TWO RESULTS FOR POSITIVE SOLUTIONS OF A FRACTIONAL DIFFERENTIAL EQUATION INVOLVING $P-$LAPLACIAN OPERATORS

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Abstract: *We identify two distinct main results regarding the existence of positive solutions for a $p-$Laplacian fractional high-order differential problem with initial conditions. To prove the results, we apply the techniques of cones, the lower and upper method, and the alternative of Leray-Schauder. An example is discussed at the end to demonstrate the accuracy of the two given existence of positive solution theorems.*

Keywords: Caputo derivative, fixed point, existence, $p-$Laplacian, positive solution, upper and lower solution.

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1 Introduction

Fractional differential equations have gained popularity over the past years and have come to light as a prominent area of study. It has impacted a wide range of many scientific disciplines that include mechanics, chemistry, biology, etc... For more information, one can consult the works in [12, 13, 19]. It was shown that the idea of the fractional theory is better than the regular theory for modeling real-world problems, see [2, 18, 20] and the cited reference. Some recent development and intensive studies on fractional differential equations (FDEs, for short) can be seen in [3, 7, 11, 13, 21]. Moreover, the investigation of p-Laplacian difficulties of FDEs occurs in various applications, including continuum mechanics, phase transition phenomena, and dynamics. For more details, see [5, 8, 22, 23, 25, 26] and the reference.

Let us cite some research papers of $p-$ Laplacian type that have motivated the present paper. We begin by citing [10], where T. Chen et al. considered the

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multiplicity of positive solutions for the following $p-$Laplacian problem:

$$\begin{cases} D^{\beta}\Big(\Phi_q\left(D^{\alpha}x\left(t\right)\right)\Big)+g\left(t,x\left(t\right)\right)=0,\ t\in\left(0,1\right) \\ x\left(0\right)=0, \\ D^{\alpha}x\left(t\right)|_{t=0}=0; x\left(1\right)+\sigma D^{\gamma}x\left(1\right)=0, \end{cases}$$

where D^{α}, D^{β} and D^{γ} denote the standard Riemann-Liouville derivatives with $1<\alpha\leq 2$, $0<\beta\leq 1$, $0<\gamma\leq 1$, and $g:[0,1]\times\mathbb{R}^{+}\times\mathbb{R}\rightarrow\mathbb{R}^{+}$ is a continuous function, the constant σ is a positive real number and the $q-$Laplacian operator is defined as $\Phi_q\left(s\right):=\left|s\right|^{q-2}s, q>1$. By means of the fixed point theorems on cones, some existence and multiplicity results of positive solutions are obtained.

In [1], M. Acikgoz et al. studied the following equation by applying Schauder fixed point theorem:

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}\chi\left(t\right)\right)\Big)+a(t)f\left(\chi\right)=0, t\in\left(0,1\right) \\ \chi\left(0\right)=\gamma\chi\left(h\right)+\lambda, \chi'\left(0\right)=\mu, \\ \Phi_p\left(D^{\alpha}\chi\left(t\right)\right)=\left(\Phi_p\left(D^{\alpha}\chi\left(t\right)\right)\right)'=\left(\Phi_p\left(D^{\alpha}\chi\left(t\right)\right)\right)''=\left(\Phi_p\left(D^{\alpha}\chi\left(t\right)\right)\right)''', \end{cases}$$

where $1<\alpha\leq 2, 3<\beta\leq 4$ are real numbers, D^{α} and D^{β} denote the Caputo derivatives and $f:\mathbb{R}^{+}\rightarrow\mathbb{R}^{+}$, $a:(0,1)\rightarrow\mathbb{R}^{+}$ are continuous functions and $0\leq\gamma<1, 0\leq h\leq 1, \lambda,\mu>0$ are real parameters.

We shall cite also the work in [24], where J. Xu et al. studied the problem:

$$\begin{cases} D^{\alpha}\Big(\Phi_p\left(D^{\beta}x\left(t\right)\right)\Big)+f\left(t,x\left(t\right)\right)=0, t\in\left(1,e\right) \\ x\left(1\right)=x\left(e\right)=\delta x\left(1\right)=\delta x\left(e\right)=0, \\ D^{\beta}x\left(1\right)=0; D^{\beta}x\left(e\right)=bD^{\beta}\left(\eta\right), \end{cases}$$

where $\alpha\in\left(1,2\right],\beta\in\left(3,4\right]$, D^{α}, D^{β} Hadamard derivatives; $\delta>0$ and $f:[1,e]\times\mathbb{R}^{+}\times\mathbb{R}\rightarrow\mathbb{R}^{+}$ is a continuous function.

Recently, using Caputo approach, the authors of the paper [6] proved the existence of positive solutions for the problem:

$$\begin{aligned} D^{\alpha}\left(\rho\left(t\right)\phi\left(D^{\beta}x\left(t\right)\right)\right)+q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right) &= 0, 0<t<1, \\ \phi\left(D^{\beta}x\left(0\right)\right) &= x\left(0\right)=0, \\ x'\left(1\right)+\sum_{i=1}^{k}\sigma_i x'\left(\zeta_i\right) &= 0, \end{aligned}$$

where $0<\alpha\leq 1, 1\leq\beta<2$, with some other conditions on the date of the considered problem.

In the present article, we focus our attention on the following $p-$Laplacian problem with integral conditions:

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathrm{t}\right)\right)\Big)=f\left(\mathrm{t},x\left(\mathrm{t}\right)\right),\ \mathrm{t}\in\left(0,1\right) \\ D^{\alpha}x\left(0\right)=\int\limits_{0}^{1}\hbar\left(\mu,x\left(\mu\right)\right)\psi\left(\mu\right)d\mu, \\ x^{(j)}\left(0\right)=0, j=0,...,n-2; x^{(n-1)}\left(1\right)=\delta x^{(n-1)}\left(0\right). \end{cases} \tag{1}$$

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We suppose that D^{α} and D^{β} are in the sense of Caputo, with $n-1<\alpha<n$, $m-1<\beta<m$, $\delta>1$, $\Phi_p:(-r,r)\rightarrow\mathbb{R}$, $r>0$, is a general increasing homeomorphism with $\Phi_p(0)=0$ with $\Phi_p(\mathfrak{s})=|\mathfrak{s}|^{p-2}\mathfrak{s}$, $p>1$, $\Phi_p^{-1}(\mathfrak{s})=\Phi_q(\mathfrak{s})$, $\frac{1}{p}+\frac{1}{q}=1$ and $f,\hbar\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$. Additionally, we assume that $\int_0^1\hbar(\mu,x(\mu))\psi(\mu)\,d\mu$ and $\sup_{\mu\in[0,1]}\psi(\mu)$ are finite.

So, the aim is to investigate positive solutions to (1) by applying the cone techniques, the procedure of lower and upper solutions, some characteristics of the Green function, and the alternative of Leray-Schauder. For more information on the applied techniques, one can consult the papers [4, 5, 9, 16].

The paper is structured in the following way. In section 2, we recall and derive some basic definitions that will be used throughout the paper. In section 3, we prove our main results on positive solutions. Further, we discuss an illustrative example for verifying the results in Section 4. Finally, in Section 5 we present the conclusions for our paper.

2 Preliminaries

In this part, we go over some of the fundamental definitions and auxiliary results that will be used in this work, see [14, 15, 17].

Definition 1 *For a continuous function* $\mathfrak{f}$ *on* $[0,\infty)$, *the fractional integral operator of order* $\alpha>0$, *in Riemann-Liouville sense is defined as:*

$$J^{\alpha}\mathfrak{f}(s)=\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\mathfrak{f}(\tau)\,d\tau;\alpha>0,s>0.$$

Definition 2 *For* $\alpha>0$ *and* $\mathfrak{f}\in C^n([0,\infty[)$, *the Caputo derivative of order* α *of* f *is given by:*

$$D^{\alpha}\mathfrak{f}(s)=\frac{1}{\Gamma(n-\alpha)}\int_0^s(s-\tau)^{n-\alpha-1}\mathfrak{f}^{(n)}(\tau)\,d\tau,n-1<\alpha<n,n\in N^*.$$

We recall also the following results [17]:

Lemma 3 *The solutions of* $D^{\alpha}\varkappa=0$, *for any* $\alpha>0$, *are given by*

$$\varkappa(t)=\varsigma_0+\varsigma_1 s+\varsigma_2 s^2+...+\varsigma_{n-1}s^{n-1},$$

where $\varsigma_i\in\mathbb{R},i=0,1,2,..,n-1,n=[\alpha]+1$.

Lemma 4 *For any* $\alpha>0$, *the property*

$$J^{\alpha}D^{\alpha}\varkappa(s)=\varkappa(s)+\varsigma_0+\varsigma_1 s+\varsigma_2 s^2+...+\varsigma_{n-1}s^{n-1},$$

with $\varsigma_i\in\mathbb{R},i=0,1,2,...,n-1$ *is valid.*

Lemma 5 *Taking* $\beta>\alpha>0$ *and* $[a,b]\subset\mathbb{R}$, *hence, we have*

$$D^{\alpha}J^{\beta}f(s)=J^{\beta-\alpha}f(s),\ s\in[a,b].$$

3 Main Results

We proceed by stating a couple of auxiliary lemmas on the Green function for (1). Let us first introduce the space:

$$\mathfrak{E} := C\Big([0,1], \|.\| \Big), \|x\| = \max_{0\leq \mathfrak{t} \leq 1} |x(\mathfrak{t})|,$$

and define the cone $P \subset \mathfrak{E}$:

$$P := \{x \in \mathfrak{E} | x(\mathfrak{t}) \geq 0,\ \mathfrak{t} \in [0,1]\}.$$

Then, we consider:

$$\sigma(\mu) = \hbar(\mu, x(\mu))\, \psi(\mu),\ \mu \in [0,1].$$

The first lemma that is to prove is given by:

Lemma 6 *Let $n-1 < \alpha < n$, $m-1 < \beta < m$ and $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$. Then,* (1) *possesses a distinct solution that can be represented as:*

$$x(\mathfrak{t}) = \int_0^1 \mathfrak{G}(\mathfrak{t}, \mathfrak{s})\, \Phi_q(H(\mathfrak{s}))\, d\mathfrak{s},$$

where

$$\mathfrak{G}(\mathfrak{t}, \mathfrak{s}) = \begin{cases} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mathfrak{t}^{n-1}(1-\mathfrak{s})^{\alpha-n}}{(\delta-1)\Gamma(n)\Gamma(\alpha)} & 0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1, \\ \frac{\mathfrak{t}^{n-1}(1-\mathfrak{s})^{n}}{(\delta-1)\Gamma(n)\Gamma(\alpha)} & 0 \leq \mathfrak{t} \leq \mathfrak{s} \leq 1 \end{cases} \tag{2}$$

and

$$H(\mathfrak{t}) = \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\tau)^{\beta-1} f(\tau, x(\tau))\, d\tau + \Phi_p\left(\int_0^1 \sigma(\mu)\, d\mu \right).$$

Proof. We remark first that

$$\Phi_p(D^\alpha x(\mathfrak{t})) = J^\beta f(\mathfrak{t}, x(\mathfrak{t})) + a_0 + a_1 \mathfrak{t} + a_2 \mathfrak{t}^2 + ... + a_{m-1}\mathfrak{t}^{m-1},$$

for $a_i \in \mathbb{R}; 0 \leq i \leq m-1$. Now, the condition $D^\alpha x(0) = \int_0^1 \sigma(\mu)\, d\mu$ permits us to obtain

$$a_0 = \Phi_p\left(\int_0^1 \sigma(\mu)\, d\mu \right).$$

Therefore,

$$\Phi_p(D^\alpha x(\mathfrak{t})) = J^\beta f(\mathfrak{t}, x(\mathfrak{t})) + \Phi_p\left(\int_0^1 \sigma(\mu)\, d\mu \right).$$

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Hence, we can see that (1) is equivalent to:

$$\begin{cases} D^{\alpha}x\left(\mathfrak{t}\right)=\Phi_q\left(J^{\beta}f\left(\mathfrak{t},x\left(\mathfrak{t}\right)\right)+\Phi_p\left(\int\limits_0^1\sigma\left(\mu\right)d\mu\right)\right),\ \mathfrak{t}\in\left(0,1\right) \\ \\ x^{(j)}\left(0\right)=0,\ j=0,...,n-2,\ x^{(n-1)}\left(1\right)=\delta x^{(n-1)}\left(0\right). \end{cases}$$

The application of Lemma 5 leads to

$$x\left(\mathfrak{t}\right)=J^{\alpha}\Phi_q\left(J^{\beta}f\left(\mathfrak{t},x\left(\mathfrak{t}\right)\right)+\Phi_p\left(\int\limits_0^1\sigma\left(\mu\right)d\mu\right)\right)+d_0+d_1\mathfrak{t}+...+d_{n-1}\mathfrak{t}^{n-1},$$

for $d_{l'}\in\mathbb{R};0\leq l^{'}\leq n-1$.
By the rest of conditions in (1), we can write

$$\begin{aligned} x\left(\mathfrak{t}\right) &= \frac{1}{\Gamma\left(\alpha\right)}\int_0^{\mathfrak{t}}\left(\mathfrak{t}-s\right)^{\alpha-1} \\ &\times\left(\Phi_q\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^{\mathfrak{s}}\left(\mathfrak{s}-\tau\right)^{\beta-1}f\left(\tau,x\left(\tau\right)\right)d\tau+\Phi_p\left(\int\limits_0^1\sigma\left(\mu\right)d\mu\right)\right)\right)d\mathfrak{s} \\ &+\frac{\mathfrak{t}^{n-1}}{\left(\delta-1\right)\Gamma\left(n\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\mathfrak{s}\right)^{\alpha-n} \\ &\times\Phi_q\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^{\mathfrak{s}}\left(\mathfrak{s}-\tau\right)^{\beta-1}f\left(\tau,x\left(\tau\right)\right)d\tau+\Phi_p\left(\int\limits_0^1\sigma\left(\mu\right)d\mu\right)\right)d\mathfrak{s}. \end{aligned}$$

This ends the proof of Lemma 6. ∎

We also present the following second auxiliary result:

Lemma 7 *Let $n-1<\alpha<n$, $m-1<\beta<m$. The function $\mathfrak{G}(\mathfrak{t},\mathfrak{s})$ defined by (2) is continuous on $[0,1]\times[0,1]$ and it satisfies the properties:*
(1♠) *For $\mathfrak{t},\mathfrak{s}\in[0,1]$, $\mathfrak{G}(\mathfrak{t},\mathfrak{s})\geq 0$;*
(2♠) *For $\mathfrak{t},\mathfrak{s}\in[0,1]$, $\mathfrak{G}(\mathfrak{t},\mathfrak{s})\leq\mathfrak{G}(1,\mathfrak{s})$;*
(3♠) *For $\mathfrak{t},\mathfrak{s}\in(0,1)$, $\frac{\mathfrak{G}(\mathfrak{t},\mathfrak{s})}{\mathfrak{G}(1,\mathfrak{s})}\geq\mathfrak{t}^{\alpha-1}$.*

Proof. We have $\mathfrak{G}(\mathfrak{t},\mathfrak{s})\geq 0;\mathfrak{t},\mathfrak{s}\in[0,1]$. On the other hand, one can state that

$$\frac{\partial\mathfrak{G}\left(\mathfrak{t},\mathfrak{s}\right)}{\partial\mathfrak{t}}=\begin{cases}\frac{(\alpha-1)(\mathfrak{t}-\mathfrak{s})^{\alpha-2}}{\Gamma(\alpha)}+\frac{\mathfrak{t}^{n-2}(1-\mathfrak{s})^{\alpha-n}}{(\delta-1)\Gamma(\alpha)\Gamma(n-1)} & 0\leq\mathfrak{s}\leq\mathfrak{t}\leq 1, \\ \frac{\mathfrak{t}^{n-2}(1-\mathfrak{s})^{\alpha-n}}{(\delta-1)\Gamma(\alpha)\Gamma(n-1)} & 0\leq\mathfrak{t}\leq\mathfrak{s}\leq 1.\end{cases}$$

Since $\frac{\partial \mathfrak{G}(\mathfrak{t},\mathfrak{s})}{\partial \mathfrak{t}} \geq 0$, for $\mathfrak{t}, \mathfrak{s} \in [0,1]$, hence, $\mathfrak{G}(\mathfrak{t},\mathfrak{s})$ is increasing with respect to $\mathfrak{t}$. Therefore, $\mathfrak{G}(\mathfrak{t},\mathfrak{s}) \leq \mathfrak{G}(1,\mathfrak{s})$; $\mathfrak{t}, \mathfrak{s} \in [0,1]$. For (3♠), we obtain

$$\begin{aligned} \frac{\mathfrak{G}(\mathfrak{t},\mathfrak{s})}{\mathfrak{G}(1,\mathfrak{s})} &= \frac{(\delta-1)\,\Gamma(n)\,(\mathfrak{t}-\mathfrak{s})^{\alpha-1} + \mathfrak{t}^{n-1}(1-\mathfrak{s})^{\alpha-n}}{(\delta-1)\,\Gamma(n)\,(1-\mathfrak{s})^{\alpha-1} + (1-\mathfrak{s})^{\alpha-n}} \\ &\geq \frac{\mathfrak{t}^{\alpha-1}\left[(\delta-1)\,\Gamma(n)\,(1-\mathfrak{s})^{\alpha-1} + (1-\mathfrak{s})^{\alpha-n}\right]}{(\delta-1)\,\Gamma(n)\,(1-\mathfrak{s})^{\alpha-1} + (1-\mathfrak{s})^{\alpha-n}} \\ &= \mathfrak{t}^{\alpha-1}. \end{aligned}$$

When $\mathfrak{t} \leq \mathfrak{s}$, we can write

$$\frac{\mathfrak{G}(\mathfrak{t},\mathfrak{s})}{\mathfrak{G}(1,\mathfrak{s})} \geq \mathfrak{t}^{\alpha-1}$$

which implies that,

$$\mathfrak{G}(\mathfrak{t},\mathfrak{s}) \geq \mathfrak{t}^{\alpha-1}\mathfrak{G}(1,\mathfrak{s}),$$

where $\mathfrak{t}, \mathfrak{s} \in [0,1]$. ■

Let us now pass to prove the completely continuous of the application $\mathfrak{T}$.

Lemma 8 *The operator $\mathfrak{T} : P \to P$ is completely continuous; where $\mathfrak{T} : P \to \mathfrak{E}$ is given by:*

$$\mathfrak{T}x(\mathfrak{t}) := \int_0^1 \mathfrak{G}(\mathfrak{t},\mathfrak{s})\,\Phi_q(H(\mathfrak{s}))\,d\mathfrak{s}.$$

Proof. Let $\Omega \subset P$ be a bounded set and consider Z

$$Z := \max_{0 \leq \mathfrak{t} \leq 1, x \in \Omega} f(\mathfrak{t}, x).$$

So, for $x \in \Omega$, we have

$$\|\mathfrak{T}x\| \leq \left| \int_0^1 \mathfrak{G}(1,\mathfrak{s})\,\Phi_q\left(\frac{Z}{\Gamma(\beta+1)} + \Phi_p\left(\int\limits_0^1 \sigma(\mu)\,d\mu\right)\right) d\mathfrak{s} \right| < \infty.$$

Thus, $\mathfrak{T}(\Omega)$ is uniformly bounded set. Next, we show that the operator $\mathfrak{T}$ is equicontinuous, since $\mathfrak{G}(\mathfrak{t},\mathfrak{s})$ is continuous on $[0,1] \times [0,1]$, we know that it is uniformly continuous on $[0,1] \times [0,1]$. Therefore, for a fixed $\mathfrak{s} \in [0,1]$ and for any $\epsilon > 0$, there is a constant $\eta > 0$, such that for any $\widetilde{\mathfrak{t}}, \widetilde{\widetilde{\mathfrak{t}}} \in [0,1]$; $\left|\widetilde{\mathfrak{t}} - \widetilde{\widetilde{\mathfrak{t}}}\right| \leq \eta$,

$$\left|\mathfrak{G}\left(\widetilde{\mathfrak{t}},\mathfrak{s}\right) - \mathfrak{G}\left(\widetilde{\widetilde{\mathfrak{t}}},\mathfrak{s}\right)\right| \leq \frac{\epsilon}{\Phi_q\left(\frac{Z}{\Gamma(\beta+1)} + \Phi_p\left(\int\limits_0^1 \sigma(\mu)\,d\mu\right)\right)}.$$

Then,

$$\begin{aligned} &\left|\mathfrak{T}x\left(\widetilde{\mathfrak{t}}\right) - \mathfrak{T}x\left(\widetilde{\widetilde{\mathfrak{t}}}\right)\right| \\ \leq\ & \Phi_q\left(\frac{Z}{\Gamma(\beta+1)} + \Phi_p\left(\int\limits_0^1 \sigma(\mu)\,d\mu\right)\right) \int_0^1 \left|\mathfrak{G}\left(\widetilde{\mathfrak{t}},\mathfrak{s}\right) - \mathfrak{G}\left(\widetilde{\widetilde{\mathfrak{t}}},\mathfrak{s}\right)\right| d\mathfrak{s} \\ \leq\ & \epsilon. \end{aligned}$$

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As a consequence, $\mathfrak{T}(\Omega)$ is equicontinuous. So, from the Arzela-Ascoli theorem, we may conclude that $\mathfrak{T} : P \to P$ is completely continuous operator. ■

Now, we introduce the following hypotheses:

($\mathbf{H_1}$): Suppose that f is monotone nondecreasing in its second variable, $f, \hbar \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$.

($\mathbf{H_2}$): For any constant $a > 0$, $f(\mathfrak{t}, a) \neq 0$ and

$$0 < \frac{1}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\beta-1} f\left(\tau, \tau^{\alpha-1}\right) d\tau < \infty.$$

The first main result is the following theorem which is valid under the above two hypotheses.

Theorem 9 *Let consider* ***(H_1)*** *and* ***(H_2)*** *are valid. Then,* (1) *has at least a single solution that is positive on* $[0,1]$.

Proof. We proceed into three steps:

Step 1: Let

$$\mathfrak{q}(\mathfrak{t}) = \min\{y(\mathfrak{t}), \mathfrak{T}y(\mathfrak{t})\}$$

and

$$\mathfrak{p}(\mathfrak{t}) = \max\{y(\mathfrak{t}), \mathfrak{T}y(\mathfrak{t})\},$$

where $y(\mathfrak{t}) = \mathfrak{t}^{\alpha-1} \in P$. Obviously from **($\mathbf{H_2}$)**, $\mathfrak{p}(\mathfrak{t})$ and $\mathfrak{q}(\mathfrak{t})$ make sense and $\mathfrak{q}(\mathfrak{t}) \leq \mathfrak{p}(\mathfrak{t})$. Now, we establish that

$$v(\mathfrak{t}) := \mathfrak{T}\mathfrak{p}(\mathfrak{t}),$$

and

$$u(\mathfrak{t}) := \mathfrak{T}\mathfrak{q}(\mathfrak{t})$$

are the upper and lower solution of the problem (1), respectively. Thanks to **($\mathbf{H_1}$)**, we confirm that $\mathfrak{T}$ is non-decreasing with respect to x. So, we write

$$\mathfrak{T}\mathfrak{p}(\mathfrak{t}) \geq \mathfrak{T}\mathfrak{q}(\mathfrak{t}); \mathfrak{t} \in [0,1].$$

Therefore,

$$u(\mathfrak{t}) \leq v(\mathfrak{t}), \mathfrak{t} \in [0,1].$$

Alternatively, it is possible to demonstrate that

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}\mathfrak{T}x(\mathfrak{t})\right)\Big) = f(\mathfrak{t}, x(\mathfrak{t})), \ \mathfrak{t} \in (0,1) \\ D^{\alpha}\mathfrak{T}x(0) = \int\limits_0^1 \hbar(\mu, x(\mu))\psi(\mu)\, d\mu, \\ \mathfrak{T}x^{(j)}(0) = 0, j = 0, ..., n-2, \mathfrak{T}x^{(n-1)}(1) = \delta\mathfrak{T}x^{(n-1)}(0). \end{cases} \tag{3}$$

Using the fact that u and v are in P and thanks to (3), we have

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}v(\mathfrak{t})\right)\Big) - f(\mathfrak{t}, v(\mathfrak{t})) \leq D^{\beta}\Big(\Phi_p\left(D^{\alpha}\mathfrak{T}\mathfrak{p}(\mathfrak{t})\right)\Big) - f(\mathfrak{t}, \mathfrak{p}(\mathfrak{t})) = 0, \ \mathfrak{t} \in (0,1) \\ D^{\alpha}v(0) = \int\limits_0^1 \hbar(\mu, v(\mu))\psi(\mu)\, d\mu, \\ v^{(j)}(0) = 0, j = 0, ..., n-2, v^{(n-1)}(1) = \delta v^{(n-1)}(0) \end{cases}$$

and

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}u\left(\mathfrak{t}\right)\right)\Big) - f\left(\mathfrak{t}, u\left(\mathfrak{t}\right)\right) \geq D^{\beta}\Big(\Phi_p\left(D^{\alpha}\mathfrak{T}\mathfrak{q}\left(\mathfrak{t}\right)\right)\Big) - f\left(\mathfrak{t}, \mathfrak{q}\left(\mathfrak{t}\right)\right), \ \mathfrak{t} \in (0,1) \\ D^{\alpha}u\left(0\right) = \int_0^1 \hbar\left(\mu, u\left(\mu\right)\right)\psi\left(\mu\right)d\mu, \\ u^{(j)}\left(0\right) = 0, j = 0, ..., n-2, u^{(n-1)}\left(1\right) = \delta u^{(n-1)}\left(0\right). \end{cases}$$

This means that u and v are, respectively, lower and upper solution of (1).

Step 2: We prove that

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathfrak{t}\right)\right)\Big) = \mathfrak{h}\left(\mathfrak{t}, x\left(\mathfrak{t}\right)\right), \ \mathfrak{t} \in (0,1) \\ D^{\alpha}x\left(0\right) = \int_0^1 \hbar\left(\mu, x\left(\mu\right)\right)\psi\left(\mu\right)d\mu, \\ x^{(j)}\left(0\right) = 0, j = 0, ..., n-2, x^{(n-1)}\left(1\right) = \delta x^{(n-1)}\left(0\right) \end{cases} \tag{4}$$

has a positive solution, where we shall take in consideration that

$$\mathfrak{h}\left(\mathfrak{t}, x\left(\mathfrak{t}\right)\right) = \begin{cases} f\left(\mathfrak{t}, u\left(\mathfrak{t}\right)\right); x\left(\mathfrak{t}\right) \leq v\left(\mathfrak{t}\right), \\ f\left(\mathfrak{t}, x\left(\mathfrak{t}\right)\right); u\left(\mathfrak{t}\right) \leq x\left(\mathfrak{t}\right) \leq v\left(\mathfrak{t}\right), \\ f\left(\mathfrak{t}, v\left(\mathfrak{t}\right)\right); u\left(\mathfrak{t}\right) \leq x\left(\mathfrak{t}\right). \end{cases}$$

So, we consider $\Lambda : C[0,1] \to C[0,1]$ defined by:

$$\Lambda x\left(\mathfrak{t}\right) = \int_0^1 \mathfrak{G}\left(\mathfrak{t}, \mathfrak{s}\right)\Phi_q\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^{\mathfrak{s}}\left(\mathfrak{s}-\tau\right)^{\beta-1}\mathfrak{h}\left(\mathfrak{t}, x\left(\tau\right)\right)d\tau + \Phi_p\left(\int_0^1 \sigma\left(\mu\right)d\mu\right)\right)d\mathfrak{s}.$$

We can see that $\Lambda x \geq 0; \forall x \in P$, and a fixed-point of Λ is a solution of (4). Similarly to Lemma 8, we can say that Λ is compact. By Schauder fixed point theorem, Λ has a fixed point, and then (4) has a positive solution.

Step 3: We show that (1) has a positive solution. Consider the case: $\varrho(\mathfrak{t})$ is a solution of (1). In order to complete the proof, we must prove that

$$u\left(\mathfrak{t}\right) \leq \varrho\left(\mathfrak{t}\right) \leq v(\mathfrak{t}), \mathfrak{t} \in [0,1].$$

Suppose, by contradiction, that $\varrho\left(\mathfrak{t}\right) > v(\mathfrak{t})$. So, considering the definition of $\mathfrak{h}$, we have

$$\mathfrak{h}\left(\mathfrak{t}, \varrho\left(\mathfrak{t}\right)\right) = f\left(\mathfrak{t}, v\left(\mathfrak{t}\right)\right). \tag{5}$$

Consequently, we get

$$D^{\beta}\Big(\Phi_p\left(D^{\alpha}\varrho\left(\mathfrak{t}\right)\right)\Big) = f\left(\mathfrak{t}, v\left(\mathfrak{t}\right)\right), \ \mathfrak{t} \in (0,1). \tag{6}$$

It is understood that v serves as a higher solution for (1). So, we have

$$\begin{cases} D^{\beta}\Big(\Phi_p\left(D^{\alpha}v\left(\mathfrak{t}\right)\right)\Big) \geq f\left(\mathfrak{t}, v\left(\mathfrak{t}\right)\right), \ \mathfrak{t} \in (0,1) \\ D^{\alpha}v\left(0\right) = \int_0^1 \hbar\left(\mu, v\left(\mu\right)\right)\psi\left(\mu\right)d\mu, \\ v^{(j)}\left(0\right) = 0, j = 0, ..., n-2, v^{(n-1)}\left(1\right) = \delta v^{(n-1)}\left(0\right). \end{cases}$$

By (5) and (6), we get

$$D^{\beta}\Big(\Phi_p\left(D^{\alpha}v\left(\mathfrak{t}\right)\right)\Big)-D^{\beta}\Big(\Phi_p\left(D^{\alpha}\varrho\left(\mathfrak{t}\right)\right)\Big)\geq f\left(\mathfrak{t},v\left(\mathfrak{t}\right)\right)-f\left(\mathfrak{t},v\left(\mathfrak{t}\right)\right)=0,\mathfrak{t}\in\left[0,1\right].$$

Thanks to (2), we attain

$$\Phi_p\left(D^{\alpha}v\left(\mathfrak{t}\right)\right)\geq\Phi_p\left(D^{\alpha}\varrho\left(\mathfrak{t}\right)\right),\mathfrak{t}\in\left[0,1\right].$$

We confirm that

$$D^{\alpha}\left(v-\varrho\right)\geq 0$$

is valid by remarking that Φ_p is monotone and increasing. Thanks to Lemma 6, we have $v-\varrho\geq 0$, which is a contradiction. By the same arguments, we prove that $u\leq\varrho$. Consequently, ϱ is a positive solution for (1).

A second main result is to be proved. Before doing this, we consider the hypotheses:

($\mathbf{H}_1^{\clubsuit}$): $f:[0,1]\times\mathbb{R}^{+}\to\mathbb{R}$ is continuous and there exist positive functions $u^{\clubsuit},v^{\clubsuit}$, such that for $\mathfrak{t}\in[0,1]$, $u^{\clubsuit}\left(\mathfrak{t}\right)\leq v^{\clubsuit}(\mathfrak{t})$ and $f\left(\mathfrak{t},v^{\clubsuit}\left(\mathfrak{t}\right)\right)<0<f\left(\mathfrak{t},u^{\clubsuit}\left(\mathfrak{t}\right)\right).$

($\mathbf{H}_2^{\clubsuit}$): The existence of a function that is both continuous and increasing $\Psi:[0,\infty)\to[1,\infty)$, such that $\left|f\left(\mathfrak{t},x\right)\right|\leq\Psi\left(x\right)$ and

$$\int_0^{\infty}\frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\right)\Big)dx}{\Psi\left(x\right)}<+\infty.$$

■

Theorem 10 *Let us consider the hypotheses:* ***($H_1^{\clubsuit}$)-($H_2^{\clubsuit}$)****. Then,* (1) *has at least one positive solution* x *on* $[0,1]$.

Proof. First, we need to prove that $u^{\clubsuit}\left(\mathfrak{t}\right)\leq x\left(\mathfrak{t}\right)\leq v^{\clubsuit}(\mathfrak{t}),\mathfrak{t}\in[0,1]$. To do this task, we define $\Delta:C[0,1]\to\left[u^{\clubsuit},v^{\clubsuit}\right]$:

$$\Delta x=\min\left\{v^{\clubsuit},\max\left\{u^{\clubsuit},x\right\}\right\}=\max\left\{u^{\clubsuit},\min\left\{v^{\clubsuit},x\right\}\right\}.$$

It is clear that Δ is a bounded operator and $u^{\clubsuit}\left(\mathfrak{t}\right)\leq\Delta x\left(\mathfrak{t}\right)\leq v^{\clubsuit}(\mathfrak{t})$, for each $\mathfrak{t}\in[0,1]$. Also, $\Delta x\left(\mathfrak{t}\right)=v^{\clubsuit}(\mathfrak{t})$, when $\Delta x\left(\mathfrak{t}\right)>v^{\clubsuit}(\mathfrak{t})$, and $\Delta x\left(\mathfrak{t}\right)=u^{\clubsuit}(\mathfrak{t})$, when $\Delta x\left(\mathfrak{t}\right)<u^{\clubsuit}(\mathfrak{t})$.

To prove that $x\leq v^{\clubsuit}$, we shall take $\lambda\in(0,1)$ and then consider the problem

$$\left\{\begin{array}{c} D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathfrak{t}\right)\right)\Big)=\lambda f\left(\mathfrak{t},x\left(\mathfrak{t}\right)\right),\ \mathfrak{t}\in(0,1),\\ D^{\alpha}x\left(0\right)=\lambda\int_0^1\hbar\left(\mu,x\left(\mu\right)\right)\psi\left(\mu\right)d\mu+\left(1-\lambda\right)D^{\alpha}x\left(0\right),\\ x^{(j)}\left(0\right)=0,j=0,...,n-2;x^{(n-1)}\left(1\right)=\lambda\delta x^{(n-1)}\left(0\right)+\left(1-\lambda\right)x^{(n-1)}\left(1\right).\end{array}\right.\tag{7}$$

Suppose that $x > v^{\clubsuit}$. Then there is $\mathfrak{t}_0 \in [0,1]$, such that

$$x(\mathfrak{t}_0) > v^{\clubsuit}(\mathfrak{t}_0).$$

Let now define $z(\mathfrak{t}) := x(\mathfrak{t}) - v^{\clubsuit}(\mathfrak{t})$, for each $\mathfrak{t} \in [0,1]$. Since $v^{\clubsuit}(\mathfrak{t}) \in P$, then, the subsequent inequality is true

$$\begin{aligned} & D^{\beta}\Big(\Phi_p\left(D^{\alpha}x(\mathfrak{t})\right)\Big) - D^{\beta}\Big(\Phi_p\left(D^{\alpha}v^{\clubsuit}(\mathfrak{t})\right)\Big) \\ \geq\ & \lambda f\left(\mathfrak{t}, v^{\clubsuit}(\mathfrak{t})\right) - \mathfrak{h}\left(\mathfrak{t}, \Delta x(\mathfrak{t})\right) \\ \geq\ & \lambda f\left(\mathfrak{t}, v^{\clubsuit}(\mathfrak{t})\right) - f\left(\mathfrak{t}, \Delta x(\mathfrak{t})\right) \\ \geq\ & (\lambda - 1) f\left(\mathfrak{t}, v^{\clubsuit}(\mathfrak{t})\right) > 0. \end{aligned}$$

This presents a contradiction. Further, if $\mathfrak{t} = 0$, then we can write

$$\begin{aligned} & D^{\alpha}v^{\clubsuit}(0) \\ <\ & D^{\alpha}x(0) = \lambda \int_0^1 \hbar\left(\mu, \Delta x(\mu)\right)\psi(\mu)\,d\mu + (1-\lambda)\,D^{\alpha}\Delta x(0) \\ \leq\ & \lambda \int_0^1 \hbar\left(\mu, v^{\clubsuit}(\mu)\right)\psi(\mu)\,d\mu + (1-\lambda)\,D^{\alpha}v^{\clubsuit}(0) \\ \leq\ & \lambda D^{\alpha}v^{\clubsuit}(0) + (1-\lambda)\,D^{\alpha}v^{\clubsuit}(0) = D^{\alpha}v^{\clubsuit}(0). \end{aligned}$$

Also, this is a contradiction. Using the identical reasoning as previously stated, we can demonstrate that $u^{\clubsuit} \leq x$. Next, let $r_1 = \left\|v^{\clubsuit} - u^{\clubsuit}\right\|$, then there exists $r_2 > 0$ independent of λ, such that every solution $x \in \left[u^{\clubsuit}, v^{\clubsuit}\right]$ of (7) verifies $|x'(\mathfrak{t})| \leq r_2$, for every $\mathfrak{t} \in [0,1]$. By taking $r_2 > 0$ so that

$$\int_0^{r_2} \frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\right)\Big)dx}{\Psi(x)} \geq r_1,$$

we have to prove that $|x'(\mathfrak{t})| \leq r_2$, for every $\mathfrak{t} \in [0,1]$. Suppose, by contradiction, this is not true. So, there is $\mathfrak{t}_1 \in [0,1]$, such that $|x'(\mathfrak{t}_1)| > r_2$. Then, there is an interval $[\zeta, \xi] \subset [0,1]$, such that the following situations occur:

(i) $x'(\zeta) = 0,\quad x'(\xi) = r_2,\quad 0 < x'(\mathfrak{t}) < r_2\quad$ for all $\mathfrak{t} \in (\zeta, \xi)$,
(ii) $x'(\zeta) = r_2,\quad x'(\xi) = 0,\quad 0 < x'(\mathfrak{t}) < r_2\quad$ for all $\mathfrak{t} \in (\zeta, \xi)$,
(iii) $x'(\zeta) = 0,\quad x'(\xi) = -r_2,\quad -r_2 < x'(\mathfrak{t}) < 0\quad$ for all $\mathfrak{t} \in (\zeta, \xi)$,
(iv) $x'(\zeta) = -r_2,\quad x'(\xi) = 0,\quad -r_2 < x'(\mathfrak{t}) < 0\quad$ for all $\mathfrak{t} \in (\zeta, \xi)$.

To prove (i), using $(\mathbf{H_2^{\clubsuit}})$, the problem (7) implies that

$$\begin{aligned} D^{\beta}\Big(\Phi_p\left(D^{\alpha}x(\mathfrak{t})\right)\Big) &\leq \lambda f\left(\mathfrak{t}, x(\mathfrak{t})\right) \\ &\leq \left|f\left(\mathfrak{t}, x(\mathfrak{t})\right)\right| \\ &\leq \Psi\left(x(\mathfrak{t})\right). \end{aligned}$$

Therefore,

$$\frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathfrak{t}\right)\right)\Big)x'\left(\mathfrak{t}\right)}{\Psi\left(x\left(\mathfrak{t}\right)\right)}\leq x'\left(\mathfrak{t}\right),$$

for all $\mathfrak{t}\in\left(\zeta,\xi\right)$, which indicates that

$$\begin{aligned}&\int_{\zeta}^{\mathfrak{t}}\frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathfrak{s}\right)\right)\Big)x'\left(\mathfrak{s}\right)d\mathfrak{s}}{\Psi\left(x\left(\mathfrak{s}\right)\right)}\\ \leq&\int_{\zeta}^{\mathfrak{t}}x'\left(\mathfrak{s}\right)d\mathfrak{s}=x\left(\mathfrak{t}\right)-x\left(\zeta\right)\leq v^{\clubsuit}\left(\mathfrak{t}\right)-u^{\clubsuit}\left(\mathfrak{t}\right)\leq r_1.\end{aligned}$$

Hence

$$\int_{0}^{x'(\mathfrak{t})}\frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}\theta\right)\Big)d\theta}{\Psi\left(\theta\right)}\leq r_1\leq\int_{0}^{r_2}\frac{D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\right)\Big)dx}{\Psi\left(x\right)}.$$

That is $x'\left(\mathfrak{t}\right)\leq r_2$, for all $\mathfrak{t}\in\left(\zeta,\xi\right)$. Taking into consideration the above four cases, we can say that $\left|x'\left(\mathfrak{t}\right)\right|\leq r_2$, for all $\mathfrak{t}\in\left[0,1\right]$. Furthermore, since f is continues there exists $r_3>0$, such that $\left\|D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\right)\Big)\right\|\leq r_3$, for all $\mathfrak{t}\in[0,1]$. Consequently, we have shown that all possible solutions x of (7) satisfy: $\|x\|\leq r$, where $r=\max\left\{r_1,r_2,r_3\right\}$.

Finally, it remains to show that (7) has a solution.

So let

$$L\left(x\right)=\mathcal{F}\left(\lambda,x\right),\tag{8}$$

where $L:C\left([0,1]\right)\rightarrow C\left([0,1]\right)\times C\left([0,1]\right)\times\mathbb{R}_{+}^{n}$

$$\begin{aligned}Lx\left(\mathfrak{t}\right)&=\Big(D^{\beta}\Big(\Phi_p\left(D^{\alpha}x\left(\mathfrak{t}\right)\right)\Big),D^{\alpha}x\left(0\right),x\left(0\right),x'\left(0\right),...,x^{(n-2)}\left(0\right),x^{(n-1)}\left(1\right)\Big),\\ \mathcal{F}\left(\lambda,x\right)\left(\mathfrak{t}\right)&=\left(-\lambda f\left(\mathfrak{t},\Delta x\left(\mathfrak{t}\right)\right),\mathcal{F}_0\left(\lambda,x\right)\left(\mathfrak{t}\right),\mathcal{F}_1\left(\lambda,x\right)\left(\mathfrak{t}\right),..,\mathcal{F}_n\left(\lambda,x\right)\left(\mathfrak{t}\right)\right),\end{aligned}$$

with,

$$\begin{gathered}\mathcal{F}_0\left(\lambda,x\right)\left(\mathfrak{t}\right)=\lambda\int_0^1\hbar\left(\mu,x\left(\mu\right)\right)\psi\left(\mu\right)d\mu+\left(1-\lambda\right)D^{\alpha}x\left(0\right),\\ \mathcal{F}_1\left(\lambda,x\right)\left(\mathfrak{t}\right)=\mathcal{F}_2\left(\lambda,x\right)\left(\mathfrak{t}\right)=...=\mathcal{F}_{n-1}\left(\lambda,x\right)\left(\mathfrak{t}\right)=0,\\ \mathcal{F}_n\left(\lambda,x\right)\left(\mathfrak{t}\right)=\lambda\delta x^{(n-1)}\left(0\right)+\left(1-\lambda\right)x^{(n-1)}\left(1\right).\end{gathered}$$

We can see that (7) is equivalent to (8). We have proved that L^{-1} exists and it is compact, also, $\mathcal{F}\left(\lambda,x\right)$ is continuous. On other hand, by employing the same techniques as in the Lemma 8 proof's, we can state that $L^{-1}\mathcal{F}\left(\lambda,x\right)$ is completely continuous. Also, it follows from the previous steps that the solutions set $x=L^{-1}\mathcal{F}\left(\lambda,x\right)$ for $0<\lambda<1$, is bounded. Then by the Leray-Schauder alternative [16], $L^{-1}\mathcal{F}\left(1,.\right)$ has a fixed point x_0. There is no doubt that $x_0\in\left[u^{\clubsuit},v^{\clubsuit}\right]$ is a solution of problem (1). The proof of the second main result is thus achieved. ■

4 An Example

Example 11 *Consider the following equation:*

$$\begin{cases} D^{\frac{5}{3}}\left(\Phi_{\frac{4}{3}}\left(D^{\frac{7}{2}}x\left(\mathrm{t}\right)\right)\right)=\frac{\exp(-\mathrm{t})x(\mathrm{t})}{(\mathrm{t}^2+2\mathrm{t}+7)^4},\ \mathrm{t}\in(0,1) \\ D^{\frac{1}{2}}x\left(0\right)=\int\limits_0^1 \hbar\left(\mu,x\left(\mu\right)\right)\psi\left(\mu\right)d\mu, \\ x^{(j)}\left(0\right)=0, j=0,...,n-2; x^{(n-1)}\left(1\right)=\delta x^{(n-1)}\left(0\right). \end{cases} \tag{9}$$

By comparison with (1), we have:

$$\alpha=\frac{7}{2},\beta=\frac{5}{3},n=3,\delta=5,f\left(\mathrm{t},x\left(\mathrm{t}\right)\right)=\frac{\exp\left(-\mathrm{t}\right)x\left(\mathrm{t}\right)}{\left(\mathrm{t}^2+2\mathrm{t}+7\right)^4}$$

and

$$\begin{aligned} & \int\limits_0^1 \hbar\left(\mu,x\left(\mu\right)\right)\psi\left(\mu\right)d\mu \\ = & \int\limits_0^1 \mu^4 x\left(\mu\right)\frac{\sin^4\mu}{\left(9\mu+6\right)^3}d\mu=\int_0^1 x\mu^5\frac{\sin^4\mu}{\left(9\mu+6\right)^3}d\mu \\ = & x\int_0^1 \mu^5\frac{\cos 4\mu-4\cos 2\mu+3}{5832\mu^3+11\,664\mu^2+7776\mu+1728}d\mu \\ = & 2.068\,0\times 10^{-5}<\infty. \end{aligned}$$

Also, we see that f is monotone nondecreasing in its second variable. And, for $a>0$, $f\left(\mathrm{t},a\right)\neq 0$, we have

$$0<\frac{1}{\Gamma\left(\frac{5}{3}\right)}\int_0^1\left(1-\tau\right)^{\frac{5}{3}-1}\frac{\exp\left(-\tau\right)\tau^{\frac{15}{2}}}{\left(\tau^2+2\tau+7\right)^4}=1.555\,4\times 10^{-6}<\infty.$$

Then, ***(H$_1$)*** *and* ***(H$_2$)*** *are satisfied.* *Therefore, by Theorem 9, we confirm that (9) has a solution on $[0,1]$.*

5 Conclusion

We have been concerned with a class of FDEs with integral boundary conditions. The class involves two Caputo fractional derivatives and a $p-$Laplacian operator. Compared to many existent FDEs, the above-considered problem is more general. We have identified two separate existing results for positive solutions to the proposed problem based on the guidance of the features of the Green function, the method of upper and lower solutions, the Leray-Schauer and other concepts theory, and eventually, one of the major findings has indeed been highlighted by a well-detailed example.

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