}^* M_z = M_z \mathbf{J}^*$, and $\mathbf{J}^*(H^2(\mathcal{H})) = H^2(\mathcal{H})$, $\mathbf{J}^* P_+ = P_+ \mathbf{J}^*$ (see [4, Section 4]).

For $\mathbf{F} \in L^\infty(\mathcal{L}(\mathcal{H}))$ and an arbitrary conjugation J in $\mathcal{H}$ let

$$\mathbf{F}_J(z) = J\mathbf{F}(z)J \quad \text{a.e on } \mathbb{T}. \tag{4.7}$$

Then $\mathbf{F}_J \in L^\infty(\mathcal{L}(\mathcal{H}))$. As observed in [4], $\mathbf{F}_J \in H^\infty(\mathcal{L}(\mathcal{H}))$ if and only if $\mathbf{F} \in H^\infty(\mathcal{L}(\mathcal{H}))$, and $\mathbf{F}_J$ is an inner function if and only if $\mathbf{F}$ is. Clearly, $(\mathbf{F}_J)_J = \mathbf{F}$. Let us also observe that if $\mathbf{F}$ is J-symmetric, that is, $J\mathbf{F}(z)J = \mathbf{F}(z)^*$ a.e on $\mathbb{T}$ (or equivalently $\mathbf{F}(\lambda)$ is J-symmetric for λ in $\mathbb{D}$, see [4]), then $\mathbf{F}_J = \widetilde{\mathbf{F}}$, where $\widetilde{\mathbf{F}}(z) = \mathbf{F}(\bar{z})^*$.

For two conjugations J_1 and J_2 in $\mathcal{H}$ let $\mathbf{J}_1^*$ and $\mathbf{J}_2^*$ denote the corresponding conjugations in Ł$^2(\mathcal{H})$ given by (4.6). For each $f \in L^2(\mathcal{H})$ we have

$$(\mathbf{J}_2^* M_{\mathbf{F}} f)(z) = J_2(\mathbf{F}(\overline{z})f(\overline{z})) = J_2\mathbf{F}(\overline{z})J_1J_1f(\overline{z}) = (M_{\mathbf{G}}\mathbf{J}_1^* f)(z) \tag{4.8}$$

where $\mathbf{G}(z) = J_2\mathbf{F}(\overline{z})J_1$ a.e. on $\mathbb{T}$. In particular

$$\mathbf{J}^* M_{\mathbf{F}} = M_{\mathbf{F}} \mathbf{J}^* \tag{4.9}$$

(see [4, Lemma 8.3]).

Note that $\mathbf{F}_J$ is also defined for $\mathbf{F} \in L^2(\mathcal{L}(\mathcal{H}))$. In that case (4.8) and (4.9) hold for $L^\infty(\mathcal{H})$.

Proposition 4.3. *[4] Let $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ be a pure inner function and let J be a conjugation on $\mathcal{H}$. Then*

(a) $\mathbf{J}^*(\Theta H^2(\mathcal{H})) = \Theta_J H^2(\mathcal{H})$;
(b) $\mathbf{J}^* P_\Theta = P_{\Theta_J} \mathbf{J}^*$;
(c) $\mathbf{J}^*(K_\Theta) = K_{\Theta_J}$;
(d) $\mathbf{J}^*(\mathbf{k}_\lambda^\Theta x) = \mathbf{k}_{\overline{\lambda}}^{\Theta_\mathbf{J}} Jx$.

Proof. Clearly, (a) and (b) are consequences of (4.9), while (c) follows from (b). To see (d) take $\lambda \in \mathbb{D}$ and $x \in \mathcal{H}$. Then a.e. on $\mathbb{T}$ we have

$$\begin{aligned}(\mathbf{J}^*(\mathbf{k}_\lambda^\Theta x))(z) &= J(\mathbf{k}_\lambda^\Theta(\overline{z})x) = J(\frac{1}{1-\overline{\lambda}\overline{z}})(I_{\mathcal{H}} - \Theta(\overline{z})\Theta(\lambda)^*)x \\ &= \frac{1}{1-\lambda z}((I_{\mathcal{H}} - J\Theta(\overline{z})JJ\Theta(\lambda)^*)Jx) \\ &= \frac{1}{1-\lambda z}((I_{\mathcal{H}} - \Theta_J(z)\Theta_J(\overline{\lambda})^*)Jx) = \mathbf{k}_{\overline{\lambda}}^{\Theta_\mathbf{J}} Jx.\end{aligned}$$

□

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If Θ is J-symmetric we obtain [4, Proposition 7.7].

Theorem 4.4. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let J_1, J_2 be two conjugations on $\mathcal{H}$. A bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\mathbf{J}_2^* A \mathbf{J}_1^*$ belongs to $\mathcal{MT}((\Theta_1)_{J_1}, (\Theta_2)_{J_2})$. More precisely, $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\mathbf{J}_2^* A \mathbf{J}_1^* = A_\Psi^{(\Theta_1)_{J_1},(\Theta_2)_{J_2}} \in \mathcal{MT}((\Theta_1)_{J_1}, (\Theta_2)_{J_2})$ with*

$$\Psi(z) = J_2 \Phi(\overline{z}) J_1 \quad \text{a.e. on} \quad \mathbb{T}. \tag{4.10}$$

Proof. Assume that $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$. Let $f \in K^\infty_{(\Theta_1)_{J_1}}$. Note that $\mathbf{J}_1^* f \in K^\infty_{\Theta_1}$. Therefore, by Proposition 4.3(b) and (4.9),

$$\begin{aligned}\mathbf{J}_2^* A \mathbf{J}_1^* f &= \mathbf{J}_2^* P_{\Theta_2} M_\Phi \mathbf{J}_1^* f = P_{(\Theta_2)_{J_2}} \mathbf{J}_2^* M_\Phi \mathbf{J}_1^* f \\ &= P_{(\Theta_2)_{J_2}} M_\Psi f = A_\Psi^{(\Theta_1)_{J_1},(\Theta_2)_{J_2}} f\end{aligned}$$

with Ψ given by (4.10). Thus $\mathbf{J}_2^* A \mathbf{J}_1^* \in \mathcal{MT}((\Theta_1)_{J_1}, (\Theta_2)_{J_2})$.

On the other hand, if $A = \mathbf{J}_2^* A_\Psi^{(\Theta_1)_{J_1},(\Theta_2)_{J_2}} \mathbf{J}_1^* \in \mathcal{MT}((\Theta_1)_{J_1}, (\Theta_2)_{J_2})$ with some $\Psi \in L^2(\mathcal{L}(\mathcal{H}))$, then $A = \mathbf{J}_2^* A_\Psi^{(\Theta_1)_{J_1},(\Theta_2)_{J_2}} \mathbf{J}_1^*$ and as above, $A = A_\Phi^{\Theta_1,\Theta_2}$ with

$$\Phi(z) = J_2 \Psi(z) J_1 \quad \text{a.e. on } \mathbb{T}.$$

□

For the scalar case Theorem 4.4 can be found in [16] (see also [11] for the symmetric case).

In the scalar case each model space K_θ is equipped with a natural conjugation C_θ defined in terms of boundary functions by $(C_\theta f)(z) = \theta(z)\overline{z f(z)}$. If $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ is an inner function and J is a conjugation in $\mathcal{H}$ we can similarly define $\mathbf{C}_\Theta^J : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ by

$$(\mathbf{C}_\Theta^J f)(z) = \Theta(z)\overline{z} J(f(z)) \quad \text{a.e. on} \quad \mathbb{T}.$$

Although $\mathbf{C}_\Theta^J$ is obviously an antilinear isometry, it is not in general an involution. A simple computation shows that $\mathbf{C}_\Theta^J$ is an involution (and so a conjugation) if and only if $\Theta(z) J \Theta(z) J = I_\mathcal{H}$ a.e. on $\mathbb{T}$, i.e., if and only Θ is J-symmetric.

If Θ is J-symmetric, then $\mathbf{C}_\Theta^J(\Theta H^2(\mathcal{H})) = H^2(\mathcal{H})^\perp$ and so

$$\mathbf{C}_\Theta^J(K_\Theta) = K_\Theta.$$

Note that in that case

$$\mathbf{C}_\Theta^J = \mathbf{J}^* \tau_\Theta.$$

By Theorem 4.2 and Theorem 4.4 we get the following.

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Theorem 4.5. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let J_1, J_2 be two conjugations in $\mathcal{H}$ such that Θ_1 is J_1-symmetric and Θ_2 is J_2-symmetric. A bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $C_{\Theta_2}^{J_2} A C_{\Theta_1}^{J_1}$ belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$. More precisely, $A = A_\Phi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $C_{\Theta_2}^{J_2} A C_{\Theta_1}^{J_1} = A_\Psi^{\Theta_1,\Theta_2} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with*

$$\Psi(z) = J_2\Theta_2(z)^*\Phi(z)\Theta_1(z)J_1 = \Theta_2(z)J_2\Phi(z)J_1\Theta_1(z)^* \quad a.e.\ on\ \mathbb{T} \tag{4.11}$$

For the scalar version of Theorem 4.5 see [16].

Remark 4.6. Recall that in the scalar case $\mathcal{H} = \mathbb{C}$ every TTO on the model space K_θ is C_θ-symmetric, i.e.,

$$C_\theta A_\varphi^\theta C_\theta = (A_\varphi^\theta)^* = A_{\overline{\varphi}}^\theta$$

(see, e.g., [27]). In that case however the only conjugation in $\mathcal{H}$ we need to consider is $J(z) = \overline{z}$ (and each $\varphi \in L^2$ is J-symmetric). In the vector valued case, the equality

$$C_\Theta^J A_\Phi^\theta C_\Theta^J = A_{\Phi^*}^\Theta. \tag{4.12}$$

is not necessarily true for an arbitrary $\Phi \in L^2(\mathcal{L}(\mathcal{H}))$ (even though we assume here that Θ is J-symmetric). It is however satisfied if also Φ is J-symmetric and commutes with Θ (see [22]). In general, using Theorem 4.5 we have that (4.12) holds if and only if $A^\Theta{}_{\Theta\widetilde{\Phi_J^*}\Theta^*} = A_{\Phi^*}^\Theta$, that is, if and only if

$$\Theta\widetilde{\Phi_J^*}\Theta^* - \Phi^* \in \Theta H^2(\mathcal{L}(\mathcal{H})) + (\Theta H^2(\mathcal{L}(\mathcal{H})))^*.$$

5. Characterizations with compressed shift operators

In [22](see Theorem 5.2 and Remark 5.4) characterizations of matrix valued truncated Toeplitz operators in $\mathcal{MT}(\Theta)$ were given by using the model operators S_Θ, S_Θ^* and the defect operators D_Θ, $\widetilde{D}_\Theta$. These characterizations generalized D. Sarason's results for the scalar case [27]. Here we obtain analogous results for matrix valued asymmetric truncated Toeplitz operators from $\mathcal{MT}(\Theta_1, \Theta_2)$. We use a reasoning analogous to that from [22] (see [16] for the scalar case).

Lemma 5.1. *If $\Phi \in H^2((\mathcal{L}(\mathcal{H}))$, then*

$$A_\Phi^{\Theta_1,\Theta_2} - S_{\Theta_2} A_\Phi^{\Theta_1,\Theta_2} S_{\Theta_1}^* = P_{\Theta_2} M_\Phi (I_{H^2(\mathcal{H})} - SS^*) \quad on\ K_{\Theta_1}^\infty.$$

Proof. Recall that $S_\Theta = P_\Theta M_{z|K_\Theta}$ and $S_\Theta^* = P_+ M_{\bar{z}|K_\Theta}$. Hence, for $f \in K_{\Theta_1}^\infty$,

$$A_\Phi^{\Theta_1,\Theta_2} f - S_{\Theta_2} A_\Phi^{\Theta_1,\Theta_2} S_{\Theta_1}^* f = P_{\Theta_2} M_\Phi f - P_{\Theta_2} M_z P_{\Theta_2} M_\Phi P_{\Theta_1} M_{\bar{z}} f$$

(note that $S^*_{\Theta_1} f \in K^\infty_{\Theta_1}$). Since $P_{\Theta_2} M_z P_{\Theta_2} = P_{\Theta_2} M_z$ on $H^2(\mathcal{H})$ (as $M_z(\Theta_2 H^2(\mathcal{H})) \subset \Theta_2 H^2(\mathcal{H})$), we have

$$\begin{aligned} A^{\Theta_1,\Theta_2}_{\Phi} f - S_{\Theta_2} A^{\Theta_1,\Theta_2}_{\Phi} S^*_{\Theta_1} f &= P_{\Theta_2} M_\Phi f - P_{\Theta_2} M_z M_\Phi P_+ M_{\bar z} f \\ &= P_{\Theta_2}(M_\Phi - M_z M_\Phi P_+ M_{\bar z}) f \\ &= P_{\Theta_2}(M_\Phi - M_\Phi M_z P_+ M_{\bar z}) f \\ &= P_{\Theta_2} M_\Phi (I_{H^2(\mathcal{H})} - SS^*) f. \end{aligned}$$

□

Recall that

$$D_\Theta = I_{K_\Theta} - S_\Theta S^*_\Theta, \quad \widetilde{D}_\Theta = I_{K_\Theta} - S^*_\Theta S_\Theta$$

and

$$\mathcal{D}_\Theta = \{(I_{\mathcal{H}} - \Theta(z)\Theta(0)^*)x : x \in \mathcal{H}\}, \quad \widetilde{\mathcal{D}}_\Theta = \{\bar z(\Theta(z) - \Theta(0))x : x \in \mathcal{H}\},$$

while the operator $\Omega_\Theta : \mathcal{D}_\Theta \to \mathcal{H} \subset H^2(\mathcal{H})$ is defined by

$$\Omega_\Theta(\mathbf{k}^\Theta_0 x) = x.$$

Theorem 5.2. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. Then A belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if there exist bounded linear operators $B_1 : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$ and $B_2 : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}$, such that*

$$A - S_{\Theta_2} A S^*_{\Theta_1} = B_1 D_{\Theta_1} + D_{\Theta_2} B^*_2. \tag{5.1}$$

Proof. The proof follows the same line of reasoning as the proof of Theorem 5.2 in [22].

Assume first that $A \in \mathcal{MT}(\Theta_1, \Theta_2)$, $A = A^{\Theta_1,\Theta_2}_{\Psi + \Xi^*}$ with $\Psi, \Xi \in H^2(\mathcal{L}(\mathcal{H}))$. Then for each $f \in K^\infty_{\Theta_1}$ (note that $S^*_{\Theta_1} f \in K^\infty_{\Theta_1}$) we have

$$\begin{aligned} &(A - S_{\Theta_2} A S^*_{\Theta_1}) f = \\ &\qquad = (A^{\Theta_1,\Theta_2}_{\Psi} - S_{\Theta_2} A^{\Theta_1,\Theta_2}_{\Psi} S^*_{\Theta_1}) f + (A^{\Theta_1,\Theta_2}_{\Xi^*} - S_{\Theta_2} A^{\Theta_1,\Theta_2}_{\Xi^*} S^*_{\Theta_1}) f. \end{aligned}$$

Since $\Psi, \Xi \in H^2(\mathcal{L}(\mathcal{H}))$, it follows from Lemma 5.1 and (3.2) that

$$\begin{aligned} (A^{\Theta_1,\Theta_2}_{\Psi} - S_{\Theta_2} A^{\Theta_1,\Theta_2}_{\Psi} S^*_{\Theta_1}) f &= P_{\Theta_2} M_\Psi (I_{H^2(\mathcal{H})} - SS^*) f \\ &= P_{\Theta_2} M_\Psi \Omega_{\Theta_1} (I_{K_{\Theta_1}} - S_{\Theta_1} S^*_{\Theta_1}) f \\ &= B_1 D_{\Theta_1} f, \end{aligned}$$

where $B_1 = P_{\Theta_2} M_\Psi \Omega_{\Theta_1} : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$. Similarly, for each $g \in K^\infty_{\Theta_2}$,

$$\begin{aligned} (A^{\Theta_2,\Theta_1}_{\Xi} - S_{\Theta_1} A^{\Theta_2,\Theta_1}_{\Xi} S^*_{\Theta_2}) g &= P_{\Theta_1} M_\Xi (I_{H^2(\mathcal{H})} - SS^*) g \\ &= B_2 D_{\Theta_2} g, \end{aligned}$$

where $B_2 = P_{\Theta_1} M_\Xi \Omega_{\Theta_2} : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}$. Note that both B_1 and B_2 are bounded since Ω_{Θ_1} and Ω_{Θ_2} are bounded. Thus

$$(A^{\Theta_1,\Theta_2}_{\Xi^*} - S_{\Theta_2} A^{\Theta_1,\Theta_2}_{\Xi^*} S^*_{\Theta_1}) f = D_{\Theta_2} B_2^* f.$$

It follows that A satisfies (5.1).

Assume now that a bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ satisfies (5.1). Then

$$S^n_{\Theta_2} A S^{*n}_{\Theta_1} - S^{n+1}_{\Theta_2} A S^{*(n+1)}_{\Theta_1} = S^n_{\Theta_2} B_1 D_{\Theta_1} S^{*n}_{\Theta_1} + S^n_{\Theta_2} D_{\Theta_2} B_2^* S^{*n}_{\Theta_1}$$

for $n = 0, 1, ...$ We thus see that for each integer $N \geq 0$,

$$A = \sum_{n=0}^{N} \left(S^n_{\Theta_2} B_1 D_{\Theta_1} S^{*n}_{\Theta_1} + S^n_{\Theta_2} D_{\Theta_2} B_2^* S^{*n}_{\Theta_1} \right) + S^{N+1}_{\Theta_2} A S^{*(N+1)}_{\Theta_1}.$$

Hence, for for all $f \in K_{\Theta_1}$, and $g \in K_{\Theta_2}$ we have

$$\begin{aligned}\langle Af, g\rangle_{L^2(\mathcal{H})} =& \sum_{n=0}^{N} \left(\langle S^n_{\Theta_2} B_1 D_{\Theta_1} S^{*n}_{\Theta_1} f, g\rangle_{L^2(\mathcal{H})} + \langle S^n_{\Theta_2} D_{\Theta_2} B_2^* S^{*n}_{\Theta_1} f, g\rangle_{L^2(\mathcal{H})} \right) \\ &+ \langle A S^{*(N+1)}_{\Theta_1} f, S^{*(N+1)}_{\Theta_2} g\rangle_{L^2(\mathcal{H})}.\end{aligned}$$

Since $S^{*N}_{\Theta_1} f \to 0$ and $S^{*N}_{\Theta_2} g \to 0$ as $N \to \infty$, we obtain

$$\begin{aligned}&\langle Af, g\rangle_{L^2(\mathcal{H})} = \\ &= \sum_{n=0}^{\infty} \left(\langle S^n_{\Theta_2} B_1 D_{\Theta_1} S^{*n}_{\Theta_1} f, g\rangle_{L^2(\mathcal{H})} + \langle f, S^n_{\Theta_1} B_2 D_{\Theta_2} S^{*n}_{\Theta_2} g\rangle_{L^2(\mathcal{H})} \right).\end{aligned} \tag{5.2}$$

Let us now define $\Psi, \Xi \in H^2(\mathcal{L}(\mathcal{H}))$ a.e. on $\mathbb{T}$ by

$$\Psi(z)x = \left(B_1 \mathbf{k}_0^{\Theta_1} x\right)(z) \quad \text{and} \quad \Xi(z)x = \left(B_2 \mathbf{k}_0^{\Theta_2} x\right)(z), \quad x \in \mathcal{H}.$$

Take $f(z) = \sum_{k=0}^{\infty} a_n z^n \in K^\infty_{\Theta_1}$ and $g(z) = \sum_{k=0}^{\infty} b_n z^n \in K^\infty_{\Theta_2}$. We then have

$$D_{\Theta_1} S^{*n}_{\Theta_1} f = \mathbf{k}_0^{\Theta_1} (S^{*n}_{\Theta_1} f)(0) = (I_\mathcal{H} - \Theta_1 \Theta_1(0)^*) a_n = \mathbf{k}_0^{\Theta_1} a_n$$

and

$$D_{\Theta_2} S^{*n}_{\Theta_2} g = \mathbf{k}_0^{\Theta_2} (S^{*n}_{\Theta_2} g)(0) = (I_\mathcal{H} - \Theta_2 \Theta_2(0)^*) b_n = \mathbf{k}_0^{\Theta_2} b_n.$$

It follows from (5.2) that

$$\begin{aligned}
\langle A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} f, g\rangle_{L^2(\mathcal{H})} &= \\
&= \langle \Psi f, g\rangle_{L^2(\mathcal{H})} + \langle f, \Xi g\rangle_{L^2(\mathcal{H})} \\
&= \sum_{n=0}^{\infty} \big(\langle z^n \Psi a_n, g\rangle_{L^2(\mathcal{H})} + \langle f, z^n \Xi b_n\rangle_{L^2(\mathcal{H})}\big) \\
&= \sum_{n=0}^{\infty} \big(\langle z^n B_1 \mathbf{k}_0^{\Theta_1} a_n, g\rangle_{L^2(\mathcal{H})} + \langle f, z^n B_2 \mathbf{k}_0^{\Theta_2} b_n\rangle_{L^2(\mathcal{H})}\big) \\
&= \sum_{n=0}^{\infty} \big(\langle S^n_{\Theta_2} B_1 D_{\Theta_1} S^{*n}_{\Theta_1} f, g\rangle_{L^2(\mathcal{H})} + \langle f, S^n_{\Theta_1} B_2 D_{\Theta_2} S^{*n}_{\Theta_2} g\rangle_{L^2(\mathcal{H})}\big) \\
&= \langle A f, g\rangle_{L^2(\mathcal{H})}
\end{aligned}$$

and so $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} \in \mathcal{MT}(\Theta_1, \Theta_2)$. □

If a bounded operator $A : K_{\Theta_1} \to K_{\Theta_2}$ satisfies (5.1), then $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with $\Psi, \Xi \in H^2(\mathcal{L}(\mathcal{H}))$ given by the following:

Corollary 5.3. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator.*

(a) If $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} \in \mathcal{MT}(\Theta_1, \Theta_2)$, then A satisfies (5.1) with

$$B_1 = P_{\Theta_2} M_\Psi \Omega_{\Theta_1} \quad \text{and} \quad B_2 = P_{\Theta_1} M_\Xi \Omega_{\Theta_2}. \tag{5.3}$$

(b) If A satisfies (5.1), then $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with

$$\Psi(z)x = \big(B_1 \mathbf{k}_0^{\Theta_1} x\big)(z) \quad \text{and} \quad \Xi(z)x = \big(B_2 \mathbf{k}_0^{\Theta_2} x\big)(z), \quad x \in \mathcal{H}. \tag{5.4}$$

Remark 5.4. (a) Assume now that $\Psi \in H^2(\mathcal{L}(\mathcal{H}))$ is given by (5.4). Then, for each $\mathbf{H} \in H^2(\mathcal{L}(\mathcal{H}))$, we have

$$\begin{aligned}
\langle \Psi, \Theta_2 \mathbf{H}\rangle_{L^2(\mathcal{H})} &= \int_{\mathbb{T}} \langle \Psi(z), \Theta_2(z)\mathbf{H}(z)\rangle_2 dm(z) \\
&= \sum_{e\in\varepsilon} \int_{\mathbb{T}} \langle \Psi(z)e, \Theta_2(z)\mathbf{H}(z)e\rangle_{\mathcal{H}} dm(z) \\
&= \sum_{e\in\varepsilon} \langle B_1 \mathbf{k}_0^{\Theta_1} e, \Theta_2(\mathbf{H}(z)e)\rangle_{L^2(\mathcal{H})} = 0
\end{aligned}$$

since $B_1 \mathbf{k}_0^{\Theta_1} e \in K_{\Theta_2}$ and $\Theta_2(\mathbf{H}e) \in \Theta_2 H^2(\mathcal{H})$ (the function $\mathbf{H}e : z \to \mathbf{H}(z)e$ belongs to $H^2(\mathcal{H})$ since $\mathbf{H} \in H^2(\mathcal{L}(\mathcal{H}))$). Hence Ψ belongs to the orthogonal complement (in $H^2(\mathcal{L}(\mathcal{H}))$) of $\Theta_2 H^2(\mathcal{L}(\mathcal{H}))$. Similarly, Ξ belongs to the orthogonal complement of $\Theta_1 H^2(\mathcal{L}(\mathcal{H}))$.

For an inner function $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ denote

$$\mathcal{M}_\Theta = H^2(\mathcal{L}(\mathcal{H})) \ominus \Theta H^2(\mathcal{L}(\mathcal{H})).$$

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Therefore, if a bounded linear operator $A : K_{\Theta_1} \to K_{\Theta_2}$ satisfies (5.1), then $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*} \in \mathcal{MT}(\Theta_1, \Theta_2)$ with $\Psi \in \mathcal{M}_{\Theta_2}$ and $\Xi \in \mathcal{M}_{\Theta_1}$ given by (5.4).

(b) Recall that $A^{\Theta_1,\Theta_2}_{\Phi} = 0$ if and only if

$$\Phi \in \Theta_2 H^2(\mathcal{L}(\mathcal{H})) + (\Theta_1 H^2(\mathcal{L}(\mathcal{H})))^*$$

(see [21]). Clearly, if $A = A^{\Theta_1,\Theta_2}_{\Psi+\Xi^*}$ with $\Psi, \Xi \in H^2(\mathcal{L}(\mathcal{H}))$, then the operators B_1 and B_2 given by (5.3) do not depend on the parts of Ψ and Ξ that belong to $\Theta_2 H^2(\mathcal{L}(\mathcal{H}))$ and $\Theta_1 H^2(\mathcal{L}(\mathcal{H}))$, respectively.

As in [22] we can use the unitary operator τ_Θ defined by (4.1) and obtain the following theorem.

Theorem 5.5. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. Then A belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if there exist bounded linear operators $\widetilde{B}_1 : \widetilde{\mathcal{D}}_{\Theta_1} \to K_{\Theta_2}$ and $\widetilde{B}_2 : \widetilde{\mathcal{D}}_{\Theta_2} \to K_{\Theta_1}$, such that*

$$A - S^*_{\Theta_2} A S_{\Theta_1} = \widetilde{B}_1 \widetilde{D}_{\Theta_1} + \widetilde{D}_{\Theta_2} \widetilde{B}^*_2. \tag{5.5}$$

Proof. Let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. By Theorem 4.2, A belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\widetilde{A} = \tau_{\Theta_2} A \tau^*_{\Theta_1}$ belongs to $\mathcal{MT}(\widetilde{\Theta}_1, \widetilde{\Theta}_2)$. By Theorem 5.2 the latter happens if and only if there exist bounded linear operators $B_1 : \mathcal{D}_{\widetilde{\Theta}_1} \to K_{\widetilde{\Theta}_2}$ and $B_2 : \mathcal{D}_{\widetilde{\Theta}_2} \to K_{\widetilde{\Theta}_1}$, such that

$$\widetilde{A} - S_{\widetilde{\Theta}_2} \widetilde{A} S^*_{\widetilde{\Theta}_1} = \tau_{\Theta_2} A \tau^*_{\Theta_1} - S_{\widetilde{\Theta}_2} \tau_{\Theta_2} A \tau^*_{\Theta_1} S^*_{\widetilde{\Theta}_1} = B_1 D_{\widetilde{\Theta}_1} + D_{\widetilde{\Theta}_2} B^*_2. \tag{5.6}$$

In other words,

$$A - \tau^*_{\Theta_2} S_{\widetilde{\Theta}_2} \tau_{\Theta_2} A \tau^*_{\Theta_1} S^*_{\widetilde{\Theta}_1} \tau_{\Theta_1} = \tau^*_{\Theta_2} B_1 D_{\widetilde{\Theta}_1} \tau_{\Theta_1} + \tau^*_{\Theta_2} D_{\widetilde{\Theta}_2} B^*_2 \tau_{\Theta_1}.$$

By (4.4) we have

$$\tau^*_{\Theta_2} S_{\widetilde{\Theta}_2} \tau_{\Theta_2} = \tau_{\widetilde{\Theta}_2} S_{\widetilde{\Theta}_2} \tau^*_{\widetilde{\Theta}_2} = S^*_{\Theta_2} \quad \text{and} \quad \tau^*_{\Theta_1} S^*_{\widetilde{\Theta}_1} \tau_{\Theta_1} = \tau_{\widetilde{\Theta}_1} S_{\widetilde{\Theta}_1} \tau^*_{\widetilde{\Theta}_1} = S_{\Theta_1},$$

while from (4.5) it follows that

$$D_{\widetilde{\Theta}_1} \tau_{\Theta_1} = \tau_{\Theta_1} \widetilde{D}_{\Theta_1} \quad \text{and} \quad \tau^*_{\Theta_2} D_{\widetilde{\Theta}_2} = \widetilde{D}_{\Theta_2} \tau^*_{\Theta_2}.$$

Thus (5.6) is equivalent to

$$A - S^*_{\Theta_2} A S_{\Theta_1} = \tau^*_{\Theta_2} B_1 \tau_{\Theta_1} \widetilde{D}_{\Theta_1} + \widetilde{D}_{\Theta_2} \tau^*_{\Theta_2} B^*_2 \tau_{\Theta_1} = \widetilde{B}_1 \widetilde{D}_{\Theta_1} + \widetilde{D}_{\Theta_2} \widetilde{B}^*_2.$$

with

$$\widetilde{B}_1 = \tau^*_{\Theta_2} B_1 \tau_{\Theta_1 | \widetilde{\mathcal{D}}_{\Theta_1}}, \quad \widetilde{B}_1 : \widetilde{\mathcal{D}}_{\Theta_1} \to K_{\Theta_2}$$

and

$$\widetilde{B}_2 = \left(\tau^*_{\Theta_2} B^*_2 \tau_{\Theta_1}\right)^* = \tau^*_{\Theta_1} B_2 \tau_{\Theta_2 | \widetilde{\mathcal{D}}_{\Theta_2}}, \quad \widetilde{B}_2 : \widetilde{\mathcal{D}}_{\Theta_2} \to K_{\Theta_1}.$$

Note that $\tau^*_{\Theta_i}\mathcal{D}_{\widetilde{\Theta_i}} = \widetilde{\mathcal{D}}_{\Theta_i}$, $i = 1, 2$. This allows us to treat $\tau^*_{\Theta_2} B^*_2 \tau_{\Theta_1}$ as an operator from K_{Θ_1} to $\widetilde{\mathcal{D}}_{\Theta_2}$. Moreover, we have

$$B_1 = \tau_{\Theta_2} \widetilde{B}_1 \tau^*_{\Theta_1|\mathcal{D}_{\widetilde{\Theta}_1}} \quad \text{and} \quad B_2 = \tau_{\Theta_1} \widetilde{B}_2 \tau^*_{\Theta_2|\mathcal{D}_{\widetilde{\Theta}_2}}. \tag{5.7}$$

□

Note from the proof of Theorem 5.5 that if $A : K_{\Theta_1} \to K_{\Theta_2}$ satisfies (5.5) with some $\widetilde{B}_1 : \widetilde{\mathcal{D}}_{\Theta_1} \to K_{\Theta_2}$ and $\widetilde{B}_2 : \widetilde{\mathcal{D}}_{\Theta_2} \to K_{\Theta_1}$, then $\widetilde{A} = \tau_{\Theta_2} A \tau^*_{\Theta_1}$ satisfies (5.6) with B_1 and B_2 given by (5.7). By Corollary 5.3, $\widetilde{A} = A^{\widetilde{\Theta}_1,\widetilde{\Theta}_2}_{\Psi+\Xi^*}$ with

$$\Psi(z)x = (B_1 \mathbf{k}_0^{\widetilde{\Theta}_1} x)(z) = (\tau_{\Theta_2} \widetilde{B}_1 \tau^*_{\Theta_1} \mathbf{k}_0^{\widetilde{\Theta}_1} x)(z) = (\tau_{\Theta_2} \widetilde{B}_1 \widetilde{\mathbf{k}}_0^{\Theta_1} x)(z)$$

and

$$\Xi(z)x = (B_2 \mathbf{k}_0^{\widetilde{\Theta}_1} x)(z) = (\tau_{\Theta_1} \widetilde{B}_2 \tau^*_{\Theta_2} \mathbf{k}_0^{\widetilde{\Theta}_2} x)(z) = (\tau_{\Theta_1} \widetilde{B}_2 \widetilde{\mathbf{k}}_0^{\Theta_2} x)(z).$$

Moreover (see Remark 5.4), $\Psi \in \mathcal{M}_{\widetilde{\Theta}_2}$ and $\Xi \in \mathcal{M}_{\widetilde{\Theta}_1}$.

It follows from Theorem 4.2 (see (4.3)) that $A = A^{\Theta_1,\Theta_2}_{\Phi}$ with

$$\begin{aligned}\Phi(z) &= \Theta_2(z)(\Psi(\overline{z}) + \Xi(\overline{z})^*)\Theta_1(z)^* \\ &= \Theta_2(z)\Psi(\overline{z})\Theta_1(z)^* + \Theta_2(z)\Xi(\overline{z})^*\Theta_1(z)^* \\ &= \Theta_2(z)\widetilde{\Xi}(z)\Theta_1(z)^* + (\Theta_1(z)\widetilde{\Psi}(z)\Theta_2(z)^*)^*.\end{aligned}$$

By Lemma 5.6 below, $\Phi = \Psi_1 + \Xi_1$ with $\Psi_1 = \Theta_2 \widetilde{\Xi} \Theta_1^* \in \Theta_2(z\mathcal{M}_{\Theta_1})^*$ and $\Xi_1 = \Theta_1 \widetilde{\Psi} \Theta_2^* \in \Theta_1(z\mathcal{M}_{\Theta_2})^*$.

Lemma 5.6. *Let $\Phi \in H^2(\mathcal{L}(\mathcal{H}))$. If $\Phi \in \mathcal{M}_\Theta$, then $\widetilde{\Phi}\widetilde{\Theta}^* \in (z\mathcal{M}_{\widetilde{\Theta}})^*$.*

Proof. We will show that if $\Phi \in \mathcal{M}_\Theta$, then $\Psi(z) = \widetilde{\Theta}(z)\overline{z}\Phi(\overline{z}) \in \mathcal{M}_{\widetilde{\Theta}}$. Let $\mathbf{H} \in H^2(\mathcal{L}(\mathcal{H}))$. Then

$$\begin{aligned}\langle \Psi, (z\mathbf{H})^* \rangle_{L^2(\mathcal{L}(\mathcal{H}))} &= \int_{\mathbb{T}} \langle \Psi, \overline{z}\mathbf{H}(z)^* \rangle_2 dm(z) \\ &= \int_{\mathbb{T}} \langle \widetilde{\Theta}(z)\overline{z}\Phi(\overline{z}), \overline{z}\mathbf{H}(z)^* \rangle_2 dm(z) \\ &= \int_{\mathbb{T}} \langle \Theta(z)^*\Phi(z), \widetilde{\mathbf{H}}(z) \rangle_2 dm(z) \\ &= \int_{\mathbb{T}} \langle \Phi(z), \Theta(z)\widetilde{\mathbf{H}}(z) \rangle_2 dm(z) \\ &= \langle \Phi, z\widetilde{\mathbf{H}} \rangle_{L^2(\mathcal{L}(\mathcal{H}))} = 0,\end{aligned}$$

Moreover,

$$\begin{aligned}\langle \Psi, \widetilde{\Theta}\mathbf{H}\rangle_{L^2(\mathcal{L}(\mathcal{H}))} &= \int_{\mathbb{T}} \langle \widetilde{\Theta}(z)\overline{z}\Phi(\overline{z}), \widetilde{\Theta}\mathbf{H}(z)\rangle_2 dm(z) \\ &= \int_{\mathbb{T}} \langle \widetilde{\Phi}(z)^*, z\mathbf{H}(z)\rangle_2 dm(z) \\ &= \langle \widetilde{\Phi}^*, z\mathbf{H}\rangle_{L^2(\mathcal{L}(\mathcal{H}))} = 0,\end{aligned}$$

which means that $\Psi \in \mathcal{M}_{\widetilde{\Theta}}$. □

As in the scalar case, we can use Theorem 5.2 and Theorem 5.5 to get the following.

Corollary 5.7. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. Then A belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if the following hold:*

(a) there exist bounded linear operators $\widehat{B}_1 : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$ and $\widehat{B}_2 : \widetilde{\mathcal{D}}_{\Theta_2} \to K_{\Theta_1}$, such that

$$S^*_{\Theta_2}A - AS^*_{\Theta_1} = \widehat{B}_1 D_{\Theta_1} + \widetilde{D}_{\Theta_2}\widehat{B}^*_2.$$

(b) there exist bounded linear operators $\widehat{B}_1 : \widetilde{\mathcal{D}}_{\Theta_1} \to K_{\Theta_2}$ and $\widehat{B}_1 : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}$, such that

$$S_{\Theta_2}A - AS_{\Theta_1} = \widehat{B}_1 \widetilde{D}_{\Theta_1} + D_{\Theta_2}\widehat{B}^*_2.$$

Proof. The proof is similar to the scalar case (see [16]). To prove (a) assume first that $A \in \mathcal{MT}(\Theta_1, \Theta_2)$. Then, by Theorem 5.2, there exist bounded linear operators $B_1 : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$ and $B_2 : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}$, such that

$$A - S_{\Theta_2}AS^*_{\Theta_1} = B_1 D_{\Theta_1} + D_{\Theta_2}B^*_2.$$

Hence

$$S^*_{\Theta_2}A - S^*_{\Theta_2}S_{\Theta_2}AS^*_{\Theta_1} = S^*_{\Theta_2}B_1 D_{\Theta_1} + S^*_{\Theta_2}D_{\Theta_2}B^*_2,$$

and since $S^*_{\Theta}S_{\Theta_2} = I_{K_{\Theta_2}} - \widetilde{D}_{\Theta_2}$, we get

$$A - S_{\Theta_2}AS^*_{\Theta_1} = S^*_{\Theta_2}B_1 D_{\Theta_1} + S^*_{\Theta_2}D_{\Theta_2}B^*_2 - D_{\Theta_2}AS^*_{\Theta_1}.$$

Observe now that $S^*_{\Theta_2}D_{\Theta_2} = \widetilde{D}_{\Theta_2}S^*_{\Theta_2}$ and $\widetilde{D}_{\Theta_2} = \widetilde{D}_{\Theta_2}P_{\widetilde{D}_{\Theta_2}}$, where $P_{\widetilde{D}_{\Theta_2}}$ is the orthogonal projection from K_{Θ_2} to $\widetilde{D}_{\Theta_2}$ (see the formula for $\widetilde{D}_{\Theta_2}$

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on page 11). It follows that

$$\begin{aligned} A - S_{\Theta_2} A S^*_{\Theta_1} &= S^*_{\Theta_2} B_1 D_{\Theta_1} + \widetilde{D}_{\Theta_2}(S^*_{\Theta_2} B^*_2 - A S^*_{\Theta_1}) \\ &= \widehat{B}_1 D_{\Theta_1} + \widehat{D}_{\Theta_2} \widehat{B}^*_2, \end{aligned}$$

where

$$\widehat{B}_1 = S^*_{\Theta_2} B_1 : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}.$$

and

$$\widehat{B}_2 = (P_{\widetilde{D}\Theta_2}(S^*_{\Theta_2} B^*_2 - A S^*_{\Theta_1}))^* : \widetilde{\mathcal{D}}_{\Theta_2} \to K_{\Theta_1}.$$

The proof of the other implication is analogous.

To prove (b) one can apply the same reasoning together with Theorem 5.5. Alternatively, one can use the fact that $A \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if $\tau_{\Theta_2} A \tau^*_{\Theta_1} \in \mathcal{MT}(\widetilde{\Theta}_1, \widetilde{\Theta}_2)$ to show that (b) is equivalent to (a). □

6. Shift invariance and MATTO's

In the scalar case the notion of shift invariance for TTO's was introduced in [27]. D. Sarason proved that a bounded linear operator $A : K_\theta \to K_\theta$ is a TTO if and only if it is shift invariant, i.e.,

$$\langle ASf, Sf\rangle_{L^2} = \langle Af, f\rangle_{L^2} \quad \text{for each } f \in K_\theta \text{ such that } \ Sf \in K_\theta.$$

In [22] we prove that the same is true for MTTO's.

For ATTO's the notion of shift invariance was introduced in [8] (see also [24]). Here we consider shift invariance of MATTO's. As in the scalar case (see [16]), we characterize MATTO's in term of four (equivalent) types of shift invariance.

Recall that for an operator valued inner function $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ and for $f \in K_\Theta$ we have

$$Sf = M_z f \in K_\Theta \text{ if and only if } f \perp \widetilde{\mathcal{D}}_\Theta \ (\tau_\Theta f(0) = 0)$$

and

$$S^* f = M_{\overline{z}} f \in K_\Theta \text{ if and only if } f \perp \mathcal{D}_\Theta \ (f(0) = 0).$$

Theorem 6.1. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions and let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded linear operator. Then A belongs to $\mathcal{MT}(\Theta_1, \Theta_2)$ if and only if it has one (and all) of the following properties:*

(a) $\langle AS^* f, S^* g\rangle_{L^2(\mathcal{H})} = \langle Af, g\rangle_{L^2(\mathcal{H})}$ for all $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \perp \mathcal{D}_{\Theta_2}$;

(b) $\langle AS^* f, g\rangle_{L^2(\mathcal{H})} = \langle Af, Sg\rangle_{L^2(\mathcal{H})}$ for all $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \perp \widetilde{\mathcal{D}}_{\Theta_2}$;

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(c) $\langle ASf, Sg\rangle_{L^2(\mathcal{H})} = \langle Af, g\rangle_{L^2(\mathcal{H})}$ *for all* $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ *such that* $f \perp \widetilde{\mathcal{D}}_{\Theta_1}$, $g \perp \widetilde{\mathcal{D}}_{\Theta_2}$*;*

(d) $\langle ASf, g\rangle_{L^2(\mathcal{H})} = \langle Af, S^*g\rangle_{L^2(\mathcal{H})}$ *for all* $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ *such that* $f \perp \widetilde{\mathcal{D}}_{\Theta_1}$, $g \perp \mathcal{D}_{\Theta_2}$*;*

Proof. (a) If $A \in \mathcal{MT}(\Theta_1, \Theta_2)$, then by Theorem 5.2,

$$A - S_{\Theta_2} A S^*_{\Theta_1} = B_1 D_{\Theta_1} + D_{\Theta_2} B_2^*$$

for some bounded linear operators $B_1 : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$ and $B_2 : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}$. It follows that for all $f \in K_{\Theta_1}$, $f \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \perp \mathcal{D}_{\Theta_2}$, we have

$$\begin{aligned}\langle AS^*f, S^*g\rangle_{L^2(\mathcal{H})} &= \langle AS^*_{\Theta_1} f, S^*_{\Theta_2} g\rangle_{L^2(\mathcal{H})} = \langle S_{\Theta_2} AS^*_{\Theta_2} f, g\rangle_{L^2(\mathcal{H})} \\ &= \langle Af, g\rangle_{L^2(\mathcal{H})} - \langle B_1 D_{\Theta_1} f, g\rangle_{L^2(\mathcal{H})} - \langle D_{\Theta_2} B_2^* f, g\rangle_{L^2(\mathcal{H})}.\end{aligned}$$

Since $D_{\Theta_1} f = 0$ and $D_{\Theta_2} B_2^* f \in \mathcal{D}_{\Theta_2}$, we get

$$\langle AS^*f, S^*g\rangle_{L^2(\mathcal{H})} = \langle Af, g\rangle_{L^2(\mathcal{H})} \tag{6.1}$$

On the other hand, if (6.1) holds for all $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \perp \mathcal{D}_{\Theta_2}$, we have

$$\langle (A - S_{\Theta_2} AS^*_{\Theta_1}) f, g\rangle_{L^2(\mathcal{H})} = \langle Af, g\rangle_{L^2(\mathcal{H})} - \langle AS^*f, S^*g\rangle_{L^2(\mathcal{H})} = 0.$$

This means that the operator $\mathbf{T}_A = A - S_{\Theta_2} AS^*_{\Theta_1}$ maps $\mathcal{D}^\perp_{\Theta_1}$ into $\mathcal{D}_{\Theta_2}$, or in other words,

$$(I_{K_{\Theta_2}} - P_{\mathcal{D}_{\Theta_2}}) \mathbf{T}_A (I_{K_{\Theta_1}} - P_{\mathcal{D}_{\Theta_1}}) = 0, \tag{6.2}$$

where $P_{\mathcal{D}_{\Theta_i}}$ is the orthogonal projection from K_{Θ_i} onto $\mathcal{D}_{\Theta_i}$, $i = 1, 2$. Recall now that

$$\text{Range} P_{\mathcal{D}_{\Theta_i}} = \mathcal{D}_{\Theta_i} = \text{Range} D_{\Theta_i},\ i = 1, 2,$$

and so there exist bounded linear operators $R_i : K_{\Theta_i} \to K_{\Theta_i}$, $i = 1, 2$, such that

$$P_{\mathcal{D}_{\Theta_i}} = D_{\Theta_i} R_i = R_i^* D_{\Theta_i},\ i = 1, 2$$

(the second equality follows from the fact that $P^*_{\mathcal{D}_{\Theta_i}} = P_{\mathcal{D}_{\Theta_i}}$). Together with (6.2) this gives

$$\begin{aligned}A - S_{\Theta_2} AS^*_{\Theta_1} = \mathbf{T}_A &= P_{\mathcal{D}_{\Theta_2}} \mathbf{T}_A + \mathbf{T}_A P_{\mathcal{D}_{\Theta_2}} - P_{\mathcal{D}_{\Theta_2}} \mathbf{T}_A P_{\mathcal{D}_{\Theta_1}} \\ &= D_{\Theta_2} R_2 \mathbf{T}_A + (I_{K_{\Theta_2}} - P_{\mathcal{D}_{\Theta_2}}) \mathbf{T}_A R_1^* D_{\Theta_1}\end{aligned}$$

and so A satisfies (5.1) with

$$B_1 = (I_{K_{\Theta_2}} - P_{\mathcal{D}_{\Theta_2}}) \mathbf{T}_A R^*_{1|\mathcal{D}_{\Theta_1}} : \mathcal{D}_{\Theta_1} \to K_{\Theta_2}$$

and

$$B_2 = (P_{\mathcal{D}_{\Theta_2}} R_2 \mathbf{T}_A)^* = \mathbf{T}_A^* R^*_{2|\mathcal{D}_{\Theta_2}} : \mathcal{D}_{\Theta_2} \to K_{\Theta_1}.$$

By Theorem 5.2, $A \in \mathcal{MT}(\Theta_1, \Theta_2)$.

(b) Here we show that (b) is equivalent to (a). Assume first that A satisfies (a) and take $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \in \widetilde{\mathcal{D}}_{\Theta_2}$. Clearly, $g = S^*Sg$ and $Sg \in K_{\Theta_2}$, $Sg \perp \mathcal{D}_{\Theta_2}$. Hence

$$\langle AS^*f, g\rangle_{L^2(\mathcal{H})} = \langle AS^*f, S^*Sg\rangle_{L^2(\mathcal{H})} = \langle Af, Sg\rangle_{L^2(\mathcal{H})}$$

and A satisfies (b).

Similarly, if A satisfies (b), then for each $f \in K_{\Theta_1}$, $g \in K_{\Theta_2}$ such that $f \perp \mathcal{D}_{\Theta_1}$, $g \in \mathcal{D}_{\Theta_2}$ we have

$$\langle AS^*f, S^*g\rangle_{L^2(\mathcal{H})} = \langle Af, SS^*g\rangle_{L^2(\mathcal{H})} = \langle Af, g\rangle_{L^2(\mathcal{H})}$$

since here $S^*f \perp \widetilde{\mathcal{D}}_{\Theta_2}$ and $S^*Sg = g$.

The proof of (c) and (d) is analogous to the proof of (a) and (b). □

7. Characterization with modified compressed shift operators

Modified compressed shifts were introduced by Sarason in [27, section 10]. For any nonconstant inner function Θ, suppose that $X_\Theta : \widetilde{\mathcal{D}}_\Theta \to \mathcal{D}_\Theta$, and consider $\widehat{X}_\Theta \in \mathcal{L}(K_\Theta)$ defined by $\widehat{X}_\Theta f = X_\Theta P_{\widetilde{\mathcal{D}}_\Theta} f$. The operator modified shift is defined by

$$S_{\Theta,X_\Theta} = S_\Theta + (\widehat{X}_\Theta - S_\Theta)P_{\widetilde{\mathcal{D}}_\Theta},$$

or

$$S_{\Theta,X_\Theta} = S_\Theta + P_{\mathcal{D}_\Theta} Y_\Theta P_{\widetilde{\mathcal{D}}_\Theta},$$

which implies that

$$S_\Theta = S_{\Theta,X_\Theta} - P_{\mathcal{D}_\Theta} Y_\Theta P_{\widetilde{\mathcal{D}}_\Theta}$$

where $Y_\Theta = \widehat{X}_\Theta - S_\Theta$.

Theorem 7.1. *Let $\Theta_1, \Theta_2 \in H^\infty(\mathcal{L}(\mathcal{H}))$ be two pure inner functions. Let $A : K_{\Theta_1} \to K_{\Theta_2}$ be a bounded operator. Then $A \in \mathcal{MT}(\Theta_1, \Theta_2)$ if and only if*

$$A - S_{\Theta_2,X_{\Theta_2}} A S^*_{\Theta_1,X_{\Theta_1}} = BP_{\mathcal{D}_{\Theta_1}} + P_{\mathcal{D}_{\Theta_2}} B^{'*}. \tag{7.1}$$

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Proof. Consider

$$\begin{aligned}
A - S_{\Theta_2} A S_{\Theta_1}^* &= A - (S_{\Theta_2, X_{\Theta_2}} - P_{\mathcal{D}_{\Theta_2}} Y_{\Theta_2} P_{\widetilde{\mathcal{D}}_{\Theta_2}}) A (S_{\Theta_1, X_{\Theta_1}}^* - P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}}) \\
&= A - S_{\Theta_2, X_{\Theta_2}} A S_{\Theta_1, X_{\Theta_1}}^* + S_{\Theta_2, X_{\Theta_2}} P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}} + P_{\mathcal{D}_{\Theta_2}} Y_{\Theta_2} P_{\widetilde{\mathcal{D}}_{\Theta_2}} A S_{\Theta_1, X_{\Theta_1}}^* \\
&- P_{\mathcal{D}_{\Theta_2}} Y_{\Theta_2} P_{\widetilde{\mathcal{D}}_{\Theta_2}} A P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}} \\
&= A - S_{\Theta_2, X_{\Theta_2}} A S_{\Theta_1, X_{\Theta_1}}^* + S_{\Theta_2, X_{\Theta_2}} P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}} \\
&+ P_{\mathcal{D}_{\Theta_2}} [Y_{\Theta_1}^* P_{\widetilde{\mathcal{D}}_{\Theta_2}} A S_{\Theta_1, X_{\Theta_1}}^* - Y_{\Theta_2}^* P_{\widetilde{\mathcal{D}}_{\Theta_2}} A P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}}] \\
&= B P_{\mathcal{D}_{\Theta_1}} + P_{\mathcal{D}_{\Theta_2}} B'^* + T_1 P_{\mathcal{D}_{\Theta_1}} + P_{\mathcal{D}_{\Theta_2}} T_2 \\
&= (B + T_1) P_{\mathcal{D}_{\Theta_1}} + P_{\mathcal{D}_{\Theta_2}} (B'^* + T_2),
\end{aligned}$$

where $T_1 = S_{\Theta_2, X_{\Theta_2}} P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^*$ and $T_2 = Y_{\Theta_1}^* P_{\widetilde{\mathcal{D}}_{\Theta_2}} A S_{\Theta_1, X_{\Theta_1}}^* - Y_{\Theta_2}^* P_{\widetilde{\mathcal{D}}_{\Theta_2}} A P_{\widetilde{\mathcal{D}}_{\Theta_1}} Y_{\Theta_1}^* P_{\mathcal{D}_{\Theta_1}}$. From equation (3.8) of [22] it follows that there is an operator $J_{\Theta_1} \in \mathcal{L}(K_{\Theta_1})$ such that

$$P_{\mathcal{D}_{\Theta_1}} = (I - S_{\Theta_1} S_{\Theta_1}^*) J_{\Theta_1} = D_{\Theta_1} J_{\Theta_1} = J_{\Theta_1}^* D_{\Theta_1},$$

and similarly there is $J_{\Theta_2} \in \mathcal{L}(K_{\Theta_2})$ such that

$$P_{\mathcal{D}_{\Theta_2}} = (I - S_{\Theta_2} S_{\Theta_2}^*) J_{\Theta_2} = D_{\Theta_2} J_{\Theta_2} = J_{\Theta_2}^* D_{\Theta_2}.$$

Then we have

$$\begin{aligned}
A - S_{\Theta_2} A S_{\Theta_1}^* &= (B + T_1) J_{\Theta_1}^* D_{\Theta_1} + D_{\Theta_2} J_{\Theta_2} (B'^* + T_2) \\
&= (B + T_1) J_{\Theta_1}^* D_{\Theta_1} + D_{\Theta_2} [(B' + T_2^*) J_{\Theta_2}^*]^* \\
&= \mathbf{B} D_{\Theta_1} + D_{\Theta_2} \mathbf{B}'^*
\end{aligned}$$

where $\mathbf{B} = (B + T_1) J_{\Theta_1}^*$ and $\mathbf{B}' = (B' + T_2^*) J_{\Theta_2}^*$. The required result follows from this and Theorem 5.2. □

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