CONVEXITY OF CHEBYSHEV SETS IN HILBERT SPACES

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ABSTRACT

Perhaps one of the major hitherto hardest unsolved problem in approximation theory is: Must every Chebyshev subset of a Hilbert space be convex? Whereas, the problem has been solved in finite dimensional Hilbert spaces, it is still open in infinite dimensional Hilbert spaces. Many partial answers to the problem in this case are available in the literature. Several research articles, including some survey articles have appeared in the past on the problem. This article in in continuation of those survey articles. In this article, we have given a brief historical and expository account of the problem, and have made an attempt to discuss the developments which have taken place on the problem. Future directions for solving the problem have also been discussed.

1. INTRODUCTION AND HISTORICAL BACKGROUND

For a non-empty subset G of a metric space (X,d) and $x \in X$, an element $g_0 \in G$ is called a best approximation to x in G if $d(x,g_0) = d(x,G) \equiv inf\{d(x,g) : g \in G\}$. The set G is said to be Chebyshev if each element of X has a unique best approximation in $G.$ It is well known that if G is a closed convex subset of a Hilbert space $H,$ then G is a Chebyshev set. This result dates back to F. Riesz [1934] who adapted an argument due to B.Levi [1906]. However the converse to this result remains an unanswered question. That is, must every Chebyshev set in a Hilbert space be convex? (it is easy to see that every Chebyshev set is always closed because otherwise elements of $\bar{G}\backslash G$ would have no best approximation in G).This problem, known as the 'problem of convexity of Chebyshev sets' is one of the hardest unsolved problem in the theory of best approximation. This problem has attracted the attention of many mathematicians, renowned specialists in geometrical functional analysis and in abstract approximation theory.

This problem in the finite dimensional spaces was independently considered by Bunt [1934], Motzkin [1935], Kritikos [1938] and Jessen [1940], who gave the following affirmative answer to the problem:

Theorem 1 If H is finite dimensional Hilbert space, then every Chebyshev set in H is convex.

Specifically, Bunt [1934], Motzkin [1935], Kritikos [1938], independently working, proved that every Chebyshev set in the Euclidean space $\mathbb{R}^n$ is convex. Jessen [1940], aware of Kritikos's proof, gave still another proof of this result. Recently, K. Deka and M. Varivoda [2022] have also given a novel proof of this result. Motzkin proved this result for $\mathbb{R}^2$ and his

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proof does not work in $\mathbb{R}^n$ for $n > 2$. Busemann [1947] noted that Jessen's proof could be extended to 'straight line spaces' and in 1955, he showed how it could be done. Since a finite dimensional normed linear space is a 'straight line space' if and only if it is strictly convex, Busemann's result is that in a smooth strictly convex finite dimensional normed linear space, every Chebyshev set is convex. Valentine [1964] independently gave essentially the same proof as Busemann. Klee [1961] and Vlasov [1961] showed that the requirement of strict convexity could be dropped. The smoothness condition cannot be dropped (see Kelly [1978]).

It is easy to show (see Efimov and Steckin [1958]) that if each Chebyshev set in a finite dimensional normed linear space X is convex, then each exposed point of the unit sphere $S(X) = \{x \in X : \|x\| = 1\}$ must be a smooth point. This gives a necessary condition for the convexity of Chebyshev set in the space X. On the other hand, if X is a smooth finite dimensional normed linear space, then each Chebyshev set in X is convex (Klee [1961], Vlasov [1961]). Thus smoothness is a sufficient condition. However Brondsted [1965] showed that smoothness is not a necessary condition. It is known (see Deutseh [1993]) that if $\dim X \leq 4$, then each Chebyshev set in X is convex if and only if each exposed point of the unit sphere $S(X)$ is a smooth point. Brondsted [1966] proved this characterization theorem if $\dim X \leq 3$. Whether or not this characterization is valid when $\dim X > 4$ is unknown. The characterization of finite dimensional normed linear spaces in which every bounded Chebyshev set is convex was given by Tsar′kov [1984]. He showed that every bounded Chebyshev set in a finite dimensional normed linear space X is convex if and only if the extreme points in $S(X^*)$ are dense in $S(X)$. Brown [1986] gave a short survey and exposition of results concerning Chebyshev sets in finite dimensional normed linear spaces and discussed in detail results of Tsar′kov [1984].

It took almost 15 years for the first generalization of Bunt's, Motzkin's and Kritikos's results to infinite dimensional spaces, and that too under additional assumptions. It is still an open question whether their result can be extended to infinite dimensional Hilbert spaces without further hypothesis. The convexity of the Chebyshev sets in infinite dimensional Hilbert spaces has been considered by several researchers who have given various partial answers of the problem. Ficken [1951] showed that in any Hilbert space, every compact Chebyshev set is convex. This result, which was published in an article of Klee [1961], appears to be the first result on the convexity of Chebyshev sets in infinite dimensional spaces. Thereafter, several attempts have been made by various researchers to weaken the compactness condtion imposed by Ficken, or to prove the convexity of Chebyshev sets under additional assumptions. The surveys of such results have been made by Singer [1970], Vlasov [1973], Narang [1977], Deutsch [1993], Balaganski and Vlasov [1996], Kanellopoulos [2000], Assadi et al. [2014], Fletcher and Moors [2015] and may be some others. An excellent exposition of the problem in finite dimensional Hilbert space was presented by Kelly [1978]. The present article is in continuation of these surveys. In this article we have made an attempt to discuss the developments which have been made on the problem (Section 3). Future directions for solving the problem have also been discussed (Section 4).

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2. NOTATIONS AND DEFINITIONS

In this section, we give certain notations and recall few definitions which are required in the sequel. All those notations and definitions which are not given here can either be found in Singer [1974] or in the respective cited references.

In this article, $X\backslash A$ denotes complement of the set A in X, $\bar{A}$ will stand for closure of the set A, $\overline{co}\,(A)$ for closed convex hull of the set A, $\mathbb{R}^n$ will denote the n-dimensional Euclidean space, $B(x,r)$ and $B[x,r]$ respectively denote open and closed balls with centre x and radius r.

Now, we recall few definitions -

Let G be a non empty subset of a metric space (X,d) and $x \in X$. An element $g_0 \in G$ is called a **best approximation** to x in G if $d(x,g_0) = d(x,G) \equiv inf\,\{d(x,g) : g \in G\}$. The set G is said to be

(i) **proximinal** if each element of X has a best approximation in G

(ii) **semi-Chebyshev** if each element of X has at most one best approximation in G

(iii) **Chebyshev** if each element of X has a unique best approximation in G

(iv) **strongly proximinal** if for each $x \in X\backslash G$ $there\ exists$ $g_0 \in G$ and $r > 0$ $(r \leq 1)$ such that $d(x,g) \geq d(x,g_0) + rd(g_{0,}g)\ for\ all\ g \in G\ \ i.e\ \ if\ g$ moves in G from g_0, then the approximation of x worsens with the rate of distance from x.

(v) **approximately compact** if for each $x \in X$ and each sequence $\langle g_n \rangle$ in G with $d(x,g_n) \to d(x,G)$, $\langle g_n \rangle$ has a convergent subsequence in G.

(vi) **boundedly compact** if every bounded sequence in G has a subsequence converging to an element of G

(vii) **boundedly connected** if for each open ball $B(x,r)$ in X, $B(x,r) \cap G$ is empty or connected

$(viii)$ **locally compact** if for any $x \in G\ there\ exists\ an\ r > 0$ such that $G \cap B[x,r]$ is compact.

The set-valued mapping $P_G: X \to 2^G \equiv$ collection of all subsets of G, which takes each $x \in X$ to its set of best approximations $P_G(x) \equiv \{g_0 \in G: d(x,g_0) = d(x,G)\}$ in G is called **metric projection**. For Chebyshev sets G, the mapping P_G is single-valued.

For a non-empty bounded subset K of a metric space (X,d) and $x \in X$, an element $k_0 \in K$ is said to be a **farthest point** to x in K if $d(x,k_0) = sup\{\,d(x,k): k \in K\,\}$. The set K is said to be

(i) **remotal** if each element of X has a farthest point in K

(ii) **uniquely remotal** if each element of X has a unique farthest point in K

A subset G of a normed linear space X is called

(i) a **sun** if G is Chebyshev and for each $x \in X, P_G\big[x + \lambda\big(x - P_G(x)\big)\big] = P_G(x)$ for any $\lambda \geq 0$

(ii) **approximatively convex** if G is proximinal and $P_G(x)$ is convex for each $x \in X$

(iii) **weakly compact** if each sequence in G contains a subsequence converging weakly to some element of G

(iv) **weakly closed** if $x \in G$ whenever $\langle x_n \rangle$ is a sequence in G and $x_n \to x$ weakly.

A mapping $f : X \to G$ is called **weakly continuous** if for any sequence $\langle x_n \rangle$ in X with $x_n \to x$ weakly, we have $f(x_n) \to f(x)$ weakly.

Remarks

(i) The word 'proximinal' a combination of the words proximity and minimal was proposed by Raymond Killgrove (see R.R.Phelps [1957])

(ii) The term 'Chebyshev' set goes back to N.V. Efimov and S.B. Steckin [1958] in honour of the founder of best approximation theory P.L. Chebyshev.

(iii) the term 'metric projection' goes back to V. Klee [1961].

(iv) the idea of 'sun' was first developed and used by V. Klee [1961], but the terminology 'sun' was first proposed by N.V. Efimov and S.B. Steckin [1958].

(v) strong proximinality in normed linear spaces was introduced by D.J.Newman and Harold S. Shapiro [1963].

3. THE INFINITE DIMENSIONAL CASE

The problem of convexity of Chebyshev sets in infinite dimensional Hilbert spaces has been considered by several researchers. Various sufficient conditions under which a Chebyshev subset of an infinite dimensional Hilbert space is convex are known in the literature. Many researchers have raised the question when such conditions are also necessary for the convexity of Chebyshev sets. Some such results (see Vlasov [1973]) are also available in the literature. In the following theorem, we summarize the main known results (to the best of our knowledge) under which Chebyshev subset of any infinite dimensional Hilbert space is convex. It may be mentioned that parts of this theorem are valid in more general spaces, and the interested reader may refer to the respective quoted paper for the exact result.

Theorem 2: Let G be a Chebyshev subset of an infinite dimensional Hilbert space H. Then each of the following condition is equivalent to G being convex.

$(i) G$ is compact (Ficken [1951]- result published in an article of Klee [1961])

$(ii) G$ is boundedly compact (Klee [1953], [1961], Vlasov [1961])

$(iii) P_G$ is non-expansive (Phelps [1957]

$(iv) G$ is weakly closed (Klee [1961])

(v) Every $x \in H \backslash G$ admits a neighborhood $N(x)$ on which the (restricted) metric projection

is both continuous and weakly continuous (Klee [1961])

$(vi) G$ is a sun (Efimov and Steckin [1961])

(vii) G is approximatively compact (Efimov and Steckin [1961], Vlasov [1961])

$(viii)$ P_G is continuous or demicontinuous, i.e., norm to weak continuous (Vlasov [1967], Asplund [1969])

$(ix) P_G$ is weakly outer radial continuous (Vlasov [1967], [1973])

$(x) P_G$ is radially continuous (Vlasov [1967a])

(xi) $\lim_{\varepsilon \to 0^+} \frac{d(x_\varepsilon, G) - d(x,G)}{\|x_\varepsilon - x\|} = 1 \; for \; every \; x \in H \backslash G, \; where \; x_\varepsilon = x + \varepsilon\,[x - P_G(x)]$ (Vlasov [1967a])

$(xiii) G$ is locally compact and boundedly connected (Vlasov [1968], Wulbert [1968])

(xii) P_G is weakly continuous (Blatter et al. [1968])

$(xiv) P_G$ is lower semi-continuous (Blatter [1969])

$(xv) G$ intersects each closed half space in a proximinal set (Asplund [1969])

(xvi) for each $x \in H \backslash G$, the function $g \to \|x - g\|$ defined on G, has a strong minimum at $P_G(x)$ (Asplund [1969], Raymond [2013])

$(xvii) G$ is approximatively convex and P_G is weakly upper semi continuous (Brown [1969-70])

$(xviii) G$is weakly compact (Vlasov [1973])

$(xix) P_G$ is outer radial lower continuous (Brosowski and Deutsch [1974])

$(xx) P_G$ is maximally monotone (Berens and Westphal [1977])

(xxi) the distance function $x \to d(x, G)$ defined on H is Frechet differentiable at each point of $H \backslash G$ (Fitzpatrik [1980])

$(xxii)$ the set of points of discontinuity of P_G is countable (Balaganski [1982])

$(xxiii)$ the distance function $x \to d(x, G)$ defined on H is a convex function (Borwein and Preiss [1987])

$(xxiv)$ the distance function $x \to d(x, G)$ $(\equiv d_G)$, defined on H is Gateaux differentiable at each $x \in H \backslash G$, or d_G is regular on $H \backslash G$ or d_G is strictly differentiable on $H \backslash G$, or $\partial d_G(x) = \left\{\frac{x - \bar{x}}{\|x - \bar{x}\|}\right\}$ for each $x \in H \backslash G$ and $\bar{x} \in P_G(x)$, or the Gateaux derivative of d_G exists and equals $\frac{x - \bar{x}}{\|x - \bar{x}\|}$ for each $x \in H \backslash G$ and $\bar{x} \in P_G(x)$, or Frechet sub-differential of d_G, $\partial_F d_G(x) = \left\{\frac{x - \bar{x}}{\|x - \bar{x}\|}\right\}$ for each $x \in H \backslash G$ and $\bar{x} \in P_G(x)$ (Wu [2002]).

While every Chebyshev set in a finite dimensional Hilbert space is convex, it is still unknown whether the same is true in infinite dimensional Hilbert spaces. Several attempts have been made to solve the problem but it seems that one needs some additional assumptions in order that every Chebyshev subset of an infinite dimensional Hilbert space to be convex. Frederick A. Ficken in 1951 showed that in any Hilbert space every compact Chebyshev set is convex. This result appeared in an article of Klee [1961] and was perhaps the first result on convexity of Chebyshev sets in infinite dimensional spaces. Thereafter, there have been several attempts to weaken the compactness assumption imposed by Ficken on the Chebyshev set, or to study additional assumptions to be imposed in order that every Chebyshev subset of an infinite dimensional Hilbert space is convex. Klee [1953] considered the problem of the existence, in a Hilbert space, of a Chebyshev set with a convex complement (called a 'Klee Cavern'). Asplund [1969] showed that existence of a 'Klee Cavern' is equivalent to the existence of non-convex Chebyshev set. The problem of convexity of Chebyshev sets was considered by N.V Efimov and S.B Steckin in a series of papers published between 1958 and 1961 in Doklady, and by many others.

Klee [1965] made a conjecture (and also provided supporting evidence) that in an infinite dimensional Hilbert space, possibly non-separable, non-convex Chebyshev set must exist. He constructed a semi-Chebyshev set the complement of which is open, bounded, and convex. Clearly, such a set does not exist in R^n. Asplund [1969] proved that if Klee's conjecture is true, then there exist a Chebyshev set having an open, bounded and convex complement. This suggests that if a non convex Chebyshev set exists, then one exists that should look 'something like' the complement of the open unit ball (see Deutsch [1993]). Berens [1980] remarked that if Klee's conjecture is true, then there exist a proximinal, even a Chebyshev set G in Hilbert space H such that for some element $x \in H$, $\overline{co}[P_G(x)] \subseteq \bigcap_{r>d(x,G)} \overline{co}[B(x,r) \bigcap G]$. To support Klee's conjecture, Berens tried unsuccessfully to construct such a set.

Dunham [1975] constructed an example of a Chebyshev set in the space C[0,1]≡ the space of all continuous functions on [0,1], which is not a sun. The existence of such a set in a Hilbert space would answer the problem of convexity of Chebyshev sets. Berens and Westphal [1978] have shown that maximal monotonicity of P_G is equivalent to convexity of the Chebyshev set G. Franchetti and Papini [1981] remarked that if a non-convex Chebyshev set exists in a Hilbert space H, then for a subset G of H, the metric projection P_G, which is always a monotone map must lack 'almost' continuity property. Balaganski [1982] has shown that if there exist a non-convex Chebyshev set in a Hilbert space H, then H contains a non-convex bounded star shaped Chebyshev set.

Johnson [1987] gave the first example of a non-convex Chebyshev set in an incomplete inner product space. He constructed a subset M of the real incomplete inner product space X of all real sequences in l^2 having at most a finite number of non-zero terms such that

(i) M is closed and non-convex

(ii) M is Chebyshev

(iii) M is not a sun

(iv) P_M is continuous

This example contained two errors which were identified and corrected by Jiang [1993]. Jiang remarked that Johnson's example may provide a framework for resolving the problem of convexity of Chebyshev sets, for answering and strongly supporting Klee's conjecture. If such an example can be constructed in a Hilbert space, then the problem is solved. Johnson's example also supports that in Theorem 2, completeness of the space appears to be essential. Johnson [2005] himself raised the question: If $H$is the completion of the space$X$, then will the closure of M remains a Chebyshev set in H. Johnson's proof that the set he constructed is Chebyshev is ingenious, lengthy and complicated. Jiang's corrections provided a concise proof of that of Johnson. Balaganski and Vlasov [1996], Fletcher and Moors [2015] also presented simplified versions of Johnson's construction.
Faraci and Iannizzoto [2008] presented a conjecture aiming for the construction of a non-convex Chebyshev set in a Hilbert space. They believe that their conjecture is likely to be proved.

4. FUTURE DIRECTIONS

The main obstacles in proving that every Chebyshev subset G of a Hilbert space H is convex have been

(i) removing the compactness like conditions imposed on the set G, or

(ii) removing the continuity like assumptions on the metric projection P_G (see Balaganski [1982], Westphal and Freking [1989], or

(iii) constructing a Johnson's like set in the Hilbert space l^2 or any other Hilbert space (perhaps this is not possible as just taking the completion in Johnson's example does not seem to work as pointed out by Deutsch [1993]).

Several attempts have been made to remove these obstacles and some progress has been made.

(iv) There have been many other approaches to tackle the problem

(a) by exploiting the theory of monotone operators (see e.g. Singer [1970], Berens and Westphal [1978], Berens [1980], Phelps [1993], Fletcher and Moors [2015])

(b) by applying methods from non linear convex analysis (see Balaganski and Vlasov [1996])

(c) by considering the differentiability of the associated distance function $x \to d(x, G), x \in H$ (see e.g, Fitzpatrik [1980], Giles [1988], Wu [2002])

(d) by trying to find a solution to the following problem in the theory of farthest points: Must a set K which is uniquely remotal in a Hilbert space H consists of a single point ? Klee [1961] pointed out that in a Hilbert space if the answer to this problem is affirmative, then every Chebyshev subset of the Hilbert space will be convex (For literature on the farthest point problem we refer to the survey articles by Narang [1977], [1991] and references cited therein).

(e) by trying to show that Chebyshev subsets of Hilbert spaces are strongly proximinal (It was proved by Narang [2017] that strongly proximinal sets in a Hilbert space are convex. So, if we can show that the Chebyshev subsets of a Hilbert space are strongly proximinal, then we shall obtain an affirmative answer of the problem of convexity of Chebyshev sets).

5 FINAL REMARKS

Several researchers have discussed the problem of convexity of Chebyshev sets in a Hilbert space. Whereas the problem has been completely solved in the finite dimensional Hilbert spaces, it has neither been proved nor disapproved till date that Chebyshev subsets of infinite dimensional Hilbert spaces are always convex.

This problem has a very strong connection with many geometrical properties of Banach spaces e.g. smoothness, uniform convexity and its weaker forms, sun properties, and above all with the problem of singletonness of uniquely remotal sets in the theory of farthest points. F.A. Ficken [1951] (see Klee [1961] has shown that there is a close connection between the convexity of Chebyshev sets and singletonness of uniquely remotal sets in a Hilbert space. Klee [1961] showed that the solution of one will lead to solution of the other. Asplund [1969] showed that a Hilbert space contains a non-convex Chebyshev set if and only if it contains a non-singleton uniquely remotal set.

The problem will be solved if we are able to remove compactness like conditions on the set, or remove the continuity like assumptions on the associated metric projection, or construct a non convex Chebyshev set in any Hilbert space, or are able to show that every uniquely remotal set in a Hilbert space is singleton.

This survey is an effort to provide bibliography of the available literature on the topic, motivate, inspire and benefit the next generation of researchers working in the area, help them to tackle, and perhaps even solve the long outstanding open problem.

We conclude the paper with a hope that the problem, which appeared to be far from solution earlier, will be solved in the near future.

Additional Information

Recently, the following article on the topic has appeared:

Convexity of Chebyshev sets revisited by Konard Deka and Marin Varivoda in American Mathematical Monthly, 129 (2022), 763-774.

In this article they have given a novel proof of the known result: Every Chebyshev set in n-dimensional Euclidean space is convex (which they have called ‘Bunt-Motzkin Theorem’).

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