Analysis of a more general class of (k,ψ)-Hilfer impulsive fractional Dirichlet problem☆

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Abstract

In this paper, we consider (k,ψ)-Hilfer fractional differential equations involving instantaneous and non-instantaneous impulses, supplemented with Dirichlet boundary conditions. By establishing the variational structure of the stated problem and combining with the Ekeland's variational principle, the existence result is obtained. Finally, an example is given to illustrate the application of our main result.

Keywords: (k,ψ)-Hilfer Fractional differential equation, Dirichlet problem, Instantaneous impulses, Non-instantaneous impulses, Ekeland's variational principle

2000 MSC: 34A08, 34B15, 34A37

1. Introduction

In the past few decades, the subject of fractional calculus has received great interest. As an important branch of mathematical analysis, fractional calculus effectively describe inherited properties of physical phenomena and systems occurring in physical sciences and engineering problems [1, 2]. With the development of fractional calculus theory, there are more than ten definitions of fractional calculus, and these definitions are closely related. However, due to the different application scope and initial value conditions involved in the definition, there are some uncertainties in the application. Therefore, the classification and unification of the definition of fractional calculus is a very meaningful work. Addressing it, some scholars had tried to explore possible solutions and proposed some definitions of generalized fractional derivatives. For example, Oliveira and Capelas de Oliveira [3] proposed a new fractional derivative, the Hilfer-Katugampola fractional derivative. Almeida [4] using the idea of the fractional derivative in the Caputo sense, proposed a new fractional derivative called ψ-Caputo derivative. Sousa and Capelas de Oliveira [5] applying the idea of the fractional derivative in the Hilfer sense, introduced a new fractional derivative with respect to another function the so-called ψ-Hilfer fractional derivative. Kucche and Mali [6] utilizing the definition of k-gamma function, presented the most generalized variant of the Hilfer derivative so-called (k,ψ)-Hilfer fractional derivative.

On the other hand, in 2011, Jiao and Zhou [7] proved the fractional derivative space $E_0^{\alpha,p}$ and established the variational structure of the fractional differential equations with Dirichlet boundary conditions. In 2017, Tian and Nieto [8] proved the fractional derivative space $E^{\alpha,p}$ and established the variational structure of the fractional differential equations with Sturm-Liouville boundary conditions. These two pioneering works make critical point theory an effective tool for studying fractional boundary value problems involving Riemann-Liouville and Caputo fractional derivatives. For theoretical applications of fractional differential equations, for instance, see [9-12]. More recently, some scholars have innovatively applied the critical point theory to study the fractional boundary value problems involving generalized fractional derivative by proving the new fractional derivative spaces, and obtained some interesting results, see [13-22]. For example, Sousa et al. [13] proved a fractional derivative space $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$ and used the mountain pass theorem

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for the study of a ψ-Hilfer fractional differential equations with fractional integral boundary conditions. They also studied existence results for ψ-Hilfer fractional boundary value problem with p-Laplacian operator via variational methods [14-18]. Ezati and Nyamoradi [19] investigated the existence and multiplicity results for Kirchhoff ψ-Hilfer fractional p-Laplacian equations with fractional integral boundary conditions by using the genus theory. Li et al. [20] considered multiplicity results for a class of instantaneous and non-instantaneous impulsive fractional Dirichlet problem involving ψ-Caputo fractional derivative with the help of critical point theorem. Recently, Ledesma and Nyamoradi [21] introduced a new fractional derivative space ${}^kE_0^{\alpha,v,\psi}[a,b]$, and established the variational structure for the following fractional Dirichlet problem involving (k,ψ)-Hilfer fractional derivative under this functional space

$$\begin{cases} {}^{k,H}D_{b-}^{\alpha,v;\psi}({}^{k,H}\mathbb{D}_{a+}^{\alpha,v;\psi}u(t)) = f(t,u(t)),\ t\in(a,b), \\ u(a)=u(b)=0, \end{cases} \tag{1.1}$$

where ${}^{k,H}D_{b-}^{\alpha,v;\psi}$ is the right-sided (k,ψ)-Hilfer fractional derivative, ${}^{k,H}\mathbb{D}_{a+}^{\alpha,v;\psi}$ is the left-sided (k,ψ)-Hilfer-Caputo fractional derivative, $\alpha\in(k/2,1)$, $k\in[1,2)$, $v\in[0,1)$, $f\in C([a,b]\times\mathbb{R},\mathbb{R})$. By using the variational methods and critical point theory, they prove the existence of weak solutions for problem (1.1).

Very recently, in [22], Torres Ledesma and Nyamoradi also discussed the following (k,ψ)-Hilfer fractional Dirichlet problem with impulses

$$\begin{cases} {}^{k,H}D_{T-}^{\alpha,v;\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)) + a(t)u(t) = f(t,u(t)),\ t\neq t_j,\ a.e.\ t\in[0,T], \\ \Delta{}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t_j)){}^kI_{0+}^{v(k-\alpha);\psi}u(t_j) = I_j(u(t_j)),\ j=1,2,\cdots,n, \\ u(0)=u(T)=0, \end{cases} \tag{1.2}$$

where ${}^{k,H}D_{T-}^{\alpha,v;\psi}$ is the right-sided (k,ψ)-Hilfer fractional derivative, ${}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}$ is the left-sided (k,ψ)-Hilfer-Caputo fractional derivative, ${}^kI_{0+}^{(\cdot);\psi}$, ${}^kI_{T-}^{(\cdot);\psi}$ are the left-sided and right-sided (k,ψ)-Riemann-Liouville fractional integral, respectively, $\alpha\in(k/2,1)$, $k\in[1,2)$, $v\in[0,1)$, $f\in C([a,b]\times\mathbb{R},\mathbb{R})$, $I_j\in C(\mathbb{R},\mathbb{R})$, $1,2,\cdots,n$. They obtain the existence of solutions for problem (1.2) by using variational methods and critical point theory.

The study of fractional differential equations (FDEs) with impulsive effects has received great attention because of impulsive FDEs become increasingly essential in physical engineering, economics, population dynamics, and social sciences [23]. In [24], Hernandez and O'Regan introduced the concept of non-instantaneous impulse. Thereafter, two main approaches of impulsive effects to FDEs are proposed such as instantaneous and non-instantaneous impulses. Instantaneous impulses: the duration of these changes is relatively short compared to the overall duration of the whole process. Non-instantaneous impulses: an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval [25]. In resent years, a vast number of study has been made for FDEs with impulses, for instance, see [26, 27]. Interestingly, some scholars have considered the fractional boundary value problems with both instantaneous and non-instantaneous impulsive effects by using the critical point theory, see [20, 28-32].

Looking in the above-mentioned contributions, a natural question is asked: Can we investigate the existence of solutions for (k,ψ)-Hilfer fractional Dirichlet problem generated by instantaneous and non-instantaneous impulsive effects by using the critical point theory? In the present paper, we give an positive answer for this question. More precisely, we considering the existence of solutions for the following FDEs with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}^{k,H}D_{T-}^{\alpha,v;\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)) = f_j(t,u(t)),\ t\in(s_j,t_{j+1}],\ j=0,1,2,\cdots,n, \\ \Delta{}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t_j)) = I_j({}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)),\ j=0,1,2,\cdots,n, \\ {}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)) = {}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t_j^+)),\ t\in(t_j,s_j],\ j=1,2,\cdots,n, \\ {}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(s_j^-)) = {}^kI_{T-}^{(1-v)(k-\alpha);\psi}({}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(s_j^+)),\ j=1,2,\cdots,n, \\ u(0)=u(T)=0, \end{cases} \tag{1.3}$$

where ${}^{k,H}D^{\alpha,v;\psi}_{T-}$ is the right-sided (k,ψ)-Hilfer fractional derivative, ${}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}$ is the left-sided (k,ψ)-Hilfer-Caputo fractional derivative, ${}^{k}I^{(\cdot);\psi}_{0+}$, ${}^{k}I^{(\cdot);\psi}_{T-}$ are the left-sided and right-sided (k,ψ)-Riemann-Liouville fractional integral, respectively, $\alpha \in (k/2, 1)$, $k \in [1, 2)$, $v \in [0, 1)$, $f_j \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_n < t_{n+1} = T$,

$$\Delta {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t_j)) = {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t_j^+)) - {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t_j^-)),$$

$${}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t_j^\pm)) = \lim_{t\to t_j^\pm} {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)),$$

$${}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(s_j^\pm)) = \lim_{t\to t_j^\pm} {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)).$$

In this problem, the instantaneous impulses start abruptly at point t_j and the non-instantaneous impulses continue during the intervals $(t_j, s_j]$. In order to state our main result, we shall present the following assumptions:

(A_1) There exist $a_j, b_j>0, (j=0, 1, 2, \ldots, n)$ and $\sigma\in[0, 1)$ such that $|f_j(t, u)|\leq a_j + b_j|u|^\sigma, \forall(t, u)\in[0, T]\times\mathbb{R}$.

(A_2) There exist $c_j, d_j > 0$, $\delta_i \in [0, 1)$, $(j=1, 2, \ldots, n)$ such that $|I_j(u)| \leq c_j + d_j|u|^{\delta_j}$, $\forall u \in \mathbb{R}$.

In this study, we focus on establishing the existence of solutions for problem (1.3) by using the Ekeland's variational principle. Compared with the existing literature, the main innovative contributions can be summarized as follows: First, this type of differential model with impulsive effects is more general than the normal case ($k = 1$, $t_j = s_j$), so our result becomes more extensive. Second, we present a new velocity pulse term $\Delta {}^{k}I^{(1-v)(k-\alpha);\psi}_{T-}({}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t_j)) = I_j({}^{k}I^{v(k-\alpha);\psi}_{0+}u(t_j))$, which is more reasonable than that given in problem (1.2). Third, we provide a new energy functional under the influence of impulsive effects, so the critical point theory can be used to deal with this type of problem.

The rest of this article is arranged as follows: In Sect. 2, we introduce notations, definitions, and some preliminary notions about (k,ψ)-Hilfer fractional calculus and fractional derivative space. Also we give the Ekeland's variational principle. In Sect. 3, we study the existence of weak solutions for problem (1.3). An illustrative example is presented in the last section.

2. Preliminaries

In this section, we recall some definitions and propositions for the (k,ψ)-Hilfer fractional calculus, fractional derivative space ${}^{k}E^{\alpha,v;\psi}_{0}[0, T]$ and the Ekeland's variational principle.

Definition 2.1. ([21, 22]) Let $p \in [1, \infty)$. The space of p-integrable functions with respect to a function ψ is defined as:

$$L^p_\psi(a, b) = \left\{u : (a, b) \to \mathbb{R} : \int_a^b |u(t)|^p\psi'(t)dt < \infty\right\}.$$

This space endowed with the norm

$$||u||_{L^p_\psi(a,b)} = \left(\int_a^b |u(t)|^p\psi'(t)dt\right)^{1/p},$$

is a Banach space.

Definition 2.2. ([21, 22]) Let $\psi : [a, b] \to \mathbb{R}$ be an increasing continuous function and continuous derivative with $\psi'(t) \neq 0$ for all $t \in [a, b]$. For $u \in L^1[a, b]$ and $k \in \mathbb{R}^+$, the left and right (k, ψ)-Riemann-Liouville fractional integrals of order $\alpha > 0$ of the function u is given by

$${}^{k}I^{\alpha;\psi}_{a+}u(t) = \frac{1}{k\Gamma_k(\alpha)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{\alpha}{k}-1}u(s)ds,$$

and

$${}^{k}I^{\alpha;\psi}_{b-}u(t) = \frac{1}{k\Gamma_k(\alpha)}\int_t^b \psi'(s)(\psi(s) - \psi(t))^{\frac{\alpha}{k}-1}u(s)ds,$$

where $\Gamma_k(\alpha)$ is the k-gamma function is defined as

$$\Gamma_k(\alpha) = \int_0^\infty s^{\alpha-1} e^{-\frac{s^k}{k}} ds.$$

Definition 2.3. ([21, 22]) Let $\alpha \in (0,1)$, $k \in (0,\infty)$, $v \in [0,1]$, $m = \lceil \frac{\alpha}{k} \rceil$, $\psi \in C^m[a,b]$ be an increasing function with $\psi'(t) \neq 0$, $t \in [a,b]$ and $u \in C^m[a,b]$. Then, the left and right (k,ψ)-Hilfer fractional derivative of a function u of order α and type v is defined by

$${}^{k,H}D_{a+}^{\alpha,v;\psi} u(t) = {}^k I_{a+}^{v(mk-\alpha);\psi} \Big(\frac{k}{\psi'(t)} \frac{d}{dt}\Big)^m {}^k I_{a+}^{(1-v)(mk-\alpha);\psi} u(t),$$

and

$${}^{k,H}D_{b-}^{\alpha,v;\psi} u(t) = {}^k I_{b-}^{v(mk-\alpha);\psi} \Big(-\frac{k}{\psi'(t)} \frac{d}{dt}\Big)^m {}^k I_{b-}^{(1-v)(mk-\alpha);\psi} u(t).$$

Definition 2.4. ([21, 22]) The fractional derivative space ${}^k E_0^{\alpha,v;\psi}[0,T]$ defined as

$${}^k E_0^{\alpha,v;\psi}[0,T] = \overline{C_0^\infty([0,T],\mathbb{R})}^{||\cdot||_{\alpha,v}},$$

where

$$||u||_{\alpha,v} = \bigg(\int_0^T |u(t)|^2 \psi'(t) dt + \int_0^T |{}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi} u(t)|^2 \psi'(t) dt\bigg)^{1/2}.$$

Remark 2.1. Let $\alpha \in (k/2,1)$, $k \geq 1$ and $v \in [0,1)$. The fractional derivative space ${}^k E_0^{\alpha,v;\psi}[0,T]$ defined as

$${}^k E_0^{\alpha,v;\psi}[0,T] = \{u \in L_\psi^2[0,T] : {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi} u \in L_\psi^2[0,T] \text{ and } u(0) = u(T) = 0\}.$$

Proposition 2.1. ([21, 22]) The fractional derivative space ${}^k E_0^{\alpha,v;\psi}[0,T]$ is a reflexive and separable Hilbert space.

Proposition 2.2. ([21, 22]) Let $u \in {}^k E_0^{\alpha,v;\psi}[0,T]$, then

$${}^k I_{0+}^{\alpha;\psi} {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi} u(t) = u(t), \ a.e. \text{ in } [0,T],$$

and

$$||u||_{L_\psi^2[0,T]} \leq \frac{(\psi(T) - \psi(0))^{\alpha/k}}{\Gamma_k(\alpha+k)} ||{}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi} u||_{L_\psi^2[0,T]}.$$

Remark 2.2. As a consequence of Proposition 2.2, the space ${}^k E_0^{\alpha,v;\psi}[0,T]$ can endowed with the equivalent norm

$$||u|| = \Big(\int_0^T |{}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi} u(t)|^2 \psi'(t) dt\Big)^{1/2}.$$

Proposition 2.3. ([21, 22]) If $\alpha \in (k/2,1)$, $k \geq 1$ and $v \in [0,1)$, then the embedding ${}^k E_0^{\alpha,v;\psi}[0,T] \hookrightarrow C[0,T]$ is continuous. Moreover

$$||u||_\infty \leq \frac{(\psi(T) - \psi(0))^{\frac{\alpha}{k} - \frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}} ||u||.$$

Proposition 2.4. ([21, 22]) Let $u, w \in L_\psi^2(a,b)$. Then

$$\int_a^b {}^k I_{a+}^{\alpha;\psi} u(t) w(t) \psi'(t) dt = \int_a^b u(t) {}^k I_{b-}^{\alpha;\psi} w(t) \psi'(t) dt.$$

Proposition 2.5. ([21, 22]) Let $\alpha \in (0,1)$ and $k \in \mathbb{R}^+$. If $u \in C[0,T]$, then

$$\lim_{t \to 0+} {}^k I_{0+}^{\alpha;\psi} u(t) = 0 \text{ and } \lim_{t \to T-} {}^k I_{T-}^{\alpha;\psi} u(t) = 0.$$

Proposition 2.6. ([21, 22]) Let $\alpha \in (0,1)$, $k \in (0,+\infty)$. If $u \in C[a,b]$, then ${}^k I_{a+}^{\alpha;\psi} u, {}^k I_{b-}^{\alpha;\psi} u \in C[a,b]$. Moreover,

$$||{}^k I_{a+}^{\alpha;\psi} u||_\infty \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)} ||u||_\infty \text{ and } ||{}^k I_{b-}^{\alpha;\psi} u||_\infty \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)} ||u||_\infty.$$

Proposition 2.7. ([21, 22]) Let $p > 1$, $k \in [1, p)$, $0 < \frac{k}{p} < \alpha < 1$. Then, for any $u \in L^p_\psi[a, b]$ we have ${}^kI^{\alpha;\psi}_{a+}u, {}^kI^{\alpha;\psi}_{b-}u \in C[a, b]$. Moreover

$$\lim_{t\to a+} {}^kI^{\alpha;\psi}_{a+}u(t) = 0 \text{ and } \lim_{t\to b-} {}^kI^{\alpha;\psi}_{b-}u(t) = 0.$$

Proposition 2.8. ([21, 22]) Let $\alpha \in (\frac{k}{2}, 1)$, $k \geq 1$ and $v \in [0, 1)$. Suppose that $(u_n)_{n\in\mathbb{N}}$ be a sequence that converges weakly to u in ${}^kE^{\alpha,v;\psi}_0[0, T]$. Then, up to a subsequence it holds that

$$\lim_{n\to+\infty} ||u - u_n||_\infty = 0.$$

Theorem 2.1. ([33]) Let M be a complete metric space and let let $\Phi : M \to (-\infty, +\infty]$ be a lower semi-continuous function, bounded from below and not identically equal to $+\infty$. Let $\varepsilon > 0$ be given and $u \in M$ be such that $\Phi(u) \leq \inf_M \Phi+\varepsilon$. Then there exists $w \in M$ such that $\Phi(w) \leq \Phi(u)$, $d(u, w) \leq 1$, and for each $z \neq w$ in M, $\Phi(z) > \Phi(w)-\varepsilon d(w, z)$, where $d(\cdot, \cdot)$ denotes the distance between two elements in M.

3. Main result

In this section we construct the energy functional to problem (1.3) and establish the existence theorem of classical solutions. Through this section we assume that $k \in [1, 2), \alpha \in (\frac{k}{2}, 1)$ and $v \in [0, 1)$.

Lemma 3.1. *A function $u \in {}^kE^{\alpha,v;\psi}_0[0, T]$ is a solution of problem (1.3), then the following identity:*

$$\begin{aligned}&\int_0^T {}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}w(t)\psi'(t)dt\\&= -k\sum_{j=1}^{n} I_j({}^kI^{v(k-\alpha);\psi}_{0+}u(t_j)){}^kI^{v(k-\alpha);\psi}_{0+}w(t_j) + \sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} f_j(t, u(t))w(t)\psi'(t)dt,\end{aligned} \tag{3.1}$$

holds for any $w \in {}^kE^{\alpha,v;\psi}_0[0, T]$.

Proof. For $w \in {}^kE^{\alpha,v;\psi}_0[0, T]$, one has $w(0) = w(T) = 0$. By Proposition 2.4, we have

$$\begin{aligned}&\int_0^T {}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}w(t)\psi'(t)dt\\&= \int_0^T {}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^kI^{(1-v)(k-\alpha);\psi}_{0+}\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right){}^kI^{v(k-\alpha);\psi}_{0+}w(t)\psi'(t)dt\\&= k\int_0^T {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)\frac{d}{dt}{}^kI^{v(k-\alpha);\psi}_{0+}w(t)dt\\&= k\Big[\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)\frac{d}{dt}{}^kI^{v(k-\alpha);\psi}_{0+}w(t)dt\\&\quad+\sum_{j=1}^{n}\int_{t_j}^{s_j} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)\frac{d}{dt}{}^kI^{v(k-\alpha);\psi}_{0+}w(t)dt\Big].\end{aligned} \tag{3.2}$$

On the one hand,

$$\begin{aligned}&k\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)\frac{d}{dt}{}^kI^{v(k-\alpha);\psi}_{0+}w(t)dt\\&= k\sum_{j=0}^{n} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^kI^{v(k-\alpha);\psi}_{0+}w(t)\Big|^{t^-_{j+1}}_{s^+_j}\\&\quad+\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}}\left(\frac{-k}{\psi'(t)}\frac{d}{dt}\right){}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^kI^{v(k-\alpha);\psi}_{0+}w(t)\psi'(t)dt\\&= k\sum_{j=0}^{n}\left[\lim_{t\to t^-_{j+1}} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^kI^{v(k-\alpha);\psi}_{0+}w(t) - \lim_{t\to s^+_j} {}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t){}^kI^{v(k-\alpha);\psi}_{0+}w(t)\right]\\&\quad+\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} {}^kI^{v(k-\alpha);\psi}_{T-}\left(\frac{-k}{\psi'(t)}\frac{d}{dt}\right){}^kI^{(1-v)(k-\alpha);\psi}_{T-}{}^{k,H}\mathbb{D}^{\alpha,v;\psi}_{0+}u(t)w(t)\psi'(t)dt\end{aligned}$$

$$= k\sum_{j=0}^{n}\left[\lim_{t\to t_{j+1}^-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) - \lim_{t\to s_j^+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)\right]$$
$$+\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} {}^{k,H}D_{T-}^{\alpha,v;\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)w(t)\psi'(t)dt. \tag{3.3}$$

On the other hand,

$$k\sum_{j=1}^{n}\int_{t_j}^{s_j} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\frac{d}{dt}\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)dt$$
$$= k\sum_{j=1}^{n} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)\Big|_{t_j^+}^{s_j^-}$$
$$- k\sum_{j=1}^{n}\int_{t_j}^{s_j}\frac{d}{dt}\, {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)dt \tag{3.4}$$
$$= k\sum_{j=1}^{n}\left[\lim_{t\to s_j^-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) - \lim_{t\to t_j^+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)\right].$$

Combining with (3.2), (3.3) and (3.4), we have

$$\int_0^T {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}w(t)\psi'(t)dt$$
$$= k\sum_{j=1}^{n}\Big[\lim_{t\to s_j^-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) - \lim_{t\to s_j^+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)$$
$$+ \lim_{t\to t_j^-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) - \lim_{t\to t_j^+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)\Big]$$
$$+ \lim_{t\to T-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) - \lim_{t\to 0+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t)$$
$$+\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} f_j(t,u(t))w(t)\psi'(t)dt. \tag{3.5}$$

Note that $w\in C_0^\infty(0,T)$, Proposition 2.5 gives that

$$\lim_{t\to 0+} {}^kI_{0+}^{v(k-\alpha);\psi}w(t) = 0. \tag{3.6}$$

In view of $u\in {}^kE_0^{\alpha,v;\psi}[0,T]$, which combined with Proposition 2.7, we obtain

$$\lim_{t\to T-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t) = 0. \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\lim_{t\to T-} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) = 0, \tag{3.8}$$
$$\lim_{t\to 0+} {}^kI_{T-}^{(1-v)(k-\alpha);\psi}\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t) = 0. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.5), we obtain the desired result (3.1). The proof is completed. □

Definition 3.1. A function $u\in {}^kE_0^{\alpha,v;\psi}[0,T]$ is called a weak solution of problem (1.3), if (3.1) holds for any $w\in {}^kE_0^{\alpha,v;\psi}[0,T]$.

Define the functional $\Phi: {}^kE_0^{\alpha,v;\psi}[0,T]\to\mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_0^T |{}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)|^2\psi'(t)dt - \sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} F_j(t,u(t))\psi'(t)dt + k\sum_{j=1}^{n}\int_0^{{}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)} I_j(s)ds, \tag{3.10}$$

where $F_j(t,u) = \int_0^u f_j(t,s)ds$. As in [22], under our assumption, we can obtain $\Phi\in C^1({}^kE_0^{\alpha,v;\psi}[0,T],\mathbb{R})$ with

$$\langle\Phi'(u),w\rangle = \int_0^T {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)\, {}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}w(t)\psi'(t)dt - \sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} f_j(t,u(t))w(t)\psi'(t)dt$$
$$+ k\sum_{j=1}^{n} I_j({}^kI_{0+}^{v(k-\alpha);\psi}u(t_j))\, {}^kI_{0+}^{v(k-\alpha);\psi}w(t_j). \tag{3.11}$$

Thus, the weak solutions of problem (1.3) are the critical points of Φ.

Lemma 3.2. *Suppose (A_1) and (A_2) are satisfied. Then there exists $r > 0$ such that $\Phi(u) > 0$ for $u \in {}^kE_0^{\alpha,v;\psi}[0,T]$ with $||u|| = r$.*

Proof. From (A_1), we obtain

$$F_j(t,u) \le a_j|u| + \frac{b_j}{\sigma+1}|u|^{\sigma+1}, \quad \sigma \in [0,1),$$

and then, by Proposition 2.3, we derive

$$\begin{aligned}
&\sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} F_j(t,u(t))\psi'(t)dt \\
&\le (\psi(T)-\psi(0))\sum_{j=0}^{n}\Big(a_j||u||_\infty + \frac{b_j}{\sigma+1}||u||_\infty^{\sigma+1}\Big) \\
&\le (\psi(T)-\psi(0))\sum_{j=0}^{n}\left[a_j\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}||u||}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}} + b_j\frac{||u||^{\sigma+1}}{\sigma+1}\left(\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}\right)^{\sigma+1}\right].
\end{aligned} \tag{3.12}$$

On the other hand, by (A_2) and Proposition 2.3, we have

$$\begin{aligned}
&\left|\int_0^{{}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)} I_j(s)ds\right| \\
&\le c_j\left|{}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)\right| + \frac{d_j}{1+\delta_j}\left|{}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)\right|^{1+\delta_j} \\
&\le \frac{c_j(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}||u||_\infty + \frac{d_j}{1+\delta_j}\left[\frac{(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}\right]^{1+\delta_j}||u||_\infty^{1+\delta_j} \\
&\le \frac{c_j(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}||u|| \\
&\quad + \frac{d_j}{1+\delta_j}\left[\frac{(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}\right]^{1+\delta_j}\left[\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}\right]^{1+\delta_j}||u||^{1+\delta_j}.
\end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_0^T |{}^{k,H}\mathbb{D}_{0+}^{\alpha,v;\psi}u(t)|^2\psi'(t)dt - \sum_{j=0}^{n}\int_{s_j}^{t_{j+1}} F_j(t,u(t))\psi'(t)dt + k\sum_{j=1}^{n}\int_0^{{}^kI_{0+}^{v(k-\alpha);\psi}u(t_j)} I_j(s)ds \\
&\ge \frac{1}{2}||u||^2 - (\psi(T)-\psi(0))\sum_{j=0}^{n}\left[a_j\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}||u||}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}} + b_j\frac{||u||^{\sigma+1}}{\sigma+1}\left(\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}\right)^{\sigma+1}\right] \\
&\quad - \sum_{j=1}^{n}\frac{kc_j(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}||u|| \\
&\quad - \sum_{j=1}^{n}\frac{kd_j}{1+\delta_j}\left[\frac{(\psi(T)-\psi(0))^{\frac{v(k-\alpha)}{k}}}{v(k-\alpha)\Gamma_k(v(k-\alpha))}\right]^{1+\delta_j}\left[\frac{(\psi(T)-\psi(0))^{\frac{\alpha}{k}-\frac{1}{2}}}{k\Gamma_k(\alpha)\big(\frac{\alpha+k}{2k}\big)^{\frac{1}{2}}}\right]^{1+\delta_j}||u||^{1+\delta_j}.
\end{aligned} \tag{3.14}$$

Hence, there exists $r > 0$ such that $\Phi(u) > 0$ for $u \in {}^kE_0^{\alpha,v;\psi}[0,T]$ with $||u|| = r$. □

Theorem 3.1. *Suppose that (A_1) and (A_2) hold. Then problem (1.3) has at least one weak solution.*

Proof. We shall apply Theorem 2.1 to prove the theorem. Let r be defined in Lemma 3.2, $M = \overline{B_r(0)} \subset {}^kE_0^{\alpha,v;\psi}[0,T]$. Since, Φ is continuous, thus Φ is lower semi-continuous. In view of (3.14), we can show that Φ is bounded from below.

Next, we divide our proof in four steps.

(i). We claim that there exists $w \in M$ such that

$$c - \varepsilon < \Phi(w) \leq c + \varepsilon, \quad c = \inf_{u \in M} \Phi(u). \tag{3.15}$$

By Lemma 3.2, we have $\inf_{u \in \partial B_r(0)} \Phi(u) > 0$. Recalling the definition (3.10) of Φ, a direct computation shows that $\Phi(0) = 0$. Hence, $\inf_{u \in B_r(0)} \Phi(u) \leq 0$ and $\inf_{u \in B_r(0)} \Phi(u) = \inf_{u \in M} \Phi(u)$. Let $0 < \varepsilon < \inf_{u \in \partial B_r(0)} \Phi(u) - c$, it is easy to see that there exists $z \in M$ such that $\Phi(z) \leq \varepsilon + \inf_{u \in M} \Phi(u)$. By Theorem 2.1, there exists $w \in M$ such that

$$\inf_{u \in M} \Phi(u) - \varepsilon < \Phi(w) \leq \Phi(z) \leq \varepsilon + \inf_{u \in M} \Phi(u),$$

then (3.15) is proved. One the other hand, by Theorem 2.1, for any $u \neq w$ in M, we have

$$\Phi(w) < \Phi(u) + \varepsilon ||u - w||. \tag{3.16}$$

(ii). We prove that

$$||\Phi'(w)||_{({}^k E_0^{\alpha,v;\psi})^*} \leq \varepsilon. \tag{3.17}$$

To see this, we define functional

$$J(u) = \Phi(u) + \varepsilon ||u - w||. \tag{3.18}$$

From (3.17) and (3.18), it follows

$$J(w) = \Phi(w) < \Phi(u) + \varepsilon ||u - w|| = J(u), \quad \text{for all } u \neq w.$$

Hence, w is the minimum point of (3.18). Therefore,

$$J(w + tu) - J(w) \geq 0, \quad \text{for all } u \in B_r(0).$$

By using the function limit property, we have

$$\begin{aligned} 0 \leq \lim_{t \to 0+} \frac{J(w+tu) - J(w)}{t} &= \lim_{t \to 0+} \frac{\Phi(w+tu) + \varepsilon ||w + tu - w|| - \Phi(w)}{t} \\ &= < \Phi'(w), u > + \varepsilon ||u||, \end{aligned}$$

and

$$\begin{aligned} 0 \geq \lim_{t \to 0-} \frac{J(w+tu) - J(w)}{t} &= \lim_{t \to 0-} \frac{\Phi(w+tu) + \varepsilon ||w + tu - w|| - \Phi(w)}{t} \\ &= < \Phi'(w), u > - \varepsilon ||u||. \end{aligned}$$

Thus, (3.17) holds.

(iii). We have to show the existence of weak solution u_0 of problem (1.3). In fact, by (3.15) and (3.17), there exists sequence $\{u_n\} \subset B_r(0)$ such that

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0.$$

Since (3.14) yields $\{u_n\}$ is bounded. In view of the reflexivity of ${}^k E_0^{\alpha,v;\psi}[0,T]$, the sequence $\{u_n\}$ weakly converges to u_0 in ${}^k E_0^{\alpha,v;\psi}[0,T]$. It follows from Proposition 2.8 that the sequence $\{u_n\}$ converges uniformly to u_0 in $C[0,T]$. We now prove that $\{u_n\}$ is strongly converges to u_0 in ${}^k E_0^{\alpha,v;\psi}[0,T]$. In fact,

$$\begin{aligned} &\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle \to 0, \\ &\sum_{j=0}^{n} \int_{s_j}^{t_{j+1}} (f_j(t, u_n(t)) - f_j(t, u_0(t)))(u_n(t) - u_0(t))\psi'(t)dt \to 0, \\ &\sum_{j=1}^{n} (I_j({}^k I_{0+}^{v(k-\alpha);\psi} u_n(t_j)) - I_j({}^k I_{0+}^{v(k-\alpha);\psi} u_0(t_j)))({}^k I_{0+}^{v(k-\alpha);\psi} u_n(t_j) - {}^k I_{0+}^{v(k-\alpha);\psi} u_0(t_j)) \to 0, \end{aligned}$$

as $n \to +\infty$. Recalling the Eq. (3.11), a direct computation shows that

$$\begin{aligned}
&\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle \\
&= ||u_n - u_0||^2 - \sum_{j=0}^{n} \int_{s_j}^{t_{j+1}} (f_j(t, u_n(t)) - f_j(t, u_0(t)))(u_n(t) - u_0(t))\psi'(t)dt \\
&\quad + k \sum_{j=1}^{n} (I_j({}^k I_{0+}^{v(k-\alpha);\psi} u_n(t_j)) - I_j({}^k I_{0+}^{v(k-\alpha);\psi} u_0(t_j)))({}^k I_{0+}^{v(k-\alpha);\psi} u_n(t_j) - {}^k I_{0+}^{v(k-\alpha);\psi} u_0(t_j)).
\end{aligned}$$

So, $||u_n - u|| \to 0$ as $n \to +\infty$. That is, $\{u_n\}$ converges strongly to u_0 in ${}^k E_0^{\alpha,v;\psi}[0,T]$. Consequently,

$$\Phi(u_0) = c, \quad \Phi'(u_0) = 0.$$

Therefore, u_0 is a weak solution of problem (1.3). □

Example 3.1. Let $\alpha = \frac{4}{5}$, $v = \frac{1}{2}$, $k = \frac{3}{2}$, $T > 0$, $a_j, b_j, c_j, d_j > 0$, $j = 0, 1, 2, \cdots, n$. Consider the following problem

$$\begin{cases}
{}^{\frac{3}{2},H} D_{T-}^{\frac{4}{5},\frac{1}{2};\psi} ({}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(t)) = f_j(t, u(t)), \ \ t \in (s_j, t_{j+1}], \ \ j = 0, 1, 2, \cdots, n, \\
\Delta {}^{\frac{3}{2}} I_{T-}^{\frac{7}{20};\psi} {}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(t_j) = I_j({}^{\frac{3}{2}} I_{0+}^{\frac{7}{20};\psi} u(t_j)), \ \ j = 0, 1, 2, \cdots, n, \\
{}^{\frac{3}{2}} I_{T-}^{\frac{7}{20};\psi} {}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(t) = {}^{\frac{3}{2}} I_{T-}^{\frac{7}{20};\psi} {}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(t_j^+), \ \ t \in (t_j, s_j], \ \ j = 1, 2, \cdots, n, \\
{}^{\frac{3}{2}} I_{T-}^{\frac{7}{20};\psi} {}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(s_j^-) = {}^{\frac{3}{2}} I_{T-}^{\frac{7}{20};\psi} {}^{\frac{3}{2},H}\mathbb{D}_{0+}^{\frac{4}{5},\frac{1}{2};\psi} u(s_j^+), \ \ j = 1, 2, \cdots, n, \\
u(0) = u(T) = 0,
\end{cases} \tag{3.19}$$

where $\psi : [0, T] \to \mathbb{R}$ is an increasing function, $\psi'(x) \neq 0$ for all $t \in [0, T]$, $f_j(t, u(t)) = a_j \sin |u(t)| + b_j(u(t))^{\frac{3}{5}} \cos u(t)$, $I_j(u) = c_j + d_j u^{\frac{2}{3}}$. Easily, we can check that (A_1) and (A_2) hold. Consequently, by Theorem 3.1, problem (3.19) has a weak solution.

Availability of data and material

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Funding

This research is supported by the Anhui Provincial Natural Science Foundation (2208085QA05) and National Natural Science Foundation of China (11601007).

Authors' contributions

The authors have made equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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