Fréchet-Urysohn property of quasicontinuous functions

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Abstract

The aim of this paper is to study the Fréchet-Urysohn property of the space $Q_p(X,\mathbb{R})$ of real-valued quasicontinuous functions, defined on a Hausdorff space X, endowed with the pointwise convergence topology.

It is proved that under *Suslin's Hypothesis*, for an open Whyburn space X, the space $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn if and only if X is countable. In particular, it is true in the class of first-countable regular spaces X.

In ZFC, it is proved that for a metrizable space X, the space $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn if and only if X is countable.

Key words: quasicontinuous function, Fréchet-Urysohn, Lusin space, open Whyburn space, k-Fréchet-Urysohn, γ-space, Suslin's Hypothesis, selection principle
2010 MSC: 54C35, 54C40

1. Introduction

A function f from a topological space X into $\mathbb{R}$ is *quasicontinuous*, $f \in Q(X,\mathbb{R})$, if for every $x \in X$ and open sets $U \ni x$ and $V \ni f(x)$ there exists a nonempty open $W \subseteq U$ with $f(W) \subseteq V$.

The condition of quasicontinuity can be found in the paper of R. Baire [2] in study of continuity point of separately continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$. The formal definition of quasicontinuity were introduced by Kempisty in 1932 in [7]. Quasicontinuous functions were studied in many papers, see for examples [3, 14, 15, 16, 17, 18], [20, 26, 22] and other. They found applications in the study of topological groups [4, 23, 25], in the study of dynamical systems [5], in the the study of CHART groups [24] and also used in the study of extensions of densely defined continuous functions [19] and of extensions to separately continuous functions on the product of pseudocompact spaces [27], etc.

Levine [10] studied quasicontinuous maps under the name of semi-continuity using the terminology of semi-open sets. A subset A of X is *semi-open* if $A \subset \overline{Int(A)}$. A function $f : X \to Y$ is called *semi-continuous* if $f^{-1}(V)$ is semi-open in X for every open set V of Y. A map $f : X \to \mathbb{R}$ is quasicontinuous if and only if f is semi-continuous [10].

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Preprint submitted to ... *March 8, 2023*

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Let X be a Hausdorff topological space, $Q(X,\mathbb{R})$ be the space of all quasicontinuous functions on X with values in $\mathbb{R}$ and τ_p be the pointwise convergence topology. Denote by $Q_p(X,\mathbb{R})$ the topological space $(Q(X,\mathbb{R}),\tau_p)$.

A subset U of a topological space X is called a *regular open set* or an *open domain* if $U = Int\overline{U}$ holds. A subset F of a topological space X is called a *regular closed* set or a *closed domain* if $F = \overline{IntF}$ holds. The family of regular open sets of (X,τ) is not a topology. But it is a base for a topology τ_s called the *semi-regularization* of τ. If $\tau_s = \tau$, then (X,τ) is called *semi-regular* (or *quasi-regular*).

In ([11], Corollary 1), it is proved that a semi-regular topology is the coarsest topology of its α-class. Note that all topologies of a given α-class on X determine the same class of qusicontinuous mappings into an arbitrary topological space (Proposition 9, [11]). Since a Hausdorff topology τ has a Hausdorff semi-regularization τ_s and $Q_p((X,\tau),\mathbb{R}) = Q_p((X,\tau_s),\mathbb{R})$, we can further assume that X is a *Hausdorff semi-regular space.*

In this paper we study the Fréchet-Urysohn property of the space $Q_p(X,\mathbb{R})$.

2. Preliminaries

Let us recall some properties and introduce new property of a topological space X.

(1) A space X is *Fréchet-Urysohn* provided that for every $A \subset X$ and $x \in \overline{A}$ there exists a sequence in A converging to x.

(2) A space X is said to be *Whyburn* if $A \subset X$ and $p \in \overline{A} \setminus A$ imply that there is a subset $B \subseteq A$ such that $\overline{B} = B \cup \{x\}$.

(3) A space X is said to be *k-Fréchet-Urysohn* if for every open subset U of X and every $x \in \overline{U}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset U$ converging to x.

Definition 2.1. A topological space X is called *open Whyburn* if for every open set $A \subset X$ and every $x \in \overline{A} \setminus A$ there is an open set $B \subseteq A$ such that $\overline{B} \setminus A = \{x\}$.

Let X be a Tychonoff topological space, $C(X,\mathbb{R})$ be the space of all continuous functions on X with values in $\mathbb{R}$ and τ_p be the pointwise convergence topology. Denote by $C_p(X,\mathbb{R})$ the topological space $(C(X,\mathbb{R}),\tau_p)$.

Let us recall that a cover $\mathcal{U}$ of a set X is called

- an *ω-cover* if each finite set $F \subseteq X$ is contained in some $U \in \mathcal{U}$;
- a *γ-cover* if for any $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

A topological space X is called a *γ-space* if each ω-cover $\mathcal{U}$ of X contains a γ-subcover of X. γ-Spaces were introduced by Gerlits and Nagy in [12] and are important in the theory of function spaces as they are exactly those X for which the space $C_p(X,\mathbb{R})$ has the Fréchet-Urysohn property [13].

Clear that $C_p(X,\mathbb{R})$ is a subspace of $Q_p(X,\mathbb{R})$. Thus, if $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn then $C_p(X,\mathbb{R})$ is Fréchet-Urysohn, too. Hence, the property Fréchet-Urysohn of $Q_p(X,\mathbb{R})$ for a Tychonoff space X implies that X is a γ-space.

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A set A is called *minimally bounded* with respect to the topology τ in a topological space (X,τ) if $\overline{IntA} \supseteq A$ and $Int\overline{A} \subseteq A$ ([1], p.101). Clearly this means A is semi-open and $X \setminus A$ is semi-open. In the case of *open* sets, minimal boundedness coincides with regular openness.

Note that if U is a minimally bounded (e.g. regular open) set of X such that U is not dense subset in X and $B \subset \overline{U} \setminus U$ then there is a quasicontinuous function $f : X \to \mathbb{R}$ such that $f(U \cup B) = 0$ and $f(X \setminus (U \cup B)) = 1$ (see Lemma 4.2 in [18]).

Proposition 2.2. *Let $Q_p(X,\mathbb{R})$ be a Fréchet-Urysohn space. Then $\overline{W} \setminus W$ is countable for every minimally bounded set W of X.*

Proof. Let W be a minimally bounded set W of X. Note that $W \cup B$ is a minimally bounded set in X for any $B \subseteq \overline{W} \setminus W$.

Let $M_K = W \cup (\overline{W} \cap K)$ for each $K \in [X]^{<\omega}$.

Suppose that $D = \overline{W} \setminus W$ is uncountable.

Consider the set $C = \{f_K : K \in [X]^{<\omega}\}$ of quasicontinuous functions f_K where

$$f_K := \begin{cases} 0 & on \ \ M_K \\ 1 & on \ \ X \setminus M_K. \end{cases}$$

Let

$$g := \begin{cases} 0 & on \ \ \overline{W} \\ 1 & on \ \ X \setminus \overline{W}. \end{cases}$$

Note that $g \in Q_p(X,\mathbb{R})$ and $g \in \overline{C}$. Since $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn there is a sequence $\{f_{K_i} : i \in \mathbb{N}\} \subset C$ such that $f_{K_i} \to g$ $(i \to \infty)$. Since D is uncountable, there is $z \in D \setminus \bigcup\limits_i K_i$. Consider $[z,(-\frac{1}{2},\frac{1}{2})] = \{f \in Q_p(X,\mathbb{R}) : f(z) \in (-\frac{1}{2},\frac{1}{2})\}$. Note that $g \in [z,(-\frac{1}{2},\frac{1}{2})]$ and $f_{K_i} \notin [z,(-\frac{1}{2},\frac{1}{2})]$ for any $i \in \mathbb{N}$ ($f_{K_i}(z) = 1$ for every $i \in \mathbb{N}$), it is a contradiction.

□

3. Main results

Lemma 3.1. *Let X be an open Whyburn space such that $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn. Then every nowhere subset in X is countable.*

Proof. Since the closure of a nowhere dense subset in X is a nowhere dense set, we can consider only closed nowhere dense sets in X.

Assume that A is an uncountable closed nowhere dense set in X. Since X is open Whyburn, for every point $a \in A$ there is a regular open set $O_a \subseteq X \setminus A$ such that $\overline{O_a} \setminus (X \setminus A) = \{a\}$.

For every a finite subset K of X we consider the set

$M_K = S_K \cup \bigcup\{O_a \cup \{a\} : a \in K \cap A\}$ where S_K is a regular open set such that $K \cap (X \setminus A) \subseteq S_K \subseteq X \setminus A$. Note that M_K is minimally bounded set in X.

Consider the set $S = \{f_K : K \in [X]^{<\omega}\}$ of quasicontinuous functions f_K where

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$$f_K := \begin{cases} 0 & on \ \ M_K \\ 1 & on \ \ X \setminus M_K. \end{cases}$$

Note that $\mathbf{0} \in \overline{S}$ where $\mathbf{0}$ denote the constant function on X with value 0. Since $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn, there is a sequence $\{f_{K_i} : i \in \mathbb{N}\} \subset S$ such that $f_{K_i} \to \mathbf{0}$ $(i \to \infty)$. Since A is uncountable, there is $z \in A \setminus \bigcup\limits_i K_i$. Consider $[z, (-\frac{1}{2}, \frac{1}{2})] = \{f \in Q_p(X,\mathbb{R}) : f(z) \in (-\frac{1}{2}, \frac{1}{2})\}$. Note that $\mathbf{0} \in [z, (-\frac{1}{2}, \frac{1}{2})]$ and $f_i \notin [z, (-\frac{1}{2}, \frac{1}{2})]$ for any $i \in \mathbb{N}$ ($f_i(z) = 1$ for every $i \in \mathbb{N}$), it is a contradiction. □

Definition 3.2. ([9]) A Hausdorff space X is called a *Lusin space* (*in the sense of Kunen*) if

(a) Every nowhere dense set in X is countable;
(b) X has at most countably many isolated points;
(c) X is uncountable.

Theorem 3.3. *Let X be an uncountable open Whyburn space such that $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn. Then X is a Lusin space.*

Proof. By Lemma 3.1, it is enough to prove that X has at most countably many isolated points.

Assume that X has uncountable many isolated points D.

Let $D = D_1 \cup D_2$ where $D_1 \cap D_2 = \emptyset$ and $|D_i| > \aleph_0$ for $i = 1, 2$. Consider the set $W = Int\overline{D_1}$. Clear that $\overline{W} \cap D_2 = \emptyset$. By Lemma 3.1, $|W \setminus D_1| \leq \omega$.

Since X is open Whyburn, for every point $d \in W \setminus D_1$ there is an open subset $O_d \subseteq D_1$ such that $\overline{O_d} \setminus D_1 = \{d\}$.

(a) Suppose that for every point $d \in W \setminus D_1$ there is a neighborhood V_d of d such that $|O_d \cap V_d| \leq \omega$. Let $W_d = O_d \cap V_d$. Then $\overline{W_d} \setminus D_1 = \{d\}$, $W_d \subset D_1$ and $|W_d| \leq \omega$.

For every a finite subset K of W we consider the set

$P_K = \bigcup\{\{d\} : d \in K \cap D_1\} \cup \bigcup\{\overline{W_d} : d \in K \cap (W \setminus D_1)\}$.

Consider the set $C = \{g_K : K \in [W]^{<\omega}\}$ of quasicontinuous functions g_K where

$$g_K := \begin{cases} 0 & on \ \ P_K \\ 1 & on \ \ X \setminus P_K. \end{cases}$$

Let

$$g := \begin{cases} 0 & on \ \ W \\ 1 & on \ \ X \setminus W. \end{cases}$$

Note that $g \in Q_p(X,\mathbb{R})$ and $g \in \overline{C}$. Since $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn there is a sequence $\{g_{K_i} : i \in \mathbb{N}\} \subset C$ such that $g_{K_i} \to g$ $(i \to \infty)$. Since D_1 is uncountable, there is $z \in D_1 \setminus \bigcup\limits_i P_{K_i}$. Consider $[z, (-\frac{1}{2}, \frac{1}{2})] = \{f \in Q_p(X,\mathbb{R}) : f(z) \in (-\frac{1}{2}, \frac{1}{2})\}$. Note that $g \in [z, (-\frac{1}{2}, \frac{1}{2})]$ and $g_i \notin [z, (-\frac{1}{2}, \frac{1}{2})]$ for any $i \in \mathbb{N}$ ($g_i(z) = 1$ for every $i \in \mathbb{N}$), it is a contradiction.

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(b) Suppose that there is a point $d \in W \setminus D_1$ such that $|O_d \cap V_d| > \omega$ for every neighborhood V_d of d. Let $O_d = O_1 \cup O_2$ such that $O_1 \cap O_2 = \emptyset$ and $|O_i| > \omega$ for $i = 1, 2$.

There are two cases:

(1) $V_d \cap O_i \neq \emptyset$ for every neighborhood V_d of d and $i = 1, 2$;

(2) $V_d \cap O_i = \emptyset$ for some neighborhood V_d of d and some $i = 1, 2$.

Suppose that the case (2) is true for $i = 1$. Note that in this case $d \in \overline{O_2}$.

Then, for cases (1) and (2), we consider the set $C = \{g_K : K \in [O_1]^{<\omega}\}$ of continuous functions g_K where

$$g_K := \begin{cases} 0 & on \ \ K \\ 1 & on \ \ X \setminus K. \end{cases}$$

Let

$$g := \begin{cases} 0 & on \ \ O_1 \\ 1 & on \ \ X \setminus O_1. \end{cases}$$

Note that $g \in Q_p(X, \mathbb{R})$ (for cases: (1) and (2)) and $g \in \overline{C}$. Since $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn there is a sequence $\{g_{K_i} : i \in \mathbb{N}\} \subset C$ such that $g_{K_i} \to g$ $(i \to \infty)$. Since O_1 is uncountable, there is $z \in O_1 \setminus \bigcup\limits_i K_i$. Consider $[z, (-\frac{1}{2}, \frac{1}{2})] = \{f \in Q_p(X, \mathbb{R}) : f(z) \in (-\frac{1}{2}, \frac{1}{2})\}$. Note that $g \in [z, (-\frac{1}{2}, \frac{1}{2})]$ and $g_i \notin [z, (-\frac{1}{2}, \frac{1}{2})]$ for any $i \in \mathbb{N}$ ($g_i(z) = 1$ for every $i \in \mathbb{N}$), it is a contradiction. □

Let $I(X)$ denote the set of isolated points of X. Note that a Lusin space has at most countably many isolated points.

Corollary 3.4. *Let X be an open Whyburn space such that $I(X)$ is an uncountable dense subset in X. Then $Q_p(X, \mathbb{R})$ is not Fréchet-Urysohn.*

Proposition 3.5. *Let X be a k-Fréchet-Urysohn regular space with countable pseudocharacter. Then X is open Whyburn.*

Proof. Let $x \in \overline{U} \setminus U$ for an open set U in X. Since X is k-Fréchet-Urysohn, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset U$ such that $x_n \to x$ $(n \to \infty)$. Since X is a regular space with countable pseudocharacter, there is a sequence $\{V_i : i \in \mathbb{N}\}$ of open neighborhoods of x such that $\bigcap V_i = \{x\}$ and $\overline{V_{i+1}} \subset V_i$ for each $i \in \mathbb{N}$. We can assume that $x_i \in V_i \setminus \overline{V_{i+1}}$. Let W_i be a neighborhood of x_i such that $\overline{W_i} \subset U \cap (V_i \setminus \overline{V_{i+1}})$. Then $W = \bigcup\{W_i : i \in \mathbb{N}\} \subset U$ and $\overline{W} \setminus U = \{x\}$.

□

Corollary 3.6. *Let X be an uncountable k-Fréchet-Urysohn (Fréchet-Urysohn) regular space with countable pseudocharacter such that $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn. Then X is a Lusin space.*

In particular, if X is an uncountable first-countable regular space such that $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn then X is a Lusin space.

Note that if X is Tychonoff and $C_p(X, \mathbb{R})$ is Fréchet-Urysohn then $C_p(X^2, \mathbb{R})$ is Fréchet-Urysohn [13]. However, this is not true for quasicontinuous functions.

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Corollary 3.7. *Let X be an uncountable first-countable regular space such that $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn. Then $Q_p(X^2,\mathbb{R})$ is not Fréchet-Urysohn space.*

Proof. Since $\mathbb{R}^\kappa$ is not Fréchet-Urysohn for any $\kappa \geq \omega_1$, X is not discrete space provided that X is an uncountable and $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn. Clear that $X^2 = X \times X$ is not Lusin space provided that X is a Lusin space and X with a non-isolated point. □

By Theorem 3.3 and results in [9] (Lemmas 1.2 and 1.5), we get that if X is an uncountable open Whyburn Hausdorff semi-regular space such that $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn then X is hereditarily Lindelöf (hence, X is perfect normal (see 3.8.A. in [6])) and X is zero-dimensional.

Since a Lusin space X is hereditarily Lindelöf and Hausdorff, it has cardinality at most $\mathfrak{c} = 2^\omega$ (de Groot, [28]).

Corollary 3.8. *Let X be an open Whyburn space of cardinality $> \mathfrak{c}$. Then $Q_p(X,\mathbb{R})$ is not Fréchet-Urysohn space.*

In particular, if X is first-countable regular space of cardinality $> \mathfrak{c}$ then $Q_p(X,\mathbb{R})$ is not Fréchet-Urysohn space.

Let us note however that Kunen (Theorem 0.0. in [9]) has shown that under *Suslin's Hypothesis* (**SH**) there are no Lusin spaces at all. K.Kunen proved that under $\mathbf{MA}(\aleph_1, \aleph_0$-centred) there is a Lusin space if and only if there is a Suslin line.

The Suslin Hypothesis is neither provable nor refutable in **ZFC**, even if we assume **CH** or $\neg$**CH**. A typical model of $\mathbf{ZFC} + \neg\mathbf{SH}$ is the Gödel constructible universe **L**, while a typical model of $\mathbf{ZFC} + \mathbf{SH}$ is the Solovay-Tennenbaum model of $\mathbf{ZFC} + \mathbf{MA}(\aleph_1)$ (see p.266 in [21]).

Theorem 3.9. (**SH**). *Let X be an open Whyburn space. The space $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn if and only if X is countable.*

In particular, for first-countable regular spaces, we have the following corollary.

Corollary 3.10. (**SH**). *Let X be a first-countable regular space. The space $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn if and only if X is countable.*

However, the following result holds in **ZFC**.

Theorem 3.11. *Let X be a metrizable space. The space $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn if and only if X is countable.*

Proof. Note that a Lusin subspace of a metrizable space is a Lusin set:*an uncountable subset of $\mathbb{R}$ that meets every nowhere dense set in a countable set.* Hence, if $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn then X is a Lusin set and it is a γ-space. But any γ-space $X \subset \mathbb{R}$ is *always first category* (see Definition in [29]) and a Lusin set is not *always first category* (p. 159 in [12]). Hence, X is countable.

If X is countable then $Q_p(X,\mathbb{R})$ is first countable (Theorem 4.1 in [18]) and, hence, $Q_p(X,\mathbb{R})$ is Fréchet-Urysohn.

□

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4. Selection principle S_1 and Fréchet-Urysohn at the point $\mathbf{0}$

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a topological space X.

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle that for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle U_n : n \in \mathbb{N}\rangle$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$ (see [31]).

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following covers of a space X:
Ω : the collection of open ω-covers of X.
Γ : the collection of open γ-covers of X.
Ω^s : the collection of minimally bounded ω-covers of X.
Γ^s : the collection of minimally bounded γ-covers of X.

In [13], it is proved that $C_p(X, \mathbb{R})$ is Fréchet-Urysohn if and only if X has the property $S_1(\Omega, \Gamma)$.

Lemma 4.1. *Let $Q_p(X, \mathbb{R})$ be Fréchet-Urysohn at the point $\mathbf{0}$. Then X has the property $S_1(\Omega^s, \Gamma^s)$.*

Proof. Let $\{\mathcal{V}_i : i \in \mathbb{N}\}$ be a family of minimally bounded ω-covers of X. For each $i \in \mathbb{N}$, we consider the family $A_i = \{f_{i,V} \in Q_p(X, \mathbb{R}) : V \in \mathcal{V}_i\}$ such that $f_{i,V}(V) = \frac{1}{i}$ and $f_{i,V}(X \setminus V) = 1$ for $V \in \mathcal{V}_i$. Let $A = \bigcup A_i$. Then $\mathbf{0} \in \overline{A} \setminus A$. Since $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn at point $\mathbf{0}$, there is a sequence $\{f_{i,V_i} : i \in \mathbb{N}\}$ such that $f_{i,V_i} \in A$ for each $i \in \mathbb{N}$ and $f_{i,V_i} \to \mathbf{0}$ $(i \to \infty)$. Note that $\{V_i : i \in \mathbb{N}\}$ is a minimally bounded γ-cover of X. $\square$

Theorem 4.2. *Let X be an open Whyburn space. The space $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn at the point $\mathbf{0}$ if and only if X has the property $S_1(\Omega^s, \Gamma^s)$.*

Proof. By Lemma 4.1, it is enough to prove a sufficient condition.

Let $\mathbf{0} \in \overline{A} \setminus A$ for some set $A \subset Q_p(X, \mathbb{R})$. For each $i \in \mathbb{N}$, we consider the set $\mathcal{U}_i = \{f^{-1}(-\frac{1}{i}, \frac{1}{i}) : f \in A\}$. Clear that $\mathcal{U}_i$ is a semi-open ω-cover of X for each $i \in \mathbb{N}$.

Let $U \in \mathcal{U}_i$. Since X is an open Whyburn semi-regular space, for each finite subset K of U, there is a minimally bounded set $V_{K,U,i}$ such that $K \subset V_{K,U,i} \subset U$. Thus, the family $\mathcal{V}_i = \{V_{K,U,i} : K \in [U]^{<\omega}$ and $U \in \mathcal{U}_i\}$ is a minimally bounded ω-cover of X for each $i \in \mathbb{N}$. Since X has the property $S_1(\Omega^s, \Gamma^s)$ there exists a sequence $(V_{K_i,U_i,i} : i \in \mathbb{N})$ such that for each i, $V_{K_i,U_i,i} \in \mathcal{V}_i$ and $\{V_{K_i,U_i,i} : i \in \mathbb{N}\}$ is a minimally bounded γ-cover of X. Then the sequence $(f_i : U_i = f_i^{-1}(-\frac{1}{i}, \frac{1}{i}), i \in \mathbb{N}) \to \mathbf{0}$ $(i \to \infty)$. $\square$

For any space X and maps $f, g : X \to \mathbb{R}$ such that f is continuous and g is quasicontinuous, it is easy to show that the map $f + g : X \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ is quasicontinuous (Proposition 5.4 in [8]).

Corollary 4.3. Let $Q_p(X, \mathbb{R})$ be Fréchet-Urysohn at the point $\mathbf{0}$. Then $C_p(X, \mathbb{R})$ is Fréchet-Urysohn.

Corollary 4.4. Let X be an open Whyburn space and X has the property $S_1(\Omega^s, \Gamma^s)$. Then X has the property $S_1(\Omega, \Gamma)$.

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5. Examples

Similarly the proof of Proposition 2.2, it is easy to see the following result.

Proposition 5.1. *Let X be a space with a dense subset D of isolated points such that $D = D_1 \cup D_2$ where $\overline{D_1} = X \setminus D_2$ and $\overline{D_2} = X \setminus D_1$ and let $Q_p(X, \mathbb{R})$ be a Fréchet-Urysohn space. Then D is countable.*

Proposition 5.2. *There is a compact space X such that $C_p(X, \mathbb{R})$ is Fréchet-Urysohn, but $Q_p(X, \mathbb{R})$ is not.*

Proof. Let $X = \omega_1 + 1$. Here $\omega_1 + 1$ is the space $\{\alpha : \alpha \leq \omega_1\}$ with the order topology. By Proposition 5.1, $Q_p(X, \mathbb{R})$ is not Fréchet-Urysohn. It well known that $C_p(Y, \mathbb{R})$ is Fréchet-Urysohn for a compact space Y if and only if Y is scattered [12]. Hence, $C_p(X, \mathbb{R})$ is Fréchet-Urysohn. $\square$

Proposition 5.3. *There is an uncountable separable metrizable space X such that $C_p(X, \mathbb{R})$ is Fréchet-Urysohn, but $Q_p(X, \mathbb{R})$ is not.*

By Corollary 3.11, it is enough consider any uncountable γ-space $X \subset \mathbb{R}$.

Proposition 5.4. *There is an uncountable T_1-space X such that $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn.*

Let X be an uncountable set with the cofinite (or co-countable) topology. Then $Q_p(X, \mathbb{R})$ is homeomorphic to $\mathbb{R}$ because any quasicontinuous function on X is a constant function, and, hence, $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn.

6. Open questions

Question 1. Suppose that X is a (first-countable, regular) submetrizable and $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn. Is the space X countable?

Question 2. Suppose that X is an open Whyburn T_2 semi-regular space and $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn. Is the space X countable?

Question 3. Suppose that X is a T_2 space and $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn. Is the space X Lusin?

Question 4. Suppose that X is a Lusin space and a γ-space. Is the space X countable?

Question 5. Suppose that a (an open Whyburn) space X has the property $S_1(\Omega^s, \Gamma^s)$. Will the space $Q_p(X, \mathbb{R})$ have the Fréchet-Urysohn property at each point?

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