The Cesàro-like operator on some analytic function spaces

Pengcheng Tang*

Hunan University of Science and Technology, Xiangtan, Hunan 411201, China

ABSTRACT

Let μ be a finite positive Borel measure on the interval $[0,1)$ and $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$. The Cesàro-like operator is defined by

$$\mathcal{C}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}a_k\right)z^n,\ z\in\mathbb{D},$$

where, for $n\geq 0$, μ_n denotes the n-th moment of the measure μ, that is, $\mu_n=\int_{[0,1)}t^nd\mu(t)$. Let X and Y be subspaces of $H(\mathbb{D})$, the purpose of this paper is to study the action of $\mathcal{C}_\mu$ on distinct pairs (X,Y). The spaces considered in this paper are Hardy space $H^p(0<p\leq\infty)$, Morrey space $L^{2,\lambda}(0<\lambda\leq 1)$, mean Lipschitz space, Bloch type space, etc.

Keywords: Cesàro-like operator, Carleson measure, Hardy spaces; Morrey space.

MSC 2010: 47B38, 30H10

1 Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and $dA(z)=\frac{1}{\pi}dxdy$ the normalized area Lebesgue measure.

For $0<\alpha<\infty$, the Bloch-type space, denoted by $\mathcal{B}^\alpha$, is defined as

$$\mathcal{B}^\alpha=\{f\in H(\mathbb{D}):||f||_{\mathcal{B}^\alpha}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty\}.$$

If $\alpha=1$, then $\mathcal{B}^\alpha$ is just the classic Bloch space $\mathcal{B}$.

*Corresponding Author

Pengcheng Tang: www.tang-tpc.com@foxmail.com

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

Let $0 < p \leq \infty$, the classical Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ for which

$$||f||_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \ 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

Let $I \subset \partial\mathbb{D}$ be an arc, and $|I|$ denote the length of I. The Carleson square $S(I)$ is defined as

$$S(I) = \{re^{i\vartheta} : e^{i\vartheta} \in I, \ 1 - \frac{|I|}{2\pi} \leq r < 1\}.$$

Let μ be a positive Borel measure on $\mathbb{D}$. For $0 \leq \beta < \infty$ and $0 < t < \infty$, we say that μ is a β-logarithmic t-Carleson measure (resp.a vanishing β-logarthmic t-Carleson measure) if

$$\sup_{|I| \subset \partial\mathbb{D}} \frac{\mu(S(I))(\log\frac{2\pi}{|I|})^\beta}{|I|^t} < \infty, \ \text{ resp. } \lim_{|I| \to 0} \frac{\mu(S(I))(\log\frac{2\pi}{|I|})^\beta}{|I|^t} = 0.$$

See [32] for more about logarithmic type Carleson measure.

A positive Borel measure μ on $[0, 1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\overline{\mu}$ defined by

$$\overline{\mu}(E) = \mu(E \cap [0, 1)), \ \text{ for any Borel subset } E \text{ of } \mathbb{D}.$$

In this way, a positive Borel measure μ on $[0, 1)$ is a β-logarithmic t-Carleson measure if and only if there exists a constant $M > 0$ such that

$$\log^\beta \frac{e}{1-t}\mu([s, 1)) \leq M(1-s)^t, \ \ 0 \leq s < 1.$$

Let $0 < \lambda \leq 1$, the Morrey space $L^{2,\lambda}(\mathbb{D})$ is the set of all $f \in H^2$ such that

$$\sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|^\lambda}\int_I |f(e^{i\theta}) - f_I|^2 d\theta\right)^{\frac{1}{2}} < \infty.$$

The space is $L^{2,\lambda}(\mathbb{D})$ a Banach space under the norm

$$||f||_{L^{2,\lambda}} = |f(0)| + \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|^\lambda}\int_I |f(e^{i\theta}) - f_I|^2 d\theta\right)^{\frac{1}{2}}$$

It is well known that $L^{2,1} = BMOA$. The Morrey spaces increase when the parameter λ decreases, so we have the following relation

$$BMOA \subseteq L^{2,\lambda_2} \subseteq L^{2,\lambda_1} \subseteq H^2, \ \ 0 \leq \lambda_1 \leq \lambda_2 \leq 1.$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

For $0 < \lambda \leq 1$ and any function $f \in L^{2,\lambda}$, it has the following equivalent norm

$$||f||_{L^{2,\lambda}} \asymp |f(0)| + \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_w(z)|^2) dA(z) \right)^{\frac{1}{2}},$$

where σ_w stands for the Möbius transformation $\sigma_w(z) = \frac{w-z}{1-z\overline{w}}$. See [26] for this characterization.

It is well known that functions $f \in BMOA$ have logarithmic growth,

$$|f(z)| \leq C \log \frac{2}{1-|z|}.$$

This does not remain true for $f \in L^{2,\lambda}$ when $0 < \lambda < 1$. Indeed, it follows Lemma 2 of [14] that

$$H^{\frac{2}{1-\lambda}} \subseteq L^{2,\lambda} \subseteq H^2, \;\; 0 < \lambda < 1.$$

It is known that for $0 < \lambda < 1$, $f \in L^{2,\lambda}$ satisfies

$$|f(z)| \lesssim \frac{||f||_{L^{2,\lambda}}}{(1-|z|)^{\frac{1-\lambda}{2}}}, \;\; z \in \mathbb{D} \tag{1.1}$$

By (1.1) we have that $L^{2,\lambda} \subseteq B^{\frac{3-\lambda}{2}}$ for all $0 < \lambda \leq 1$. When $\lambda = 1$, it is obvious that the inclusion is strictly. For $0 < \lambda < 1$, the function $h(z) = \sum_{k=0}^{\infty} z^{2^k} \in \mathcal{B} \subsetneq \mathcal{B}^{\frac{3-\lambda}{2}}$ shows that the inclusion is also strictly. Since h has a radial limit almost nowhere and hence $h \notin H^p$ for any $0 < p < \infty$, this implies that $h \notin L^{2,\lambda}$. The reader is referred to [10, 13, 39, 40] for more about Morrey space.

Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions $f \in H(\mathbb{D})$ having a non-tangential limit almost everywhere such that $\omega_p(t, f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is known (see [28]) that Λ^p_α is a subset of H^p and

$$\Lambda^p_\alpha = \left(f \in H(\mathbb{D}) : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}} \right), \;\text{ as } r \to 1 \right).$$

The space Λ^p_α is a Banach space with the norm $|| \cdot ||_{\Lambda^p_\alpha}$ given by

$$||f||_{\Lambda^p_\alpha} = |f(0)| + \sup_{0\leq r<1} (1-r)^{1-\alpha} M_p(r, f').$$

In [27], Bourdon, Shapiro and Sledd proved that

$$\Lambda^p_{\frac{1}{p}} \subseteq BMOA, \;\; 1 < p < \infty.$$

See [25, 11] for more about Lipschitz space and related analytic function spaces.

The mixed norm space $H^{p,q,\alpha}$, $0 < p, q \leq \infty$, $0 < \alpha < \infty$, is the space of all functions $f \in H(\mathbb{D})$, for which

$$||f||_{p,q,\alpha} = \left(\int_0^1 M_p^q(r, f)(1-r)^{q\alpha-1} dr \right)^{\frac{1}{q}} < \infty, \text{ for } 0 < q < \infty,$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

and
$$||f||_{p,\infty,\alpha}=\sup_{0\leq r<1}(1-r)^{\alpha}M_p(r,f)<\infty.$$

For $t\in\mathbb{R}$, the fractional derivative of order t of $f\in H(\mathbb{D})$ is defined by $D^tf(z)=\sum_{n=0}^{\infty}(n+1)^t\widehat{f}(n)z^n$. If $0<p,q\leq\infty$, $0<\alpha<\infty$, then $H_t^{q,p,\alpha}$ is the space of all analytic functions $f\in H(\mathbb{D})$ such that
$$||D^tf||_{p,q,\alpha}<\infty.$$

It is a well known fact that if $f\in H(\mathbb{D})$, $0<p,q\leq\infty$, $0<\alpha,\beta<\infty$, and $s,t\in\mathbb{R}$ are such that $s-t=\alpha-\beta$, then
$$||D^sf||_{p,q,\alpha}\asymp||D^tf||_{p,q,\beta}.$$
Consequently, we get $H_s^{q,p,\alpha}=H_t^{p,q,\beta}$ (see [22]).

With these notations above, for $1<p<\infty$, we see that $\Lambda^p_{\frac{1}{p}}=H^{p,\infty,1}_{1+\frac{1}{p}}$. On the other hand, by the inclusions between mixed norm spaces(see [11]), we have that $H_2^{1,\infty,1}=H^{1,\infty,\frac{1}{p}}_{1+\frac{1}{p}}\subseteq H^{p,\infty,1}_{1+\frac{1}{p}}=\Lambda^p_{\frac{1}{p}}$. Therefore, the space $H_2^{1,\infty,1}$ can be regarded as the limit case of $H^{p,\infty,1}_{1+\frac{1}{p}}=\Lambda^p_{\frac{1}{p}}$ as $p\to 1$. In view of this point, we may use the symbol $\Lambda_1^{1,*}$ to denote the space $H_2^{1,\infty,1}$. Note that $\Lambda_1^{1,*}\subseteq\Lambda^p_{\frac{1}{p}}\subseteq BMOA\subseteq\mathcal{B}$ for all $1<p<\infty$.

It is clear that the space $\Lambda_1^{1,*}$ is equivalent to the space
$$\left\{f\in H(\mathbb{D}):\sup_{0\leq r<1}(1-r)M_1(f'',r)<\infty\right\}. \tag{1.2}$$

For $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H(\mathbb{D})$, the Cesàro operator $\mathcal{C}$ is defined by
$$\mathcal{C}(f)(z)=\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\widehat{f}(k)\right)z^n=\int_0^1\frac{f(tz)}{1-tz}dt,\ z\in\mathbb{D}.$$
The Cesàro operator $\mathcal{C}$ is bounded on H^p for $0<p<\infty$. The case of $1<p<\infty$ follows from a result of Hardy on Fourier series [8] together with the Riesz transform. Siskakis [3] give an alternative proof of this result and to extend it to $p=1$ by using semigroups of composition operators. A direct proof of the boundedness on H^1 was given by Siskakis in [5]. Miao [15] proved the case $0<p<1$. Stempak [19] gave a proof valid for $0<p\leq 2$. Andersen [18] and Nowak [21] provided another proof valid for all $0<p<\infty$. In the case $p=\infty$, since $\mathcal{C}(1)(z)=\log\frac{1}{1-z}\notin H^{\infty}$, so that $\mathcal{C}(H^{\infty})\nsubseteq H^{\infty}$. Danikas and Siskakis [23] proved that $\mathcal{C}(H^{\infty})\subseteq BMOA$ and $\mathcal{C}(BMOA)\nsubseteq BMOA$. Cesàro operator $\mathcal{C}$ act on weighted Bergman spaces, Dirichlet space and general mixed normed spaces the reader is referred to [5, 34, 1, 29, 16].

Recently, Galanopoulos, Girela and Merchán [30] introduced a Cesàro-like operator $\mathcal{C}_{\mu}$ on $H(\mathbb{D})$, which is a natural generalization of the classical Cesàro operator $\mathcal{C}$. They consider the following generalization: For a positive Borel measure μ on the interval $[0,1)$ they define the operator
$$\mathcal{C}_{\mu}(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}\widehat{f}(k)\right)z^n=\int_0^1\frac{f(tz)}{(1-tz)}d\mu(t),\ z\in\mathbb{D}. \tag{1.3}$$

where μ_n stands for the moment of order n of μ, that is, $\mu_n = \int_0^1 t^n d\mu(t)$. They studied the operators $\mathcal{C}_\mu$ acting on distinct spaces of analytic functions(e.g. Hardy space, Bergman space, Bloch space, etc.).

The Cesàro-like operator $\mathcal{C}_\mu$ defined above has attracted the interest of many mathematicians. Jin and Tang [12] studied the boundedness(compactness) of $\mathcal{C}_\mu$ from one Dirichlet-type space into another one. Bao, Sun and Wulan [7] studied the range of $\mathcal{C}_\mu$ acting on H^∞. They proved that $\mathcal{C}_\mu(H^\infty) \subseteq \cap_{p>1}\Lambda^p_{\frac{1}{p}}$ if and only if μ is a 1-Carleson measure. This gives an answer to the question which was left open in [30]. In fact, they worked on a more general version. Just recently, Blasco [24] used a different method to also get the same result. Based on the previous results, it is natural to discuss the range of H^∞ under the action of $\mathcal{C}_\mu$ when μ is an α-Carleson measure with $0 < \alpha < 1$. Furthermore, what is the condition for the measure μ such that $\mathcal{C}_\mu(H^p) \subseteq \cap_{q>1}\Lambda^q_{\frac{1}{q}}$? We shall prove the following general version of the results, which give the answers to these questions. As consequences of our study, we may reproduce many of the known conclusions as well as obtain some new results.

Theorem 1.1. *Suppose $0 < p \leq \infty$, $0 < \lambda \leq 1$ and μ is a finite positive Borel measure on the interval $[0, 1)$. Let X be subspace of $H(\mathbb{D})$ with $L^{2,\lambda} \subseteq X \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}$. Then $\mathcal{C}_\mu(H^p) \subseteq X$ if and only if μ is a $\frac{1+\lambda}{2} + \frac{1}{p}$-Carleson measure.*

Theorem 1.2. *Suppose $0 < p < \infty$, $1 < q < \infty$ and μ is a finite positive Borel measure on the interval $[0, 1)$. Let X and Y be subspaces of $H(\mathbb{D})$ such that $H^p \subseteq X \subseteq \mathcal{B}^{1+\frac{1}{p}}$ and $\Lambda^q_{\frac{1}{q}} \subseteq Y \subseteq \mathcal{B}$. Then the following statements hold.*
(1) The operator $\mathcal{C}_\mu$ is bounded from X into Y if and only if μ is a $1 + \frac{1}{p}$-Carleson measure.
(2) If μ is a 1-logarithmic $1 + \frac{1}{p}$-Carleson measure, then $\mathcal{C}_\mu : X \to \Lambda^{1,*}_1$ is bounded.

If $1 \leq p < \infty$, we know that $\mathcal{C}_\mu : BMOA \to \Lambda^p_{\frac{1}{p}}$ if and only if μ is a 1-logarithmic 1-Carleson measure. Theorem 1.2 includes a characterization of those μ so that $\mathcal{C}_\mu$ maps $L^{2,\lambda}$ into $\Lambda^p_{\frac{1}{p}}$.

In [30], the authors proved that if X and Y are spaces of holomorphic functions in the unit disc $\mathbb{D}$, such that $\Lambda^2_{\frac{1}{2}} \subseteq X, Y \subseteq \mathcal{B}$, then $\mathcal{C}_\mu$ is a bounded operator from the space X into the space Y if and only if μ is a 1-logarithmic 1-Carleson measure. Since $\Lambda^2_{\frac{1}{2}} \subseteq BMOA = L^{2,1} \subseteq \mathcal{B}$, so that $\mathcal{C}_\mu$ is a bounded operator from X into the space $L^{2,1}$ if and only if μ is a 1-logarithmic 1-Carleson measure. It is natural to ask what's the condition for μ such that $\mathcal{C}_\mu$ is bounded from X into $L^{2,\lambda}$? On the other hand, whether the space $\Lambda^2_{\frac{1}{2}}$ can be extended to the space $\Lambda^{1,*}_1$? We are now ready to state our next results, which generalized the previous mentioned results. Our results also gives the range of X under the action of $\mathcal{C}_\mu$ when μ is a 1-logarithmic s-Carleson measure with $\frac{1}{2} < s < 1$.

Theorem 1.3. *Suppose $0 < \lambda \leq 1$ and μ is a finite positive Borel measure on the interval $[0, 1)$. Let X and Y be subspaces of $H(\mathbb{D})$ such that $\Lambda^{1,*}_1 \subseteq X \subseteq \mathcal{B}$ and $L^{2,\lambda} \subseteq Y \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}$. Then the following conditions are equivalent.*
(1) The operator $\mathcal{C}_\mu$ is bounded from X into Y.
(2) The measure μ is a 1-logarithmic $\frac{1+\lambda}{2}$-Carleson measure.

Theorem 1.4. *Suppose μ is a finite positive Borel measure on the interval $[0, 1)$. Let X and Y be subspaces of $H(\mathbb{D})$ such that $\Lambda^{1,*}_1 \subseteq X, Y \subseteq \mathcal{B}$. Then the following conditions are equivalent.*
(1) The operator $\mathcal{C}_\mu$ is bounded from X into Y.
(2) The measure μ is a 1-logarithmic 1-Carleson measure.

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

The boundedness of the operator $\mathcal{C}_\mu$ acting on $BMOA$ has been studied in [30, 7, 24]. The space of $BMOA$ is close related to the Morrey space $L^{2,\lambda}$. Since the Moreey space $L^{2,\lambda}$ has showed up in a natural way in our work, it seems natural to study the action of the operators $\mathcal{C}_\mu$ on the Moreey space $L^{2,\lambda}$ for general values of the parameters λ. The following result gives a complete characterization of the boundedness of $\mathcal{C}_\mu$ act between different Morrey spaces. Note that the case of $\lambda_1 = 1$ is contained in Theorem 1.3.

Theorem 1.5. *Suppose $0 < \lambda_1 < 1$, $0 < \lambda_2 \leq 1$, μ is a finite positive Borel measure on the interval $[0, 1)$. Let X and Y be subspaces of $H(\mathbb{D})$ such that $L^{2,\lambda_1} \subseteq X \subseteq \mathcal{B}^{\frac{3-\lambda_1}{2}}$ and $L^{2,\lambda_2} \subseteq Y \subseteq \mathcal{B}^{\frac{3-\lambda_2}{2}}$. Then the following statements are equivalent.*
(1) The operator $\mathcal{C}_\mu$ is bounded from X into Y.
(2) The measure μ is a $1 + \frac{\lambda_2-\lambda_1}{2}$-Carleson measure.

In section 2, we shall give some basic results that will be used in the proof. Section 3 will be devoted to present the proofs of Theorem 1.1–Theorem 1.5 and gives some relevant corollaries. It is necessary to clarify that the subspaces X and Y of $H(\mathbb{D})$ we shall be dealing with are Banach spaces continuously embedded in $H(\mathbb{D})$, to prove that the operator $\mathcal{C}_\mu$ is bounded from X into Y it suffices to show that it maps X into Y by using the closed graph theorem.

Throughout the paper, the letter C will denote a positive constant which depends only upon the displayed parameters (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, we will use the notation $Q_1 \lesssim Q_2$ if there exists a constant C such that $Q_1 \leq CQ_2$, and $Q_1 \gtrsim Q_2$ is understood in an analogous manner. In particular, if $Q_1 \lesssim Q_2$ and $Q_1 \gtrsim Q_2$, then we write $Q_1 \asymp Q_2$ and say that Q_1 and Q_2 are equivalent. This notation has already been used above in the introduction.

2 Preliminary Results

Lemma 2.1. *Let $0 < \alpha < \infty$ and $f \in \mathcal{B}^\alpha$. Then for each $z \in \mathbb{D}$, we have the following inequalities:*

$$|f(z)| \lesssim \begin{cases} ||f||_{\mathcal{B}^\alpha}, \text{ if } 0 < \alpha < 1; \\ ||f||_{\mathcal{B}^\alpha} \log \frac{2}{1-|z|}, \text{ if } \alpha = 1; \\ \frac{||f||_{\mathcal{B}^\alpha}}{(1-|z|)^{\alpha-1}}, \text{ if } \alpha > 1. \end{cases}$$

This well known Lemma can be found in [20].

Lemma 2.2. *Let $\alpha > 0$ and $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$, $\widehat{f}(n) \geq 0$ for all $n \geq 0$. Then $f \in \mathcal{B}^\alpha$ if and only if*

$$\sup_{n\geq 1} n^{-\alpha} \sum_{k=1}^{n} k\widehat{f}(k) < \infty.$$

This result follows from Corollary 3.2 in [31] or Theorem 2.6 in [9].

Lemma 2.3. *Let $0 < s < \infty$ and μ be a finite positive Borel measure on the interval $[0, 1)$. Then the following statements hold:*
(1) μ is an s-Carleson measure if and only if $\mu_n = O(\frac{1}{n^s})$.
(2) μ is a vanishing s-Carleson measure if and only if $\mu_n = o(\frac{1}{n^s})$.

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

This Lemma follows from Theorem 2.1 and Theorem 2.4 in [6].

The following integral estimates are useful. We only list the required ones. See [36] for the detailed proofs and other cases.

Lemma 2.4. *Suppose that* $r \geq 0, t \geq 0, \delta > -1, k \geq 0$. *Let*

$$J_{w,a} = \int_{\mathbb{D}} \frac{(1-|z|^2)^\delta}{|1-z\overline{w}|^t |1-z\overline{a}|^r} \log^k \frac{e}{1-|z|^2} dA(z), \quad w, a \in \mathbb{D}.$$

(1) If $t + r - \delta > 2$, $t - \delta < 2$ *and* $r - \delta < 2$, *then*

$$J_{w,a} \asymp \frac{1}{|1-w\overline{a}|^{t+r-\delta-2}} \log^k \frac{e}{|1-w\overline{a}|}.$$

(2) If $t - \delta > 2 > r - \delta$, *then*

$$J_{w,a} \asymp \frac{1}{(1-|w|^2)^{t-\delta-2}|1-w\overline{a}|^r} \log^k \frac{e}{1-|w|^2}.$$

We also need the following estimates. (See e.g. Theorem 1.12 in [20])

Lemma 2.5. *Let* α *be any real number and* $z \in \mathbb{D}$. *Then*

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^\alpha} \asymp \begin{cases} 1 & \text{if } \alpha < 1, \\ \log \frac{2}{1-|z|^2} & \text{if } \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}} & \text{if } \alpha > 1, \end{cases}$$

The following result is known to experts. We give a detailed proof by using the integral estimates with double variable points. These integral estimates are practical and have its own interests. The reader is referred to [36, 33, 37] for various integral estimates.

Lemma 2.6. *Let* $0 < \lambda < 1$, *then for any* $c \leq \frac{1-\lambda}{2}$, *we have*

$$f(z) = \frac{1}{(1-z)^c} \in L^{2,\lambda}.$$

Proof. It is suffices to prove the case of $c = \frac{1-\lambda}{2}$. For $0 < r < 1$ and $w \in \mathbb{D}$, by Proposition 3.1-(7) in [37] we have

$$\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{3-\lambda}|1-r\overline{w}e^{i\theta}|^2} \asymp \frac{1}{(1-r)^{2-\lambda}|1-r^2\overline{w}|^2} + \frac{1}{(1-r^2|w|)|1-r^2\overline{w}|^{3-\lambda}}.$$

It is easy to check that

$$\frac{1}{|1-r^2\overline{w}|^2} \lesssim \frac{1}{(1-r|w|)^2} \text{ and } \frac{1-r}{(1-r^2|w|)|1-r^2\overline{w}|^{3-\lambda}} \lesssim \frac{(1-r)^{\lambda-1}}{(1-r|w|)^2}.$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

Using the polar coordinate formula and above inequalities we get

$$
\begin{aligned}
||f||_{L^{2,\lambda}} &\asymp \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_w(z)|^2) dA(z) \right)^{\frac{1}{2}} \\
&\lesssim \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \left(\int_0^1 (1-r) \int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{3-\lambda}|1-r\overline{w}e^{i\theta}|^2} dr \right)^{\frac{1}{2}} \\
&\asymp \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \left(\int_0^1 \frac{(1-r)^{\lambda-1}}{|1-r^2\overline{w}|^2} dr + \int_0^1 \frac{1-r}{(1-r^2|w|)|1-r^2\overline{w}|^{3-\lambda}} dr \right)^{\frac{1}{2}} \\
&\lesssim \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \left(\int_0^1 \frac{(1-r)^{\lambda-1}}{(1-r|w|)^2} dr \right)^{\frac{1}{2}} \\
&\lesssim 1.
\end{aligned}
$$

The last step above we have used the integral estimate

$$
\int_0^1 \frac{(1-r)^{\lambda-1}}{(1-r|w|)^2} dr \asymp \frac{1}{(1-|w|)^{2-\lambda}},
$$

which can be found in the literature [38]. □

3 Proofs of the main results

First, we give some characterizations of positive Borel measures μ on $[0,1)$ as logarithmic type Carleson measures, this will be used in our proofs.

Proposition 3.1. *Suppose $\beta > 0$, $\gamma \geq 0$, $0 \leq q < s < \infty$ and μ is a finite positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(1) *μ is a γ-logarithmic s-Carleson measure;*

(2)

$$
S_1 := \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q (1-|w|t)^{s+\beta-q}} d\mu(t) < \infty;
$$

(3)

$$
S_2 := \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q |1-wt|^{s+\beta-q}} d\mu(t) < \infty.
$$

(4)

$$
S_3 := \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-t}}{(1-t)^q (1-|w|t)^{s+\beta-q}} d\mu(t) < \infty.
$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

Proof. The proof of $(2) \Rightarrow (1)$ is straightforward. In fact,

$$\begin{aligned} S_1 &:= \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) \\ &\geq \int_{|w|}^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) \\ &\gtrsim \frac{\mu(|w|,1) \log^\gamma \frac{e}{1-|w|}}{(1-|w|)^s}. \end{aligned}$$

This finish the proof of $(2) \Rightarrow (1)$. Similarly, we may obtain $(3) \Rightarrow (1)$ and $(4) \Rightarrow (1)$. Since $(2) \Rightarrow (3)$ is obvious, to complete the proof we have to prove that $(1) \Rightarrow (2)$ and $(1) \Rightarrow (4)$.

$(1) \Rightarrow (2)$. The proof of this implication follows closely the arguments of the proof of Proposition 2.1 in [7]. We include a detailed proof for completeness.

It is suffices to consider the case $w \in \mathbb{D}$ with $\frac{1}{2} \leq |w| < 1$ and $q > 0$. For every positive integer $n \geq 1$, let

$$Q_0(w) = \varnothing, Q_n(w) = \{t \in [0,1) : 1 - 2^n(1-|w|) \leq t < 1\}.$$

Let n_w be the minimal integer such that $1 - 2^{n_w}(1-|w|) \leq 0$. Then $Q_n(w) = [0,1)$ when $n \geq n_w$. If $t \in Q_1(w)$, then

$$1 - |w| \leq 1 - |w|t.$$

Also, for $2 \leq n \leq n_w$ and $t \in Q_n(w) \backslash Q_{n-1}(w)$, we have

$$(2^{n-1}-1)(1-|w|) = |w| - (1 - 2^{n-1}(1-|w|)) \leq |w| - t \leq 1 - |w|t.$$

Notice that $\beta > 0$, $\gamma \geq 0$, $0 < q < s < \infty$ and μ is a γ-logarithmic s-Carleson measure, these together with above inequalities we have

$$\begin{aligned} &\int_0^1 \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^\beta}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) \\ &= \sum_{n=1}^{n_w} \int_{Q_n(w)\backslash Q_{n-1}(w)} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^\beta}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) \\ &\lesssim \sum_{n=1}^{n_w} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^{q-s}}{2^{n(s+\beta-q)}} \int_{Q_n(w)\backslash Q_{n-1}(w)} \frac{1}{(1-t)^q} d\mu(t) \\ &\lesssim \sum_{n=1}^{n_w} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^{q-s}}{2^{n(s+\beta-q)}} \int_0^\infty x^{q-1} \mu\big(\{t \in [1-2^n(1-|w|),1) : 1 - \frac{1}{x} < t\}\big) dx \\ &\asymp \sum_{n=1}^{n_w} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^{q-s}}{2^{n(s+\beta-q)}} \int_0^{\frac{1}{2^n(1-|w|)}} x^{q-1} \mu\big([1-2^n(1-|w|),1)\big) dx \\ &\quad + \sum_{n=1}^{n_w} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^{q-s}}{2^{n(s+\beta-q)}} \int_{\frac{1}{2^n(1-|w|)}}^\infty x^{q-1} \mu\big([1-\frac{1}{x},1)\big) dx \\ &\lesssim \sum_{n=1}^{n_w} \frac{\log^\gamma \frac{e}{1-|w|}(1-|w|)^{q-s}}{2^{n(s+\beta-q)}} \left(\frac{2^{ns}(1-|w|)^s}{\log^\gamma \frac{e}{2^n(1-|w|)}} \int_0^{\frac{1}{2^n(1-|w|)}} x^{q-1} dx + \int_{\frac{1}{2^n(1-|w|)}}^\infty \frac{\log^{-\gamma} ex}{x^{s+1-q}} dx \right) \\ &\lesssim \sum_{n=1}^{n_w} \frac{1}{2^{\beta n}} \frac{\log^\gamma \frac{e}{1-|w|}}{\log^\gamma \frac{e}{2^n(1-|w|)}} \lesssim \sum_{n=1}^{n_w} \frac{1}{2^{\beta n}} \left(1 + \frac{n^\gamma \log 2}{\log^\gamma \frac{2}{2^n(1-|w|)}}\right) \lesssim \sum_{n=1}^{n_w} \frac{n^\gamma}{2^{\beta n}} \lesssim 1. \end{aligned}$$

This implies that

$$S_1 := \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) < \infty.$$

$(1) \Rightarrow (4)$. We only need consider the case of $\gamma > 0$. For $0 < \delta < s-q$, let

$$f(t) = (1-t)^\delta \log^\gamma \frac{e}{1-t}, \;\; 0 \le t < 1.$$

It is known that f is a normal function on $[0,1)$. Furthermore, we may choosing $b = \delta$ and $0 < a = \varepsilon < \delta$ such that

$$\frac{f(t)}{(1-t)^b} \text{is increasing}, \;\; \frac{f(t)}{(1-t)^a} \text{is decreasing}, \;\; \text{as } t \to 1^-.$$

Hence, it follows form Lemma 2.2 in [35] that

$$\frac{f(t)}{f(r)} \lesssim \left(\frac{1-t}{1-r}\right)^\varepsilon + \left(\frac{1-t}{1-r}\right)^\delta \tag{3.1}$$

for all $0 < t, r < 1$. Bearing in mind that $(1) \Leftrightarrow (2)$ we have proved already. By (3.1) we have

$$\begin{aligned}
&\int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-t}}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) \\
&= \int_0^1 \frac{(1-|w|)^{\beta+\delta} \log^\gamma \frac{e}{1-|w|}}{(1-t)^{q+\delta}(1-|w|t)^{s+\beta-q}} \cdot \frac{f(t)}{f(|w|)} d\mu(t) \\
&\lesssim \int_0^1 \frac{(1-|w|)^{\beta+\delta} \log^\gamma \frac{e}{1-|w|}}{(1-t)^{q+\delta}(1-|w|t)^{s+\beta-q}} \left\{ \left(\frac{1-t}{1-|w|}\right)^\varepsilon + \left(\frac{1-t}{1-|w|}\right)^\delta \right\} d\mu(t) \\
&\lesssim \int_0^1 \frac{(1-|w|)^\beta \log^\gamma \frac{e}{1-|w|}}{(1-t)^q(1-|w|t)^{s+\beta-q}} d\mu(t) + \int_0^1 \frac{(1-|w|)^{\beta+\delta-\varepsilon} \log^\gamma \frac{e}{1-|w|}}{(1-t)^{q+\delta-\varepsilon}(1-|w|t)^{s+\beta-q}} d\mu(t) \\
&\lesssim 1.
\end{aligned}$$

This gives (4). □

Remark 3.2. *For $\gamma \in \mathbb{R}$ and $0 < s < \infty$, we may prove the following result in a same way.*

$$\sup_{t\in[0,1)} \frac{\log^\gamma \frac{e}{1-t} \mu([t,1))}{(1-t)^s} < \infty \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).$$

We now present the proofs of Theorems 1.1–Theorem 1.5.
Proof of Theorem 1.1 (1). If $\mathcal{C}_\mu(H^p) \subseteq X$, take

$$f_a(z) = \frac{(1-a)}{(1-az)^{1+\frac{1}{p}}}, \;\; 0 < a < 1.$$

Then $f_a \in H^p$ for all $0 < p \leq \infty$ and $\sup_{0<a<1} ||f_a||_p \lesssim 1$. This implies that

$$\mathcal{C}_\mu(f_a) \in X \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}.$$

It is easy to see that

$$\mathcal{C}_\mu(f_a)'(z) = \int_0^1 \frac{tf_a'(tz)}{(1-tz)} d\mu(t) + \int_0^1 \frac{tf_a(tz)}{(1-tz)^2} d\mu(t).$$

Since $\mathcal{C}_\mu(f_a) \in X \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}$, it follows from Lemma 2.1 that

$$|\mathcal{C}_\mu(f_a)'(a)| \lesssim \frac{1}{(1-a)^{\frac{3-\lambda}{2}}}, \quad a \in (0,1).$$

Then it follows that, for $\frac{1}{2} < a < 1$,

$$\begin{aligned}
\frac{1}{(1-a)^{\frac{3-\lambda}{2}}} &\gtrsim \left| \int_0^1 \frac{(1+\frac{1}{p})ta(1-a)}{(1-ta)(1-ta^2)^{2+\frac{1}{p}}} d\mu(t) + \int_0^1 \frac{t(1-a)}{(1-ta)^2(1-ta^2)^{1+\frac{1}{p}}} d\mu(t) \right| \\
&\gtrsim \int_a^1 \frac{1}{(1-ta^2)^{2+\frac{1}{p}}} d\mu(t) \\
&\gtrsim \frac{\mu([a,1))}{(1-a)^{2+\frac{1}{p}}}.
\end{aligned}$$

This gives that

$$\mu([a,1)) \lesssim (1-a)^{\frac{1+\lambda}{2}+\frac{1}{p}} \ \text{ for all } \ \frac{1}{2} < a < 1.$$

This implies that μ is a $\frac{1+\lambda}{2} + \frac{1}{p}$-Carleson measure.

On the other hand, suppose μ is a $\frac{1+\lambda}{2} + \frac{1}{p}$-Carleson measure. Let $L^{2,\lambda} \subseteq X \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}$, to prove $\mathcal{C}_\mu(H^p) \subseteq X$ it is sufficient to prove that $\mathcal{C}_\mu : H^p \to L^{2,\lambda}$ is bounded. Without loss of generality, we may assume $f \in H^p$ and $f(0) = 0$. By (1.3), we know that

$$\mathcal{C}_\mu(f)'(z) = \int_0^1 \frac{tf'(tz)}{(1-tz)} d\mu(t) + \int_0^1 \frac{tf(tz)}{(1-tz)^2} d\mu(t), \quad z \in \mathbb{D}.$$

Let

$$\delta_p = \begin{cases} \dfrac{1}{p} & 0 < p < \infty; \\ 0 & p = \infty. \end{cases}$$

It is known that (see e.g. page 36 in [28])

$$|f(z)| \lesssim \frac{||f||_p}{(1-|z|)^{\delta_p}},$$

and hence

$$|f'(z)| \lesssim \frac{||f||_p}{(1-|z|)^{1+\delta_p}}.$$

It follows that

$$\begin{aligned}
|\mathcal{C}_\mu(f)'(z)| &\le \int_0^1 \frac{|tf'(tz)|}{|1-tz|} d\mu(t) + \int_0^1 \frac{|tf(tz)|}{|1-tz|^2} d\mu(t) \\
&\le ||f||_p \int_0^1 \frac{d\mu(t)}{|1-tz|(1-t|z|)^{1+\delta_p}} + ||f||_p \int_0^1 \frac{d\mu(t)}{(1-t|z|)^{\delta_p}|1-tz|^2} \\
&\lesssim ||f||_p \int_0^1 \frac{d\mu(t)}{|1-tz|(1-t|z|)^{1+\delta_p}}. \qquad (3.2)
\end{aligned}$$

Since $0<\lambda<1$, we can choose a positive real number $1-\lambda<\sigma<1$ such that

$$\frac{1}{(1-t|z|)^{2+2\delta_p}} \le \frac{1}{(1-t)^{2+2\delta_p-\sigma}(1-|z|)^\sigma}. \qquad (3.3)$$

By (3.2) and Minkowski's inequality, (3.3), Lemma 2.4 and Proposition 3.1, we get

$$\begin{aligned}
&||\mathcal{C}_\mu(f)||_{L^{2.\lambda}} \asymp \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |\mathcal{C}_\mu(f)'(z)|^2 (1-|\sigma_w(z)|^2) dA(z) \right)^{\frac{1}{2}} \\
&\lesssim ||f||_p \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \left(\int_{\mathbb{D}} \left(\int_0^1 \frac{d\mu(t)}{|1-tz|(1-t|z|)^{1+\delta_p}} \right)^2 \frac{1-|z|^2}{|1-z\overline{w}|^2} dA(z) \right)^{\frac{1}{2}} \\
&\le ||f||_p \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \int_0^1 \left(\int_{\mathbb{D}} \frac{(1-|z|^2)dA(z)}{(1-t|z|)^{2(1+\delta_p)}|1-tz|^2|1-z\overline{w}|^2} \right)^{\frac{1}{2}} d\mu(t) \\
&\lesssim ||f||_p \sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{2-\lambda}{2}} \int_0^1 \frac{1}{(1-t)^{1+\delta_p-\frac{\sigma}{2}}} \left(\int_{\mathbb{D}} \frac{(1-|z|)^{1-\sigma}dA(z)}{|1-tz|^2|1-z\overline{w}|^2} \right)^{\frac{1}{2}} d\mu(t) \\
&\asymp ||f||_p \sup_{w\in\mathbb{D}} \int_0^1 \frac{(1-|w|^2)^{\frac{2-\lambda}{2}}}{(1-t)^{1+\delta_p-\frac{\sigma}{2}}|1-tw|^{\frac{1+\sigma}{2}}} d\mu(t) \\
&\lesssim ||f||_p.
\end{aligned}$$

Therefore, $\mathcal{C}_\mu : H^p \to L^{2,\lambda}$ is bounded.

The proof is complete. □

Corollary 3.3. *Suppose $0<\lambda\le 1$ and μ is a finite positive Borel measure on the interval $[0,1)$. Then $\mathcal{C}_\mu : H^\infty \to L^{2,\lambda}$ is bounded if and only if μ is a $\frac{1+\lambda}{2}$-Carleson measure.*

Remark 3.4. *If $\frac{1}{2}<\alpha\le 1$, then Corollary 3.3 show that μ is an α-Carleson measure if and only if $\mathcal{C}_\mu(H^\infty) \subseteq L^{2,2\alpha-1}$. When $0<\alpha\le\frac{1}{2}$ and μ is an α-Carleson measure, by Proposition 3.1 we* *have*

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^{2-\alpha}|\mathcal{C}_\mu(f)'(z)| &\lesssim ||f||_{H^\infty}\sup_{z\in\mathbb{D}}\int_0^1\frac{(1-|z|^2)^{2-\alpha}d\mu(t)}{(1-t|z|)|1-tz|}\\ &\lesssim ||f||_{H^\infty}\sup_{z\in\mathbb{D}}\int_0^1\frac{(1-|z|^2)^{1-\alpha}d\mu(t)}{|1-tz|}\\ &\lesssim ||f||_{H^\infty}.\end{aligned}$$

This yields that $\mathcal{C}_\mu(H^\infty)\subseteq\mathcal{B}^{2-\alpha}$.$\square$

For $2<p\leq\infty$, it follows from Theorem 9 in [40] that the Cesrào operator $\mathcal{C}$ is bounded from H^p to $L^{2,1-\frac{2}{p}}$. As a consequence of Theorem 1.1, we have the following result.

Corollary 3.5. *Suppose* $2<p\leq\infty$ *and* μ *is a finite positive Borel measure on the interval* $[0,1)$. *Then* $\mathcal{C}_\mu: H^p\to L^{2,1-\frac{2}{p}}$ *is bounded if and only if* μ *is a* 1*-Carleosn measure.*

Proof of Theorem 1.2 (1). The proof of necessity is similar to that Theorem 1.1 and hence omitted. For the sufficiency, it is suffices to show that $\mathcal{C}_\mu(\mathcal{B}^{1+\frac{1}{p}})\subseteq\Lambda^q_{\frac{1}{q}}$ when μ is an $1+\frac{1}{p}$-Carleson measure.

Notice that (3.2) is remain valid for all $f\in\mathcal{B}^{1+\frac{1}{p}}$. By (3.2) and the Minkowski inequality, Lemma 2.5 and Proposition 3.1 we have

$$\begin{aligned}&\sup_{0<r<1}(1-r)^{1-\frac{1}{q}}\left(\frac{1}{2\pi}\int_0^{2\pi}|\mathcal{C}_\mu(f)'(re^{i\theta})|^q d\theta\right)^{\frac{1}{q}}\\ \lesssim&\|f\|_{\mathcal{B}^{1+\frac{1}{p}}}\sup_{0<r<1}(1-r)^{1-\frac{1}{q}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1\frac{d\mu(t)}{|1-tre^{i\theta}|(1-tr)^{1+\delta_p}}\right)^q d\theta\right)^{\frac{1}{q}}\\ \lesssim&\|f\|_{\mathcal{B}^{1+\frac{1}{p}}}\sup_{0<r<1}(1-r)^{1-\frac{1}{q}}\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-tre^{i\theta}|^q(1-tr)^{q(1+\delta_p)}}d\theta\right)^{\frac{1}{q}}d\mu(t)\\ \lesssim&\|f\|_{\mathcal{B}^{1+\frac{1}{p}}}\sup_{0<r<1}\int_0^1\frac{(1-r)^{1-\frac{1}{q}}}{(1-tr)^{2+\delta_p-\frac{1}{q}}}d\mu(t)\\ \lesssim&\|f\|_{\mathcal{B}^{1+\frac{1}{p}}}.\end{aligned}$$

This gives $\mathcal{C}_\mu:\mathcal{B}^{1+\frac{1}{p}}\to\Lambda^q_{\frac{1}{q}}$ is bounded.

(2). Suppose μ is a 1-logarithmic $1+\frac{1}{p}$-Carleson measure. Let $H^p\subseteq X\subseteq\mathcal{B}^{1+\frac{1}{p}}$ and $f\in X$, then $f\in X\subseteq\mathcal{B}^{1+\frac{1}{p}}$. Using the integral representation of $\mathcal{C}_\mu$ we see that

$$\mathcal{C}_\mu(f)''(z)=\int_0^1\frac{t^2f''(tz)}{1-tz}d\mu(t)+2\int_0^1\frac{t^2f'(tz)}{(1-tz)^2}d\mu(t)+2\int_0^1\frac{t^2f(tz)}{(1-tz)^3}d\mu(t).\tag{3.4}$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

It follows from Lemma 2.1 we have that

$$
\begin{aligned}
|\mathcal{C}_\mu(f)''(z)| &\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \int_0^1 \left(\frac{(1-t|z|)^{-2-\frac{1}{p}}}{|1-tz|} + \frac{(1-t|z|)^{-1-\frac{1}{p}}}{|1-tz|^2} + \frac{(1-t|z|)^{-\frac{1}{p}}}{|1-tz|^3} \right) d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \int_0^1 \frac{1}{(1-t|z|)^{2+\frac{1}{p}}|1-tz|} d\mu(t).
\end{aligned}
$$

By Fubini's theorem, Lemma 2.5 and Proposition 3.1, we have

$$
\begin{aligned}
&\sup_{0\le r<1} (1-r) M_1(\mathcal{C}_\mu(f)'', r) \\
&\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \sup_{0\le r<1} (1-r) \int_0^{2\pi} \int_0^1 \frac{d\mu(t)}{(1-tr)^{2+\frac{1}{p}}|1-tre^{i\theta}|} d\theta \\
&\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \sup_{0\le r<1} \int_0^1 \frac{1-r}{(1-tr)^{2+\frac{1}{p}}} \int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|} d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \sup_{0\le r<1} \int_0^1 \frac{(1-r)\log\frac{e}{1-tr}}{(1-tr)^{2+\frac{1}{p}}} d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}^{1+\frac{1}{p}}} \lesssim ||f||_X.
\end{aligned}
$$

This gives $\mathcal{C}_\mu : X \to \Lambda_1^{1,*}$ is bounded. □

Theorem 1.1 and Theorem 1.2 lead to the following result.

Corollary 3.6. *Suppose $0 < p \le \infty$, $1 < q < \infty$ and μ is a finite positive Borel measure on the interval $[0,1)$. Let X be a subspace of $H(\mathbb{D})$ with $\Lambda^q_{\frac{1}{q}} \subseteq X \subseteq \mathcal{B}$. Then $\mathcal{C}_\mu : H^p \to X$ is bounded if and only if μ is a $1+\frac{1}{p}$-Carleson measure.*

Remark 3.7. *In [24], Blasco proved that $\mathcal{C}_\eta : H^1 \to \Lambda^2_{\frac{1}{2}}$ is bounded if and only if*

$$
\sup_{n\ge0} (n+1)^3 \sum_{k=n}^{\infty} |\eta_k|^2 < \infty, \tag{3.5}
$$

where η is a complex Borel measure on $[0,1)$. See Theorem 3.7 in [24] for the detailed. If μ is a positive Borel measure on $[0,1)$, then Corollary 3.6 shows that $\mathcal{C}_\mu : H^1 \to \Lambda^2_{\frac{1}{2}}$ is bounded if and only if μ is a 2-Carleosn measure. The condition (3.5) is equivalent to μ is a 2-Carleosn measure when μ is a positive Borel measure on $[0,1)$. In fact,

$$
\infty > \sup_{n\ge0} (n+1)^3 \sum_{k=n}^{\infty} |\mu_k|^2 \gtrsim \sup_{n\ge0} (n+1)^3 \sum_{k=n}^{2n} \mu_k^2 \gtrsim \sup_{n\ge0} (n+1)^4 \mu_{2n}^2.
$$

On the other hand, if μ is a 2-Carleosn measure, we have

$$
\begin{aligned}
\sup_{n\ge0} (n+1)^3 \sum_{k=n}^{\infty} |\mu_k|^2 &\lesssim \sup_{n\ge0} (n+1)^3 \sum_{k=n}^{\infty} \frac{1}{(k+1)^4} \\
&\lesssim \sup_{n\ge0} (n+1)^3 \int_{n+1}^{\infty} \frac{1}{x^4} dx \lesssim 1. \square
\end{aligned}
$$

10 Jul 2023 01:20:07 PDT
230318-Tang Version 2 - Submitted to Rocky Mountain J. Math.

For $0<\lambda<1$, let $p=\frac{2}{1-\lambda}$ in Theorem 1.2, then we may obtain the boundedness of $\mathcal{C}_\mu$ acting from $L^{2,\lambda}$ to the mean Lipschitz space $\Lambda^q_{\frac{1}{q}}$.

Corollary 3.8. *Suppose $0<\lambda<1$, $1<p<\infty$ and μ is a finite positive Borel measure on the interval $[0,1)$. Let X be subspace of $H(\mathbb{D})$ such that $\Lambda^p_{\frac{1}{p}}\subseteq X\subseteq\mathcal{B}$. Then $\mathcal{C}_\mu:L^{2,\lambda}\to X$ is bounded if and only if μ is a $\frac{3-\lambda}{2}$-Carleson measure.*

Proof of Theorem 1.3 (1) $\Rightarrow$ (2). Let $\Lambda^{1,*}_{1,}\subseteq X\subseteq\mathcal{B}$ and $L^{2,\lambda}\subseteq Y\subseteq\mathcal{B}^{\frac{3-\lambda}{2}}$. It is easy to check that $g(z)=\log\frac{1}{1-z}\in X$ and

$$\mathcal{C}_\mu(g)(z)=\sum_{k=0}^{\infty}\mu_k\left(\sum_{n=1}^{k}\frac{1}{n}\right)z^k.$$

If $\mathcal{C}_\mu(X)\subseteq Y$, then $\mathcal{C}_\mu(g)\in Y\subseteq\mathcal{B}^{\frac{3-\lambda}{2}}$. It follows from Lemma 2.1 that

$$\sum_{k=1}^{\infty}k\mu_k\left(\sum_{n=1}^{k}\frac{1}{n}\right)r^k\lesssim\frac{1}{(1-r)^{\frac{3-\lambda}{2}}},\quad r\in(0,1).$$

For $K\geq 2$ take $r_K=1-\frac{1}{K}$. Since the sequence $\{\mu_k\}$ is decreasing, simple estimations lead us to the following

$$\begin{aligned}
K^{\frac{3-\lambda}{2}}&\gtrsim\sum_{k=1}^{\infty}k\mu_k\left(\sum_{n=1}^{k}\frac{1}{n}\right)r_K^k\\
&\gtrsim\sum_{k=1}^{K}k\mu_k\left(\sum_{n=1}^{k}\frac{1}{n}\right)r_K^k\\
&\gtrsim\sum_{k=1}^{K}k\mu_k\log k r_K^K\\
&\gtrsim\mu_K\sum_{k=1}^{K}k\log k\\
&\asymp\mu_K K^2\log K.
\end{aligned}$$

Hence $\mu_K\lesssim\frac{1}{K^{\frac{1+\lambda}{2}}\log K}$ which implies that μ is a 1-logarithmic $\frac{1+\lambda}{2}$-Carleson measure.

(2) $\Rightarrow$ (1). Assume that μ is a 1-logarithmic $\frac{1+\lambda}{2}$-Carleson measure. It suffices to show that $\mathcal{C}_\mu:\mathcal{B}\to L^{2,\lambda}$ is bounded. Let $f\in\mathcal{B}$, it is clear that

$$\begin{aligned}
|\mathcal{C}_\mu(f)'(z)|&\leq\int_0^1\frac{|tf'(tz)|}{|1-tz|}d\mu(t)+\int_0^1\frac{|tf(tz)|}{|1-tz|^2}d\mu(t)\\
&\leq||f||_{\mathcal{B}}\int_0^1\frac{d\mu(t)}{|1-tz|(1-t|z|)}+||f||_{\mathcal{B}}\int_0^1\frac{\log\frac{e}{1-t|z|}}{|1-tz|^2}d\mu(t)
\end{aligned}$$

This gives

$$
\begin{aligned}
||\mathcal{C}_\mu(f)||_{L^{2.\lambda}} &\asymp \sup_{w\in\mathbb{D}}\left((1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|\mathcal{C}_\mu(f)'(z)|^2(1-|\sigma_w(z)|^2)dA(z)\right)^{\frac{1}{2}}\\
&\lesssim ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}(1-|w|^2)^{\frac{2-\lambda}{2}}\left(\int_{\mathbb{D}}\left(\int_0^1\frac{d\mu(t)}{|1-tz|(1-t|z|)}\right)^2\frac{1-|z|^2}{|1-z\overline{w}|^2}dA(z)\right)^{\frac{1}{2}}\\
&+||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}(1-|w|^2)^{\frac{2-\lambda}{2}}\left(\int_{\mathbb{D}}\left(\int_0^1\frac{\log\frac{e}{1-t|z|}d\mu(t)}{|1-tz|^2}\right)^2\frac{1-|z|^2}{|1-z\overline{w}|^2}dA(z)\right)^{\frac{1}{2}}\\
&:= E_1+E_2.
\end{aligned}
$$

Since μ is a 1-logarithmic $\frac{1+\lambda}{2}$-Carleson measure, by the Minkowski inequality, Lemma 2.4 and Proposition 3.1, we have

$$
\begin{aligned}
E_2 &:= ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}(1-|w|^2)^{\frac{2-\lambda}{2}}\left(\int_{\mathbb{D}}\left(\int_0^1\frac{\log\frac{e}{1-t|z|}d\mu(t)}{|1-tz|^2}\right)^2\frac{1-|z|^2}{|1-z\overline{w}|^2}dA(z)\right)^{\frac{1}{2}}\\
&\le ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}(1-|w|^2)^{\frac{2-\lambda}{2}}\int_0^1\left(\int_{\mathbb{D}}\frac{\log^2\frac{e}{1-|z|}(1-|z|^2)dA(z)}{|1-tz|^4|1-z\overline{w}|^2}\right)^{\frac{1}{2}}d\mu(t)\\
&\asymp ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}\int_0^1\frac{(1-|w|^2)^{\frac{2-\lambda}{2}}\log\frac{e}{1-t}}{(1-t)^{\frac{1}{2}}|1-t\overline{w}|}d\mu(t)\\
&\le ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}\int_0^1\frac{(1-|w|^2)^{\frac{2-\lambda}{2}}\log\frac{e}{1-t}}{(1-t)^{\frac{1}{2}}(1-t|w|)^{\frac{1+\lambda}{2}+\frac{2-\lambda}{2}-\frac{1}{2}}}d\mu(t)\\
&\lesssim ||f||_{\mathcal{B}}.
\end{aligned}
$$

Note that μ is also a $\frac{1+\lambda}{2}$-Carleson measure. Arguing as the proof of Theorem 1.1 (the case of $\delta_p=0$) we may obtain that

$$
E_1 := ||f||_{\mathcal{B}}\sup_{w\in\mathbb{D}}(1-|w|^2)^{\frac{2-\lambda}{2}}\left(\int_{\mathbb{D}}\left(\int_0^1\frac{d\mu(t)}{|1-tz|(1-t|z|)}\right)^2\frac{1-|z|^2}{|1-z\overline{w}|^2}dA(z)\right)^{\frac{1}{2}}\lesssim ||f||_{\mathcal{B}}.
$$

Therefore, we deduce that

$$
||\mathcal{C}_\mu(f)||_{L^{2,\lambda}}\lesssim ||f||_{\mathcal{B}}.
$$

The proof is complete. □

Corollary 3.9. *Suppose $0<\lambda\le 1$ and μ is a finite positive Borel measure on the interval $[0,1)$. Let X be subspace of $H(\mathbb{D})$ such that $\Lambda_1^{1,*}\subseteq X\subseteq\mathcal{B}$. Then $\mathcal{C}_\mu: X\to L^{2,\lambda}$ is bounded if and only if μ is a 1-logarithmic$\frac{1+\lambda}{2}$-Carleson measure.*

Proof of Theorem 1.4 (1) $\Rightarrow$ (2). Arguing as the proof of Theorem 1.3 we may obtain that μ is a 1-logarithmic 1-Carleson measure.

(2) $\Rightarrow$ (1). Suppose μ is a 1-logarithmic 1-Carleson measure and $\Lambda_1^{1,*} \subseteq X, Y \subseteq \mathcal{B}$. Note that $f \in X \subseteq \mathcal{B}$, by (3.4) we have

$$
\begin{aligned}
|\mathcal{C}_\mu(f)''(z)| &\lesssim ||f||_{\mathcal{B}} \int_0^1 \left(\frac{(1-t|z|)^{-2}}{|1-tz|} + \frac{(1-t|z|)^{-1}}{|1-tz|^2} + \frac{\log \frac{e}{1-t|z|}}{|1-tz|^3} \right) d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}} \left(\int_0^1 \frac{d\mu(t)}{(1-t|z|)^2|1-tz|} + \int_0^1 \frac{\log \frac{e}{1-t|z|} d\mu(t)}{|1-tz|^3} \right).
\end{aligned}
$$

Using (1.2) and Fubini's theorem, Lemma 2.5 and Proposition 3.1, we have

$$
\begin{aligned}
&\sup_{0\le r<1} (1-r) M_1(\mathcal{C}_\mu(f)'', r) \\
&\lesssim ||f||_{\mathcal{B}} \sup_{0\le r<1} (1-r) \int_0^{2\pi} \int_0^1 \frac{d\mu(t)}{(1-tr)^2|1-tre^{i\theta}|} d\theta \\
&+ ||f||_{\mathcal{B}} \sup_{0\le r<1} (1-r) \int_0^{2\pi} \int_0^1 \frac{\log \frac{e}{1-tr} d\mu(t)}{|1-tre^{i\theta}|^3} d\theta \\
&\lesssim ||f||_{\mathcal{B}} \sup_{0\le r<1} \int_0^1 \frac{1-r}{(1-tr)^2} \int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|} d\mu(t) \\
&\quad + ||f||_{\mathcal{B}} \sup_{0\le r<1} \int_0^1 (1-r) \log \frac{e}{1-t} \int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^3} d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}} \sup_{0\le r<1} \int_0^1 \frac{(1-r)\log \frac{e}{1-t}}{(1-tr)^2} d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}}.
\end{aligned}
$$

This yields that $\mathcal{C}_\mu(f) \in \Lambda_1^{1,*} \subseteq Y$. □

Note that the spaces $\Lambda^p_{\frac{1}{p}}$ for $1 \le p < \infty$, the spaces Q_p for all $0 < p < \infty$ are satisfied the condition in Theorem 1.4. Therefore, we may obtain a number of results.

Proof of Theorem 1.5 (1) $\Rightarrow$ (2). Let $f(z) = (1-z)^{-\frac{1-\lambda_1}{2}}$, then Lemma 2.6 shows that $f \in L^{2,\lambda_1} \subseteq X$. Note that $\mathcal{C}_\mu(f) \in Y \subseteq \mathcal{B}^{\frac{3-\lambda_2}{2}}$ and

$$
\mathcal{C}_\mu(f)(z) = \sum_{k=0}^{\infty} \mu_k \left(\sum_{j=0}^{k} \frac{\Gamma(\frac{1-\lambda_1}{2} + j)}{\Gamma(\frac{1-\lambda_1}{2})\Gamma(j+1)} \right) z^k,
$$

By the Stirling formula,

$$
\frac{\Gamma(j + \frac{1-\lambda_1}{2})}{\Gamma(\frac{1-\lambda_1}{2})\Gamma(j+1)} \asymp (j+1)^{-\frac{1+\lambda_1}{2}}
$$

for all nonnegative integers j. This together with $\{\mu_k\}$ is decreasing with k and Lemma 2.2 we deduce that

$$
\begin{aligned}
1 &\gtrsim n^{-\frac{3-\lambda_2}{2}} \sum_{k=1}^{n} k\mu_k \left(\sum_{j=0}^{k} \frac{\Gamma(\frac{1-\lambda_1}{2}+j)}{\Gamma(\frac{1-\lambda_2}{2})\Gamma(j+1)} \right) \\
&\gtrsim n^{-\frac{3-\lambda_2}{2}} \sum_{k=1}^{n} k\mu_k \left(\sum_{j=0}^{k} (j+1)^{-\frac{1+\lambda_1}{2}} \right) \\
&\gtrsim n^{-\frac{3-\lambda_2}{2}} \mu_n \sum_{k=1}^{n} k^{\frac{3-\lambda_1}{2}} \\
&\gtrsim \mu_n n^{1+\frac{\lambda_2-\lambda_1}{2}}.
\end{aligned}
$$

Lemma 2.3 shows that μ is a $1+\frac{\lambda_2-\lambda_1}{2}$-Carleson measure.

$(2) \Rightarrow (1)$. It suffices to prove that $\mathcal{C}_\mu : \mathcal{B}^{\frac{3-\lambda_1}{2}} \to L^{2,\lambda_2}$ is bounded. Let $f \in \mathcal{B}^{\frac{3-\lambda_2}{2}}$, then (1.3) and Lemma 2.1 imply that

$$
|\mathcal{C}_\mu(f)'(z)| \lesssim ||f||_{\mathcal{B}^{\frac{3-\lambda_1}{2}}} \int_0^1 \frac{d\mu(t)}{(1-t|z|)^{\frac{3-\lambda_1}{2}}|1-tz|}.
$$

Then arguing as the proof of Theorem 1.1 we can get the desired result.

The proof is complete. □

Corollary 3.10. *Suppose $0<\lambda<1$, μ is a finite positive Borel measure on the interval $[0,1)$. Then $\mathcal{C}_\mu$ is a bounded operator on $L^{2,\lambda}$ if and only if μ is a 1-Carleson measure.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Funding

The research is support by thee Natural Science Foundation of Hunan Province (No. 2022JJ30369).

References

[1] A. Aleman and A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46(2) (1997) 337–356.

[2] A. Aleman and J. Cima, An integral operator on H^p and Hardy's inequality, J. Anal. Math. 85 (2001) 157–176.

[3] A. Siskakis, Composition semigroups and the Cesàro operator on H^p, J. Lond. Math. Soc. 236 (1) (1987) 153–164.

[4] A. Siskakis, The Cesàro operator is bounded on H^1, Proc. Am. Math. Soc. 110 (2) (1990) 461–462.

[5] A. Siskakis, The Bergman space norm of the Cesàro operator, Arch. Math. 67 (4) (1996) 312–318.

[6] G. Bao and H. Wulan, Hankel matrices acting on Dirichlet spaces, J. Math. Anal. Appl. 409 (2014) 228–235.

[7] G. Bao, F. Sun, and H. Wulan, Carleson measure and the range of Cesàro-like operator acting on H^∞, Anal. Math. Phys. 12 (2022) Paper No.142.

[8] G. Hardy, Notes on some points in the integral calculus LXVI: the arithmetic mean of a Fourier constant, Messenger Math. 58 (3–4) (1929) 50–52.

[9] H. Wulan and K. Zhu, Möbius invariant Q_K spaces, Berlin: Springer–Verlag, 2017.

[10] H. Wulan and J. Zhou, Q_K and Morrey type spaces, Ann. Acad. Sci. Fenn. Math. 28 (2013) 193–207.

[11] I. Arévalo, A characterization of the inclusions between mixed norm spaces, J. Math. Anal. Appl. 429 (2015) 942–955.

[12] J. Jin and S. Tang, Generalized Cesàro operator on Dirichlet-type spaces, Acta Math. Sci 42(B) (2022) 212–220.

[13] J. Liu and Z. Lou, Properties of analytic Morrey spaces and applications, Math. Nachr. 288 (2015) 1673–1693.

[14] J. Liu and Z. Lou, Carleson measure for analytic Morrey spaces, Nonlinear Anal. 125 (2015) 423–432.

[15] J. Miao, The Cesàro operator is bounded on H^p for $0 < p < 1$, Proc. Amer. Math. Soc. 116 (1992) 1077–1079.

[16] J. Shi and G. Ren, Boundedness of the Cesàro operator on mixed norm spaces, Proc. Amer. Math. Soc. 126(12) (1998) 3553–3560.

[17] J. Xiao, Cesàro-type operators on Hardy, BMOA and Bloch spaces, Arch. Math. 68 (1997) 398–406.

[18] K. Andersen, Cesàro averaging operators on Hardy spaces, Proc. Roy. Soc. Edinb. Sect. A 126 (3) (1996) 617–624.

[19] K. Stempak, Cesàro averaging operators, Proc. Roy. Soc. Edinb. Sect. A 124 (1) (1994) 121–126.

[20] K. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag (GTM 226), New York, 2005.

[21] M. Nowak, Another proof of boundednessof the Cesàro operator on H^p, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 54 (2000) 75–78.

[22] M. Pavlović, Function classes on the unit disc, An Introduction, De Gruyter Studies in Mathematics, vol. 52, De Gruyter, Berlin, 2014.

[23] N. Danikas and A. Siskakis, The Cesàro operator on bounded analytic functions, Analysis 13 (1993) 295–299.

[24] O. Blasco, Cesàro-type operators on Hardy spaces, J. Math. Anal. Appl. (2023) paper No. 127017. Doi:10.1016/j.jmaa.2023.127017

[25] O. Blasco, D. Girela and M. Márquez, Mean growth of the derivative of analytic functions, bounded mean oscillation, and normal functions, Indiana Univ. Math. J. 47 (1998) 893–912.

[26] P. Li, J. Liu and Z. Lou, Integral operators on analytic Morrey spaces, Sci. China Math. 57(9) (2014) 1961–1974.

[27] P. Bourdon, J. Shapiro and W. Sledd, Fourier series, mean Lipschitz spaces, and bounded mean oscillation, in: Analysis at Urbana, vol. I, Urbana, IL, 1986-1987, in: London Math. Soc. Lecture Note Ser., vol. 137, 1989, pp. 81-110.

[28] P. Duren, Theory of H^p spaces, Academic Press, New York, 1970.

[29] P. Galanopoulos, The Cesàro operator on Dirichlet spaces, Acta Sci. Math.(Szeged) 67 (2001) 411–420.

[30] P. Galanopoulos, D. Girela and N. Merchán, Cesàro-like operators acting on spaces of analytic functions, Anal. Math. Phys. 12 (2022) Paper No. 51.

[31] P. Tang, X. Zhang, Generalized integral type Hilbert operator acting on weighted Bloch spaces. (2022). arXiv:2208.11452.

[32] R. Zhao, On logarithmic Carleson measures, Acta Sci. Math. (Szeged) 69 (3–4) (2003) 605–618.

[33] S. Li, X. Zhang, S. Xu, The Bergman type operators on the $F(p,q,s)$ tyoe spaces in C^n, Chin. J. Contemp. Math. 38(4) (2017) 303–316.

[34] V. Miller and T. Miller, The Cesàro operator on the Bergman space $A^2(\mathbb{D})$, Arch. Math. 78 (2002) 409–416.

[35] X. Zhang, L. Xi, H. Fan and J. Li, Atomic decomposition of μ-Bergman space in C^n, Acta Math. Sci. 34B (3) (2014) 779–789.

[36] X. Zhang, S. Li, Q. Shang and Y. Guo, An integral estimate and the equivalent norms of $F(p,q,s,k)$ spaces in the unit ball, Acta Math. Sci. 38B (2018) 1861–1880.

[37] X. Zhang, R. Lv and P. Tang, Several equivalent characterizations of general Hardy type spaces on the unit ball in C^n. Chin. J. Contemp. Math. 40(2) (2019) 101–114.

[38] X. Zhang, Y. Guo, Q. Shang and S. Li, The Gleason's problem on $F(p,q,s)$ type spaces in the unit ball of C^n, Complex Anal. Oper. Theory 12 (2018) 1251–1265.

[39] Z. Wu and C. Xie, Q spaces and Morrey spaces, J. Funct. Anal. 201 (2003) 282–297.

[40] Z. Wu, A new characterization for Carleson measure and some applications, Integr. Equ. Oper. Theory. 71 (2011) 161–180.