ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

ON N-HAUSDORFF HOMOGENEOUS AND N-URYSOHN HOMOGENEOUS SPACES

M. BONANZINGA, N. CARLSON, D. GIACOPELLO, AND F. MAESANO

ABSTRACT. In this paper we study n-Hausdorff homogeneous and n-Urysohn homogeneous spaces. We give some upper bounds for the cardinality of these kind of spaces and give examples. Additionally we show that for every $n > 2$, there is no n-Hausdorff non Hausdorff 2-homogeneous space. Finally, for any n-Hausdorff space, where $n \geq 2$, we show X can be embedded in a homogeneous space that is the countable union of n-H-closed spaces.

Keywords: n-Hausdorff spaces, n-Urysohn spaces, homogeneous extensions, n-Katetov extensions.
AMS Subject Classification: 54A25, 54D10, 54D20, 54D35, 54D80.

1. Introduction

Thoughout the paper, n and m will always denote integers. Given a topological space X, the *Hausdorff number* $H(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a\in A} U_a = \emptyset$. A space X is said *n-Hausdorff*, $n \geq 2$, if $H(X) \leq n$. Of course, with $|X| \geq 2$, X is Hausdorff iff $H(X) = 2$ [5]; the *Urysohn number* $U(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a\in A} \overline{U_a} = \emptyset$. A space X is said *n-Urysohn*, $n \geq 2$, if $U(X) \leq n$. Of course, with $|X| \geq 2$, X is Urysohn iff $U(X) = 2$ (see [6, 7]).

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$ (see [1, 10] for surveys on homogeneous spaces).

Definition 1.1. [14] A space X is 2-homogeneous if for every $x_1, x_2, y_1, y_2 \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

In general one can give the definition of n-homogeneous space for any n. Notice that 1-homogeneity coincides with the definition of homogeneity. Of course, if a space is $(n+1)$-homogeneous, then it is m-homogeneous for every $m = 1, 2, \ldots, n+1$.

In this paper we prove that n-Hausdorff ($n > 2$) non Hausdorff spaces are not m-homogeneous ($m > 1$) and give an example (Example 2.7) of a 3-Urysohn homogeneous non Urysohn space. Also we show that even in the class of homogeneous spaces $(n+1)$-Hausdorff ($(n+1)$-Urysohn) spaces need not be n-Hausdorff (resp., n-Urysohn), with $n \geq 2$. Also we present some upper bounds on the cardinality of n-Hausdorff homogeneous and n-Urysohn homogeneous spaces (see also [5, 8] for other bounds on the cardinality of n-Hausdorff and n-Urysohn spaces). In particular, we prove the analogous

2020 *Mathematics Subject Classification.* 54A25, 54D10, 54D20, 54D35, 54D80.
Key words and phrases. n-Hausdorff spaces, n-Urysohn spaces, homogeneous extensions, n-Katetov extensions.

version of the following result for n-Urysohn spaces and a variation of the same result for n-Hausdorff spaces.

Theorem 1.2. [15] Let X be a homogeneous Hausdorff space. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.

In the last section of the paper, for any $n \geq 2$ and for any n-Hausdorff space X, we show that X can be embedded in a homogeneous space that is the countable union of n-H-closed spaces. Using this result we give an example of n-Hausdorff homogeneous space which is not n-Urysohn, for every $n \geq 2$.

For a subset A of a topological space X we will denote by $[A]^{<\lambda}$ ($[A]^{\lambda}$) the family of all subsets of A of cardinality $< \lambda (= \lambda)$.

We consider cardinal invariants of topological spaces (see [16, 20]) and all cardinal functions are multiplied by ω. In particular, given a topological space X, we will denote with $d(X)$ its density, $\chi(X)$ its character, $\pi\chi(X)$ its π-character, $\pi w(X)$ its π-weight, $c(X)$ its cellularity and $e(X)$ its extent. Recall also that, for any space X, $d(X)\pi\chi(X) = \pi w(X)$.

Recall that a family $\mathscr{U}$ of open sets of a space X is *point-finite* if for every $x \in X$, the set $\{U \in \mathscr{U} : x \in U\}$ is finite [16]. Tkachuck [26] defined $p(X) = sup\{|\mathscr{U}| : \mathscr{U}$ is a point-finite family in $X\}$. In [5], Bonanzinga introduced the following definition:

Definition 1.3. [5] A family $\mathscr{U}$ of open sets of a space X is *point-($\leq n$) finite*, where $n \in \mathbb{N}$, if for every $x \in X$, the set $\{U \in \mathscr{U} : x \in U\}$ has cardinality $\leq n$. For each $n \in \mathbb{N}$, put

$$p_n(X) = sup\{|\mathscr{U}| : \mathscr{U} \text{ is a point-}(\leq n) \text{ finite family in } X\}.$$

Proposition 1.4. [5] Let X be a topological space. Then $p_n(X) = c(X)$ for every $n \in \mathbb{N}$.

2. Examples and positive results

In [5], examples of $(n+1)$-Hausdorff spaces which are not n-Hausdorff, for every $n \geq 2$, and an example of a space X such that $H(X) = \omega$ and $H(X) \neq n$, for each $n \geq 2$, are given. Also, in [6] examples of Hausdorff $(n+1)$-Urysohn spaces which are not n-Urysohn were given for every $n \geq 2$.

Recall that a hyperconnected (or nowhere Hausdorff) space is a space such that the intersection of any two nonempty open sets is nonempty; a space is nowhere Urysohn if there is no pair of nonempty open sets with disjoint closures. Such spaces are also called "anti-Urysohn" spaces (see [23]).

Proposition 2.1. A non Hausdorff 2-homogeneous space is hyperconnected.

Proof. Let X be a non Hausdorff 2-homogeneous space. Suppose that there are two nonempty open subset V_1 and V_2 of X such that $V_1 \cap V_2 = \emptyset$. Fix two points $y_1 \in V_1$ and $y_2 \in V_2$. Since X is not Hausdorff there exist two points $x_1, x_2 \in X$ such that for every open neighbourhood U_1 of x_1 and U_2 of x_2, one has that $U_1 \cap U_2 \neq \emptyset$. Define the homeomorphism $h : X \to X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Of course $h^{\leftarrow}(V_1) \cap h^{\leftarrow}(V_2) \neq \emptyset$. Pick a point $x \in h^{\leftarrow}(V_1) \cap h^{\leftarrow}(V_2)$, then $h(x) \in V_1 \cap V_2$, a contradiction. □

Proposition 2.2. A non Urysohn 2-homogeneous space is nowhere Urysohn.

Proof. The proof is similar to the one of Proposition 2.1. One just needs to consider that if $h : X \to X$ is a homeomorphism, then $h(\overline{A}) = \overline{h(A)}$ for each $A \subseteq X$. □

The following proposition follows directly from the definition.

Proposition 2.3. A space X is hyperconnected if and only if for every finite $A \subseteq X$, $|A| = n$, $n \geq 2$, and for every choice of neighbourhoods U_a, $a \in A$, $\bigcap_{a\in A} U_a \neq \emptyset$.

By Proposition 2.3 one can easily show the following.

Proposition 2.4. Let $n \geq 2$. Any n-Hausdorff space is not hyperconnected.

Theorem 2.5. There is no n-Hausdorff non Hausdorff m-homogeneous space for every $n > 2$ and every $m > 1$.

Proof. It follows directly from Propositions 2.4 and 2.1. □

The following example shows that there exist 3-Hausdorff homogeneous spaces.

Example 2.6. A countable 3-Hausdorff homogeneous space.

Consider the space X of non-negative integers with the topology generated by the base $\{\{n, n+1\} : n \text{ is even}\}$. X is a 3-Hausdorff homogeneous space. △

Note that the space in the previous example is a homogeneous space which is not 2-homogeneous.

The analogues of Proposition 2.4 and Theorem 2.5 for n-Urysohn spaces do not hold, as the following example shows.

Example 2.7. A 3-Urysohn space that is n-homogeneous, for all $n \geq 1$, that is not Urysohn.

Consider the well known "irrational slope space", also called Bing's Tripod space (see [25, Example 75]). This space is n-homogeneous, $n \geq 1$ [2], and 3-Urysohn. △

Recall that for every $n \geq 2$ there exist examples of $(n+1)$-Hausdorff spaces which are not n-Hausdorff [5], and examples of $(n+1)$-Urysohn spaces which are not n-Urysohn [7]. Then it is natural to pose the following Questions.

Question 2.8. Is every $(n+1)$-Hausdorff homogeneous space n-Hausdorff, for each $n \geq 2$?

Question 2.9. Is every $(n+1)$-Urysohn homogeneous space n-Urysohn, for each $n \geq 2$?

Examples 2.6 and 2.7 answer negatively Questions 2.8 and 2.9, respectively, for $n = 2$. Note that the space in Example 2.6 is 3-Urysohn, and the construction can be generalized to obtain $(n+1)$-Urysohn non n-Hausdorff spaces for each $n \geq 2$.

In [5], Bonanzinga gives an example of an ω-Hausdorff space which is not n-Hausdorff for every $n \geq 2$. Now we give a countable ω-Hausdorff homogeneous space which is not n-Hausdorff for every $n \geq 2$.

Example 2.10. There is a countable T_1 hyperconnected (hence not n-Hausdorff for every $n \geq 2$) space, which is ω-Hausdorff and homogeneous.

In [9], the following space is constructed. Let $X = \mathbb{Z}\times\mathbb{Z}$ and $\mathscr{B} = \{U_{j,k}, V_{j,k} : j,k \in \mathbb{Z}\}$ is the subbase for the topology, where

$$U_{j,k} = \{(x,y) \in \mathbb{Z}^2 : x > j \text{ or } y > k\}$$
$$V_{j,k} = \{(x,y) \in \mathbb{Z}^2 : x < j \text{ or } y < k\}.$$

This is a T_1 hyperconnected, hence not n-Hausdorff space for every $n \geq 2$ which is ω-Hausdorff, homogeneous, first countable, Lindelof. △

In [7], Bonanzinga, Cammaroto and Matveev constructed a Hausdorff space with extent equal to κ, $\kappa \geq \omega$, which is not κ-Urysohn (we give this example for sake of completeness, see Example 2.12 below). The construction of such a space may be considered a modification of the irrational slope space [25, Example 75]. Since the irrational slope space is homogeneous, it is natural to ask the following.

Question 2.11. Is the space in Example 2.12 homogeneous?

Example 2.12. For every infinite cardinal κ there exists a Hausdorff space with extent equal to κ which is not κ-Urysohn.

Let $\tilde{D} = \{d_{\alpha,n} : \alpha < \kappa, n \in \omega\}$ be a discrete space of cardinality κ, and $D = \tilde{D} \cup \{p\}$ be the one point compactification of $\tilde{D}$. Put $E = D \cup \{d^*\}$ where d^* is isolated in E and is not in D. Consider κ^+ with the order topology, $D \times \kappa^+$ with the Tychonoff product topology, and denote $W = \{p\} \times \kappa^+$; then W is a subspace of $D \times \kappa^+$ homeomorphic to κ^+. Also, for $\alpha < \kappa^+$ denote $W_\alpha = \{p\} \times [\alpha, \kappa^+)$. For $\alpha < \kappa$, $\beta < \kappa^+$, denote $D_\alpha = \{d_{\alpha,n} : n \in \omega\}$, and $T_{\alpha,\beta} = D_\alpha \times [\beta, \kappa^+) \subset D \times \kappa^+$. Let $\vec{p}$ be the point in E^{κ^+} with all coordinates equal to p. Let $S = \{x \in E^{\kappa^+} : |\{\alpha < \kappa^+ : x(\alpha) \neq p\}| \leq \kappa\}$ be the Σ_k-product in E^{κ^+} with center at $\vec{p}$. It can be proved that there is a homeomorphic embedding $f : D \times \kappa^+ \to E^{\kappa^+}$ such that

(1) $f(D \times \kappa^+) \cap S = f(W)$.
(2) $f(W)$ is closed in S and homeomorphic to κ^+ with the order topology.
(3) for every distinct $\alpha, \gamma < \kappa$, the sets $f(T_{\alpha,0})$ and $f(T_{\gamma,0})$ can be separated by open neighbourhoods in E^{κ^+}.
(4) $\overline{f(T_{\alpha,\beta})} \cap S = f(W_\beta)$.

Finally, let $L = \{l_\alpha : \alpha < \kappa\}$ (where all points l_α are distinct) be a set disjoint from E^{κ^+} and topologize $X = S \cup L$ as follows: S, with the topology inherited from E^{κ^+} is open in X; a basic neighbourhood of l_α takes the form $\{l_\alpha\} \cup (U \cap S)$ where U is arbitrary neighbourhood (in E^{κ^+}) of $f(T_{\alpha,\beta})$ for some $\beta < \kappa^+$. We recall that L is closed discrete in this space, so $e(X) \geq \kappa$, and for every family $\{U_l : l \in L\}$ of neighbourhoods of points $l \in L$ in X, $\bigcap\{\overline{U_l} : l \in L\} \neq \emptyset$, so it is not κ-Urysohn. △

3. On the cardinality of n-Hausdorff homogeneous and n-Urysohn homogeneous spaces.

In [19], Hajnal and Juhász proved that, for every Hausdorff space X, $|X| \leq 2^{c(X)\chi(X)}$. In [5] Bonanzinga proved that $|X| \leq 2^{2^{c(X)\chi(X)}}$ for every 3-Hausdorff space X and asked if $|X| \leq 2^{c(X)\chi(X)}$ holds for every n-Hausdorff space X, with $n \geq 2$. In [18] Gotchev, using the cardinal function called "non Hausdorff number" introduced independently from [5], gave a positive answer to the previous question.

In [15], Carlson and Ridderbos proved the following result.

Theorem 3.1. [15] Let X be a homogeneous Hausdorff space. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.

In fact, in [15] it is proved that the previous theorem holds for power homogeneous Hausdorff spaces. Recall that a topological space X is power homogeneous if X^μ is homogeneous for some cardinal number μ. Clearly, if a space is homogeneous it is power homogeneous.

Then, it is natural to pose the following question.

Question 3.2. Is $|X| \leq 2^{c(X)\pi\chi(X)}$ true for every homogeneous space X such that $H(X)$ is finite?

In the following we give partial answers to the previous question.

Given a set A and a cardinal κ, $[X]^\kappa$ denotes the set of all subsets of A whose cardinality is κ.

Theorem 3.3. [17] Let κ be a cardinal number and $f : [(2^{2^\kappa})^+]^3 \to \kappa$ be a function. Then there exists a subset $H \in [(2^{2^\kappa})^+]^{\kappa^+}$ such that $f \restriction [H]^3$ is constant.

Theorem 3.4. Let X be a 3-Hausdorff homogeneous space. Then

$$|X| \leq 2^{2^{c(X)\pi\chi(X)}}$$

Proof. Let $c(X)\pi\chi(X) = \kappa$. Then, by Proposition 1.4, we have $p_2(X) \leq \kappa$. Suppose that $|X| \geq (2^{2^\kappa})^+$. For every triple $x_1, x_2, x_3 \in X$ of distinct points select neighbouroods $U_i(x_1,x_2,x_3)$ of x_i for $i = 1,2,3$ such that
$\bigcap_{i=1}^3 U_i(x_1,x_2,x_3) = \emptyset$. Fix a point $p \in X$ and a local π-base $\mathscr{B}$ for p with $|\mathscr{B}| = \kappa$. Since the space is homogeneous, there exists a family $\{h_x\}_{x\in X}$ of homeomorphisms $h_x : X \to X$ such that $h_x(p) = x$ for every $x \in X$. Fix distinct points $x_1,x_2,x_3 \in X$ and observe that the set $\bigcap_{i=1}^3 h_{x_i}^{\leftarrow}(U_i(x_1,x_2,x_3))$ is an open neighbourhood of p; since $\mathscr{B}$ is a π-base, there is a non empty $B(x_1,x_2,x_3) \in \mathscr{B}$ such that $B(x_1,x_2,x_3)$ is contained in it. Consider now the function $f : [X]^3 \to \mathscr{B}$ defined by $f(\{x_1,x_2,x_3\}) = B(x_1,x_2,x_3)$. Then by Theorem 3.3 there is $Z \in [X]^{\kappa^+}$ and $B \in \mathscr{B}$ such that $f \restriction [Z]^3 = \{B\}$.

Now, the family $\{h_z(B) : z \in Z\}$ is point-(≤ 2) finite in X. To see this, suppose by way of contradiction that there exists $x_0 \in X$ such that $|\{h_z(B) : x_0 \in h_z(B)\}| = 3$. So there are $z_1,z_2,z_3 \in X$ such that $x_0 \in h_{z_i}(B),\ i = 1,2,3$. This implies $x_0 \in h_{z_i}(B) \subseteq h_{z_i}(\bigcap_{i=1}^{3} h_{z_i}^{\leftarrow}(U_i(z_1,z_2,z_3))) \subseteq h_{z_i}(h_{z_i}^{\leftarrow}(U_i(z_1,z_2,z_3))) = U_i(z_1,z_2,z_3)$. Then, $x_0 \in \bigcap_{i=1}^{3} U_i(z_1,z_2,z_3) \neq \emptyset$, a contradiction.

Furthermore, $\{h_z(B) : z \in Z\}$ has cardinality exactly κ^+. Otherwise there exists $z_0 \in Z$ s.t. $|\{z \in Z : h_z(B) = h_{z_0}(B)\}| = \kappa^+$. As before, from $h_{z_0}(B) \subseteq U_i(z_1,z_2,z_3)$ for every triple of elements in $\{z \in Z : h_z(B) = h_{z_0}(B)\}$ we obtain a contradiction.

Thus $p_2(X) = \kappa^+$, a contradiction with $p_2(X) \leq \kappa$. This concludes the proof. □

We remark that Gotchev showed in [18] that if $n \geq 2$ and X is an n-Hausdorff space, then $|X| \leq 2^{c(X)\chi(X)}$.

Recall the following result (for further details, refer to [21]).

Theorem 3.5. Let κ be a cardinal number, $n \geq 3$ and $f : [(2^{2^{\cdot^{\cdot^{\cdot^{2^\kappa}}}}})^+]^n \to \kappa$ be a function (where the power is made $(n-1)$-many times). Then there exists a subset $H \in [(2^{2^{\cdot^{\cdot^{\cdot^{2^\kappa}}}}})^+]^{\kappa^+}$ such that $f \restriction [H]^n$ is constant.

Theorem 3.6. Let X be an n-Hausdorff homogeneous space, with $n \geq 2$. Then

$$|X| \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{c(X)\pi\chi(X)}}}}}}$$

where the power is made $(n-1)$-many times.

Proof. Similar to the proof of the previous theorem using Theorem 3.5 instead of Theorem 3.3. □

Next Theorem 3.12 shows that Question 3.2 has a positive answer if $H(X)$ is replaced by $U(X)$.

In [13], Carlson, Porter and Ridderbos proved the following result.

Theorem 3.7. [13] If X is an n-Hausdorff homogeneous space, with $n \geq 2$, then $|X| \leq d(X)^{\pi\chi(X)}$.

Also recall that a space is quasiregular if every nonempty open set contains a nonempty regular closed set.

Theorem 3.8. If X is an n-Hausdorff quasiregular homogeneous space with $n \geq 2$, then $|X| \leq 2^{c(X)\pi\chi(X)}$.

Proof. It was shown in [11] that if X is quasiregular then $d(X) \leq \pi\chi(X)^{c(X)}$. By Theorem 3.7 we have $|X| \leq d(X)^{\pi\chi(X)} \leq \left(\pi\chi(X)^{c(X)}\right)^{\pi\chi(X)} = 2^{c(X)\pi\chi(X)}$. □

Definition 3.9. [28] Let X be a space. For $A \subseteq X$, the θ-*closure* of A is defined by

$$cl_\theta(A) = \{x \in X : \overline{V} \cap A \neq \emptyset \text{ for every open set } V \text{containing } x\}.$$

A set $A \subseteq X$ is θ-*dense* if $cl_\theta(A) = X$. The θ-*density* of X, $d_\theta(X)$, is defined as the least cardinality of a θ-dense subset of X.

Theorem 3.10. [13] Let X be an n-Urysohn homogeneous space, where $n \geq 2$. Then $|X| \leq d_\theta(X)^{\pi\chi(X)}$.

Theorem 3.11. [11] Let X be a space. Then $d_\theta(X) \leq \pi\chi(X)^{c(X)}$.

By Theorems 3.10 and 3.11, we obtain the following result.

Theorem 3.12. Let X be an n-Urysohn homogeneous space, where $n \geq 2$. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.

Proof. As X is n-Urysohn and homogeneous, we have $|X| \leq d_\theta(X)^{\pi\chi(X)}$ by Theorem 3.10. Thus, by Theorem 3.11, we have $|X| \leq d_\theta(X)^{\pi\chi(X)} \leq \left(\pi\chi(X)^{c(X)}\right)^{\pi\chi(X)} = 2^{c(X)\pi\chi(X)}$. □

4. An embedding into a homogeneous space

In [14] Carlson, Porter and Ridderbos proved the following result.

Theorem 4.1. [14] Let X be a Hausdorff space. Then X can be embedded in a homogeneous space that is the countable union of H-closed spaces.

In Theorem 4.12 below we show that every n-Hausdorff space, $n \geq 2$ can be embedded in a homogeneous space that is the countable union of n-H-closed spaces.

Definition 4.2. [3] Let $n \geq 2$. An n-Hausdorff space X is called *n-H-closed* if X is closed in every n-Hausdorff space Y in which X is embedded.

Given a space X and an ultrafilter $\mathscr{U}$ on it, we put $a\mathscr{U} = \bigcap\{\overline{U} : U \in \mathscr{U}\}$. For an n-Hausdorff space X, with $n \geq 2$, an open ultrafilter $\mathscr{U}$ on X is said to be *full* if $|a\mathscr{U}| = n-1$.

Theorem 4.3. [3] Let $n \geq 2$, and X be a space. The following are equivalent:

(a) X is n-Hausdorff;
(b) if $\mathscr{U}$ is an open ultrafilter of X, then $|a\mathscr{U}| \leq n-1$.

Theorem 4.4. [3] Let $n \geq 2$, and X be an n-Hausdorff space. The following are equivalent:

(a) X is n-H-closed;
(b) every open ultrafilter on X is full.

Recall the following construction, made in [3]. Let $n \geq 2$, X be an n-Hausdorff space and $\mathfrak{U} = \{\mathscr{U} : \mathscr{U}$ is an open ultrafilter such that $|a\mathscr{U}| < n-1\}$. We index $\mathfrak{U}$ by $\mathfrak{U} = \{\mathscr{U}_\alpha : \alpha \in |\mathfrak{U}|\}$. For each $\alpha \in |\mathfrak{U}|$, let $k\alpha = (n-1) - |a\mathscr{U}_\alpha|$ and $\{p_{\alpha i} : 1 \leq i \leq k\alpha\}$ be a set of distinct points disjoint from X. Let $Y = X \cup \{p_{\alpha i} : 1 \leq i \leq k\alpha, \alpha \in |\mathfrak{U}|\}$. A set V is defined to be open in Y if $V \cap X$ is open in X and if $p_{\alpha i} \in V$ for $1 \leq i \leq k\alpha$, $V \cap X \in \mathscr{U}_\alpha$. The space Y is an n-Hausdorff space.

In the following results we use the notation of the previous contruction.

Proposition 4.5. [3] For every $\alpha \in |\mathfrak{U}|$,

$$\mathscr{U}_\alpha = \{V \cap X : p_{\alpha i} \in V \in \tau(Y) \text{ for some } 1 \leq i \leq k\alpha\},$$

where $\tau(Y)$ is the topology on Y.

By the previous proposition the space Y has the property that every open ultrafilter on Y is full. Indeed the points $p_{\alpha i}$, $1 \leq i \leq k\alpha$, added to the space X, are in the closure of each element of $\mathscr{U}_\alpha$. Therefore the space Y is n-H-closed.

Definition 4.6. [3] Let $n \geq 2$, S and T be n-H-closed extensions of an n-Hausdorff space X. We say S is *projectively larger* than T if there is a continuous surjection $f : S \to T$ such that $f(x) = x$ for $x \in X$.

This projectively larger function may not be unique [3].

Theorem 4.7. [3] Let $n \geq 2$, X be an n-Hausdorff space and Y be the n-H-closed extension of X constructed above. If Z is an n-H-closed extension of X, there is a continuous surjection $f : Y \to Z$ such that $f(x) = x$ for all $x \in X$.

Theorem 4.7 shows that the n-H-closed extension Y of X is projectively larger than every n-H-closed extension of X. Moreover, the space Y has an interesting unique property as it is noted in the next result.

Theorem 4.8. [3] Let $n \geq 2$, X be an n-Hausdorff space and Y be the n-H-closed extension of X described above. Let $f : Y \to Y$ be a continuous surjection such that $f(x) = x$ for all $x \in X$. Then f is a homeomorphism.

Remark 4.9. In the class of Hausdorff spaces the function in Definition 4.6 is unique [3]. Sometimes this is a problem in non-Hausdorff spaces. The n-H-closed space Y constructed before for an n-Hausdorff space X is a projective maximum, that is Y is projectively larger than every n-H-closed extention and given a continuous surjection $f : Y \to Y$ such that $f(x) = x$ for every $x \in X$, then f is a homeomorphism. For the future we denote this Y with n-kX and we call it the *n-Katětov extension* of X.

Uspenskiǐ showed in [27] that for any space X there exists a cardinal κ and a nonempty subspace $Z \subseteq X^\kappa$ such that $X \times Z$ is homogeneous. The space Z is found by selecting a set A such that $\kappa = |A| \geq |X|$ and letting $Z = \{f \in {}^AX : \text{for each } x \in X, |f^{\leftarrow}(x)| = \kappa\}$, where ${}^AX$ is the space of all functions from A to X. Both Z and $X \times Z$ are homogeneous and homeomorphic. For our construction we write $\mathbf{H}(X) = X \times Z$ and consider X as a subspace of $\mathbf{H}(X)$ [14].

Lemma 4.10. [14] Let X be a space and $h : X \to X$ be a homeomorphism and let id_Z be the identity function on Z . Then the function $h \times id_Z : \mathbf{H}(X) \to \mathbf{H}(X)$ is also a homeomorphism that extends h.

Lemma 4.11. Let $n \geq 2$, X be an n-Hausdorff space and $h : X \to X$ be a homeomorphism. Then there is a homeomorphism n-$kh : n$-$kX \to n$-kX that extends h.

Proof. Let $p \in n\text{-}kX \setminus X$, then $p = p_{\alpha i}$ for some $\alpha \in |\mathfrak{U}|$ and for some $i = 1, \dots, k\alpha$. The set $\mathscr{V} = \{h(U) : U \in \mathscr{U}_\alpha\}$ is an open ultrafilter on X and since $|a\mathscr{U}_\alpha| = |a\mathscr{V}|$, there exists $\beta \in |\mathfrak{U}|$ such that $\mathscr{V} = \mathscr{U}_\beta$. Define $n\text{-}kh(p_{\alpha i}) = p_{\beta i}$ for every $i = 1, \dots, k\alpha = k\beta$. For $x \in X$, define $n\text{-}kh(x) = h(x)$. The function n-kh is clearly a homeomorphism that extends h. □

Theorem 4.12. Let $n \geq 2$, X be an n-Hausdorff space. Then X can be embedded in a homogeneous space that is the countable union of n-H-closed spaces.

Proof. Let $H_1 = \mathbf{H}(n\text{-}kX)$. If H_m is defined, let's define $H_{m+1} = \mathbf{H}(n\text{-}kH_m)$ and $H = \bigcup_{m \in \mathbb{N}} H_m$. A subset $U \subseteq H$ is open in H if $U \cap H_m \in \tau(H_m)$ for every $m \in \mathbb{N}$. The space H is the countable union of n-H-closed spaces. We have to prove that H is homogeneous. Let $p, q \in H$. Since $H_m \subsetneq H_{m+1}$, there exists $m \in \mathbb{N}$ such that $p, q \in H_m$. Each H_m is homogeneous, then there exist a homeomorphism $h : H_m \to H_m$ such that $h(p) = q$. By Lemma 4.11 there exists a homeomorphism $n\text{-}kh : n\text{-}kH_m \to n\text{-}kH_m$ that extends h. By Lemma 4.10 the function $n\text{-}kh \times id_Z : H_{m+1} \to H_{m+1}$ is a homeomorphism. Put $n\text{-}kh = h_1$. By induction h can be extended to $h_k : H_{m+k} \to H_{m+k}$ for every $k \in \mathbb{N}$. The function $g = \bigcup_{k \in \mathbb{N}} h_k : H \to H$ extends h and it is a homeomorphism on H. Then H is homogeneous. □

Example 4.13. An example of an n-Hausdorff, homogeneous, not n-Urysohn space which is the countable union of n-H-closed spaces, for every $n \geq 2$.

Let's take an n-Hausdorff, not n-Urysohn space X (for example see [5, Example 4]), $n \geq 2$. Then, by Theorem 4.12, X can be embedded in an n-Hausdorff, homogeneous space Y which is the countable union of n-H-closed spaces. Furthermore Y is not n-Urysohn, since X is a non-n-Urysohn subset of it. $\triangle$

Acknowledgement: The research was supported by "National Group for Algebric and Geometric Structures, and their Applications" (GNSAGA-INdAM).

References

[1] A.V. Arhangel'skii, J. van Mill, *Topological Homogeneity*, in *Recent Progress in General Topology III*, K. Hart, J. van Mill, P. Simon, Atlantis Press (2014).

[2] I. Banakh, T. Banakh, O. Hryniv and Y. Stelmakh, *The connected countable spaces of Bing and Ritter are topologically homogeneous*, Top. Proc. **57** (2021), 149-158.

[3] F. A. Basile, M. Bonanzinga, N. Carlson, J. Porter, *n-H-closed spaces*, Topol. Appl. **254** (2019), 59-68.

[4] F. A. Basile, M. Bonanzinga, N. Carlson, J. Porter, *Absolutes and n-H-closed spaces*, Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat **99:2** (2021).

[5] M. Bonanzinga, *On the Hausdorff number of a topological space*, Houston J. Math. **39:3** (2013), 1013-1030.

[6] M. Bonanzinga, F. Cammaroto and M. Matveev, *On a weaker form of countable compactness*, Quaest. Math. **30:4** (2007), 407-415.

[7] M. Bonanzinga, F. Cammaroto and M. Matveev, *On the Urysohn number of a topological space*, Quaest. Math. **34:4** (2011), 441-446.

[8] M. Bonanzinga, N. Carlson, D. Giacopello, *New bounds on the cardinality of n-Hausdorff and n-Urysohn spaces*, https://arxiv.org/abs/2302.13060.

[9] M. Bonanzinga, D. Stavrova and P. Staynova, *Separation and cardinality-Some new results and old questions*, Topol. Appl. **221** (2017), 556-569.

[10] N. Carlson, *A survey of cardinality bounds on homogeneous topological spaces*, Top. Proc. **57** (2021), pp. 259-278. (E-published on October 18, 2020)

[11] N. Carlson, *Non-regular power homogeneous spaces*, Topol. Appl. **154** (2007), pp. 302-308.

[12] N. Carlson and J. Porter, *On the cardinality of Hausdorff spaces and H-closed spaces*, Topol. Appl. **241** (2018), 377-395.

[13] N. Carlson, J. Porter and G. J. Ridderbos, *On cardinality bounds for homogeneous spaces and the G_κ-modification of a space*, Topol. Appl. **159** (2012), 2932-2941.

[14] N. Carlson, J. Porter and G. J. Ridderbos, *On homogeneity and the H-closed property*, Top. Proc. **49** (2017), 153-164.

[15] N. Carlson and G. J. Ridderbos, *Partition relations and power homogeneity*, Top. Proc. **32** (2008), pp. 115-124.

[16] R. Engelking, *General Topology*, 2nd Edition, Sigma Ser. Pure Math., Vol. 6 Heldermann, Berlin (1989).

[17] P. Erdös, R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956) 427-489.

[18] I. Gotchev, *Generalization of two cardinal inequalities of Hajnal and Juhász*, Topology Appl. **221** (2017), 425–431.

[19] A. Hajnal, I Juhász, *Discrete subspaces of topological spaces*, Proc. of Nederl. Akad., Series, A **70** (1967), pp. 343-356.

[20] R.E. Hodel, *Arhangelskii's solution to Alexandroff problem: A survey*, Topol. Appl. **153** (2006), pp. 2199-2217.

[21] T. Jech, *Set Theory*, 3rd Millenium Edition, Springer-Verlag, 1997.

[22] I. Juhász, *Cardinal functions in topology-ten years later*, Math. Centre Tracts, Vol. **123**, Math. Centrum, Amsterdam (1980).

[23] I. Juhász, L. Soukup, and Z. Szentmiklóssy, *Anti-Urysohn spaces*, Top. Appl. **213** (2016), pp. 8-23.

[24] G.J.Ridderbos, *On the cardinality of power homogeneous Hausdorff spaces*, Fund. Math. **192** (2006), pp. 255-266.

[25] L.A. Steen and J.A. Seebach, *Counterexamples in Topology*, Springer-Verlag, New York (1978).

[26] V.V. Tkachuck, *On cardinal invariants of Suslin number type*, Soviet Math. Dokl. **27:3** (1983), pp. 681-684.

[27] V.V. Uspenskiĭ, *For any X, the product $X \times Y$ is homogeneous for some Y*, Proc. Amer. Math Soc. **87** (1983), pp. 187-188.
[28] N.V. Velichko, *H-Closed Topological Spaces*, Mat. Sb. (N.S.) **70** (1966), pp. 98-112.

MIFT Department, University of Messina, Italy
Email address: mbonanzinga@unime.it

Mathematics department, California Lutheran University, USA
Email address: ncarlson@callutheran.edu

MIFT Department, University of Messina, Italy
Email address: dagiacopello@unime.it

MIFT Department, University of Messina, Italy
Email address: fomaesano@unime.it