Simplified two-level algorithm for the stationary Smagorinsky model based on penalty method[☆]

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Abstract

In this work, we combine the penalty method and simplified two-level technique to solve the stationary Smagorinsky model, based on two different finite element space pairs: the $P_2 - P_0$ element and the $P_1 - P_0$ element. The simplified two-level penalty algorithm involves solving one small penalty Smagorinsky model on a coarse mesh, and one penalty Stokes equations on a fine mesh. Moreover, convergence results of the presented algorithm are proved. Then, some numerical experiments are provided to illustrate the theoretical results of the simplified two-level penalty algorithm.

Keywords: Smagorinsky model; Penalty method; Inf-sup condition; Two-level technique; Error estimate.

1. Introduction

Numerical simulation of turbulence is a great computational challenge in fluid dynamics. Large eddy simulation is one of effective methods for turbulence simulation. Moreover, the Smagorinsky model [28] is one of the most popular large eddy simulation [3, 4], which is widely used in many aspects, such as gas dynamics and geophysical flow. In this paper, we consider the following steady-state Smagorinsky model [4]:

$$\begin{aligned} -\nu\Delta\mathbf{u} - \nabla\cdot((C_S\delta)^2|\nabla\mathbf{u}|\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega,\\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and regular domain with a Lipschitz continuous boundary $\partial\Omega$. Here $\mathbf{u}$, p and $\mathbf{f}$ represent the velocity vector of a viscous incompressible fluid, the pressure and the prescribe spatially filtered forcing term, respectively. Besides, C_S is the Smagorinsky constant, δ is the radius of the spatial filter used in large eddy simulation, and ν is the viscosity. For a tensor σ, $|\sigma| = \sqrt{\sum_{i,j=1}^{2} |\sigma_{ij}|^2}$ denotes the Frobenius norm.

The Smagorinsky model adds an artificial viscosity term to the Navier-Stokes equations, and this additional term dissipates energy in a large scale structures at the same rate as the discarded small scale. Some researches have been made to solve numerically the Smagorinsky problem. Borggaard et al. [4] applied the two-level finite element method to the Smagorinsky model, which solved the Smagorinsky problem on a coarse grid and Newton's linearized Smagorinsky problem on a fine grid. Huang et al. [19] combined the two-level method with the lowest order finite element method to solve the Smagorinsky problem. Then, Shi et al. [27] proposed the nonconforming finite element method for the considered model. Furthermore, several efficient schemes [2, 29, 33] have been well further developed for solving the Smagorinsky model.

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As is known, two-level technique is an efficient numerical scheme for solving nonlinear problems, which was proposed by Xu [30, 31]. This technique can greatly reduce the calculation cost. Many researchers have explored this idea for different nonlinear equations (see Layton [20], Layton and Lenferink [23, 22], He et al. [12, 11, 13], and Huang et al. [16, 17, 18]). We notice that the Smagorinsky model is also nonlinear. Besides, the velocity and pressure are coupled by incompressible condition. In fact, the popular method to overcome this coupling is the penalty method [15, 26, 32].

From the above literatures, we know that the two-level method and the penalty method are effective for solving nonlinear problems with incompressible conditions. Therefore, the penalty method is applied to (1) which approximates the solution $(\mathbf{u}, p)$ by $(\mathbf{u}_\epsilon, p_\epsilon)$ satisfying the following equations:

$$\begin{aligned} -\nu\Delta\mathbf{u}_\epsilon - \nabla\cdot((C_S\delta)^2|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon) + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f} \quad &\text{in } \Omega,\\ \nabla\cdot\mathbf{u}_\epsilon + \frac{\epsilon}{\nu}p_\epsilon = 0 \quad &\text{in } \Omega,\\ \mathbf{u}_\epsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $0 < \epsilon < 1$ is a penalty parameter. We note that p_ϵ can be eliminated to obtain a penalty system of $\mathbf{u}_\epsilon$, which is much easier to solve than the original equations (1).

In this paper, we combine simplified two-level method and penalty method to solve the stationary S-magorinsky model based on two finite element space pairs: one is $P_2 - P_0$ element satisfying the discrete inf-sup condition, and other one is $P_1 - P_0$ element dissatisfying this condition. We first solve the penalty Smagorinsky problem on a coarse mesh with mesh size H, and then solved the penalty Stokes problem on a fine mesh with mesh size h ($h \ll H$). Secondly, we prove stability and convergence of the penalty finite element method for the Smagorinsky model, and error estimate of the simplified two-level penalty algorithm by selecting appropriate δ, ϵ and h. Moreover, if we choose $P_1 - P_0$ element and $H = O(\epsilon^{\frac{1}{4}}h^{\frac{1}{2}})$, then the approximate solution produced by the simplified two-level penalty algorithm is asymptotically as accurate as the approximation produced by solving the nonlinear system on the fine mesh; if we choose $P_2 - P_0$ element and $H = O(h^{\frac{1}{2}})$, then we provide an approximate solution with the convergence rate of same order as the penalty finite element solution obtained on the fine mesh.

The paper is organized as follows. In Section 2, we introduced some preliminary knowledge of the stationary Smagorinsky model. In Section 3, we give the penalty finite element discretization based on two finite element space pairs. Section 4 shows the simplified two-level penalty algorithm and its error estimate. In Section 5, numerical experiments are given to illustrate the accuracy and efficiency of the presented algorithm. We conclude the paper in Section 6.

2. Preliminaries

In this section, we introduce some the notations and results used in this paper. We introduce the necessary function spaces. For $\Omega \subset \mathbb{R}^2$ and $W^{k,r}(\Omega)$, $H^k(\Omega)$, $0 < r \le \infty$, $k = 0, 1, 2, ...$, denote the usual Sobolev spaces. The space $L^2(\Omega)$ is equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Let $\|\cdot\|_{L^r}$ the norm on $L^r(\Omega)$, and $\|\cdot\|_{k,r}$ the norm on $W^{k,r}(\Omega)$. Set $\|\cdot\|_k = \|\cdot\|_{k,2}$. Besides, C is a positive constant representing different values in different situations, and it depends on Ω but not on the mesh scales h, H, the parameter δ and ϵ in this paper.

We then introduce the following function spaces:

$$\mathbf{X} = \{\mathbf{v} \in W^{1,3}(\Omega)^2 : \mathbf{v} = 0 \text{ on } \partial\Omega\} \subset H_0^1(\Omega)^2, \quad M = \{q \in L^2(\Omega) : (q, 1) = 0\}.$$

Besides, we define bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ as follow:

$$a(\mathbf{u}, \mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}), \quad d(\mathbf{v}, q) = (\nabla\cdot\mathbf{v}, q) \ \ \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{X}, q \in M.$$

Obviously, the bilinear form $d(\cdot,\cdot)$ satisfies the inf-sup condition [10, 5]:

$$\beta\|q\|_0 \le \sup_{\mathbf{v}\neq 0\in H_0^1(\Omega)^2} \frac{|d(\mathbf{v}, q)|}{\|\nabla\mathbf{v}\|_0}, \quad \forall q \in M, \tag{3}$$

where β is a positive constant dependent on Ω.

We also introduce a trilinear form $b(\cdot,\cdot,\cdot)$ on $\mathbf{X}\times\mathbf{X}\times\mathbf{X}$ by

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) + 0.5((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w}) \\ &= 0.5((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - 0.5((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}. \end{aligned}$$

The above trilinear term has the following important properties [14, 8]:

$$\begin{aligned} & b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \\ & |b(\mathbf{u},\mathbf{v},\mathbf{w})| \le N\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{v}\|_0\|\nabla\mathbf{w}\|_0, \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}, \end{aligned} \tag{4}$$

where $N := \sup\limits_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}} \frac{|b(\mathbf{u},\mathbf{v},\mathbf{w})|}{\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{v}\|_0\|\nabla\mathbf{w}\|_0}$ is a positive constant.

Next, we recall some of the following inequalities throughout the paper. The Gagliardo-Nirenberg inequality [24, 25]:

$$\|\mathbf{v}\|_{L^4}^2 \le C\|\mathbf{v}\|_0\|\nabla\mathbf{v}\|_0,\ \forall\ \mathbf{v}\in\mathbf{X}, \quad \|\nabla\mathbf{v}\|_{L^4}^2 \le C\|\mathbf{v}\|_2\|\nabla\mathbf{v}\|_0,\ \forall\ \mathbf{v}\in\mathbf{X}\cap H^2(\Omega)^2, \tag{5}$$

the Agmon's inequality [1, 24]:

$$\|\mathbf{v}\|_{L^\infty}^2 \le C\|\mathbf{v}\|_0\|\mathbf{v}\|_2,\ \forall\ \mathbf{v}\in\mathbf{X}\cap H^2(\Omega)^2, \tag{6}$$

the Poincáre inequality [10]:

$$\|\mathbf{v}\|_0 \le C\|\nabla\mathbf{v}\|_0,\ \forall\ \mathbf{v}\in\mathbf{X}, \tag{7}$$

and Young's inequality:

$$(\mathbf{u},\mathbf{v}) \le \frac{\xi}{p}\|\mathbf{u}\|_{L^p}^p + \frac{\xi^{-\frac{q}{p}}}{q}\|\mathbf{v}\|_{L^q}^q,\ \xi\in(0,\infty),\ \frac{1}{p}+\frac{1}{q}=1,\ p,q\in[1,\infty]. \tag{8}$$

Furthermore, we recall the inverse inequality [21]:

$$\|\nabla\mathbf{v}_\mu\|_{L^3} \le C_{inv}\mu^{-\frac{1}{3}}\|\nabla\mathbf{v}_\mu\|_0,\ \forall\mathbf{v}_\mu\in\mathbf{X}_\mu. \tag{9}$$

Here μ is mesh size and C_{inv} is a positive constant independent on μ. In Section 3, we will give a detailed definition on μ.

Furthermore, we will also apply the following lemma concerning strong monotonicity and Lipschitz continuity of the r-Laplacian.

Lemma 2.1. *[4] For all $\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}\in W^{1,r}(\Omega)^2$, there exists a generic constant C depending on r and Ω such that the following inequalities hold*

$$\begin{aligned} (|\nabla\mathbf{u}_1|^{r-2}\nabla\mathbf{u}_1,\nabla(\mathbf{u}_1-\mathbf{u}_2)) - (|\nabla\mathbf{u}_2|^{r-2}\nabla\mathbf{u}_2,\nabla(\mathbf{u}_1-\mathbf{u}_2)) &\ge C\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{L^r}^r, \\ (|\nabla\mathbf{u}_1|^{r-2}\nabla\mathbf{u}_1,\nabla\mathbf{v}) - (|\nabla\mathbf{u}_2|^{r-2}\nabla\mathbf{u}_2,\nabla\mathbf{v}) &\le CM\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{L^r}\|\nabla\mathbf{v}\|_{L^r}, \end{aligned}$$

where $M = \max\{\|\nabla\mathbf{u}_1\|_{L^r}^{r-2}, \|\nabla\mathbf{u}_2\|_{L^r}^{r-2}\}$.

With the above notations, the variational formulation of (1): find $(\mathbf{u},p)\in(\mathbf{X},M)$ satisfying for all $(\mathbf{v},q)\in(\mathbf{X},M)$,

$$a(\mathbf{u},\mathbf{v}) + (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u},\nabla\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - d(\mathbf{v},p) + d(\mathbf{u},q) = (\mathbf{f},\mathbf{v}), \tag{10}$$

and the variational formulation of (2): find $(\mathbf{u}_\epsilon,p_\epsilon)\in(\mathbf{X},M)$ satisfying for all $(\mathbf{v},q)\in(\mathbf{X},M)$,

$$a(\mathbf{u}_\epsilon,\mathbf{v}) + (C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon,\nabla\mathbf{v}) + b(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) - d(\mathbf{v},p_\epsilon) + d(\mathbf{u}_\epsilon,q) + \frac{\epsilon}{\nu}(p_\epsilon,q) = (\mathbf{f},\mathbf{v}). \tag{11}$$

Then, we recall the well-posedness of the solution to the problem (10) in the following lemma [4, 8, 9].

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Lemma 2.2. *There exists a weak solution* $\mathbf{u} \in \mathbf{X}$ *to the problem* (10) *satisfying*

$$\|\nabla \mathbf{u}\|_{L^3} \le (C_S\delta)^{-1}\|\mathbf{f}\|_{-1,3}^{1/2}, \quad \|\mathbf{f}\|_{-1,3} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|_{L^3}},$$
$$\|\nabla\mathbf{u}\|_0 \le \Psi(\|\mathbf{f}\|_{-1}), \quad \|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in H_0^1(\Omega)^2} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|_0},$$

where Ψ *is defined as the inverse function of* $\Phi : \mathbb{R}^+ \to \mathbb{R}$

$$\Phi(x) = \nu x + (C_S\delta)^2 \left(\inf_{\mathbf{v}\in\mathbf{X}} \frac{\|\nabla\mathbf{v}\|_{L^3}}{\|\nabla\mathbf{v}\|_0}\right)^3 x^2. \tag{12}$$

Furthermore, if the following inequality holds,

$$0 < \nu^{-1}N\Psi(\|\mathbf{f}\|_{-1}) \le 1, \tag{13}$$

then the problem (10) *has a unique solution.*

Moreover, a similar argument to that used in [12], we have following results on the penalty problem (11).

Theorem 2.1. *If* ν *satisfy the condition of* (13)*, the problem* (11) *exists unique solution* $(\mathbf{u}_\epsilon, p_\epsilon) \in (\mathbf{X}, M)$*, which satisfies*

$$\|\nabla\mathbf{u}_\epsilon\|_{L^3} \le (C_S\delta)^{-1}\|\mathbf{f}\|_{-1,3}^{1/2}, \quad \|\nabla\mathbf{u}_\epsilon\|_0 \le \Psi(\|\mathbf{f}\|_{-1}).$$

Furthermore, we will give error bounds of $\mathbf{u} - \mathbf{u}_\epsilon$ and $p - p_\epsilon$ in the following theorem.

Theorem 2.2. *Assume that* $\mathbf{u}, \mathbf{u}_\epsilon \in \mathbf{X} \cap W_0^{1,\infty}(\Omega)^2$*. If* ν *and* $\mathbf{f}$ *satisfy* (13)*, then we have the following error result*

$$\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0 + \|p - p_\epsilon\|_0 \le C(\epsilon + \epsilon\delta^2 + \epsilon\delta^4).$$

Proof. Subtracting (11) from (10), one gets

$$\begin{aligned} &a(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{v}) + (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u},\nabla\mathbf{v}) - (C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon,\nabla\mathbf{v}) + b(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u},\mathbf{v}) + b(\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{v}) \\ &- d(\mathbf{v}, p - p_\epsilon) + d(\mathbf{u}-\mathbf{u}_\epsilon, q) + \frac{\epsilon}{\nu}(p - p_\epsilon, q) = \frac{\epsilon}{\nu}(p, q). \end{aligned} \tag{14}$$

Use Lemma 2.1 to have

$$\begin{aligned} &(C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u},\nabla\mathbf{v}) - (C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon,\nabla\mathbf{v}) \\ &\le |(C_S\delta)^2(|\nabla\mathbf{u}|\nabla(\mathbf{u}-\mathbf{u}_\epsilon),\nabla\mathbf{v})| + |(C_S\delta)^2((|\nabla\mathbf{u}| - |\nabla\mathbf{u}_\epsilon|)\nabla\mathbf{u}_\epsilon,\nabla\mathbf{v})| \\ &\le C(C_S\delta)^2(\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{u}_\epsilon\|_{L^\infty})\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0\|\nabla\mathbf{v}\|_0 \le C\delta^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0\|\nabla\mathbf{v}\|_0. \end{aligned} \tag{15}$$

In addition, taking $q = 0$ in (14), we apply Lemma 2.2, Theorem 2.1, (3), (4) and (15) to arrive at

$$\begin{aligned} \|p - p_\epsilon\|_0 &\le \beta^{-1}(\nu\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0 + N(\|\nabla\mathbf{u}\|_0 + \|\nabla\mathbf{u}_\epsilon\|_0)\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0 + C\delta^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0) \\ &\le C(\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0 + \delta^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0). \end{aligned} \tag{16}$$

Now, by choosing $(\mathbf{v}, q) = (\mathbf{u}-\mathbf{u}_\epsilon, p - p_\epsilon)$ in (14), using (4) and Lemma 2.1, 2.2, we obtain

$$\begin{aligned} \nu\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0^2 + C(C_S\delta)^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_{L^3}^3 + \frac{\epsilon}{\nu}\|p - p_\epsilon\|_0^2 &\le N\|\nabla\mathbf{u}\|_0\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0^2 + \frac{\epsilon}{\nu}\|p\|_0\|p - p_\epsilon\|_0 \\ &\le N\Psi(\|\mathbf{f}\|_{-1})\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0^2 + \frac{\epsilon}{\nu}\|p\|_0\|p - p_\epsilon\|_0, \end{aligned} \tag{17}$$

which is rearranged as

$$\nu(1 - \nu^{-1}N\Psi(\|\mathbf{f}\|_{-1}))\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0^2 + C(C_S\delta)^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_{L^3}^3 + \frac{\epsilon}{\nu}\|p - p_\epsilon\|_0^2 \le \frac{\epsilon}{\nu}\|p\|_0\|p - p_\epsilon\|_0. \tag{18}$$

Finally, combining (18) with (16), and using(13), we arrive at

$$\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0^2 \le C\frac{\epsilon}{\nu}\|p\|_0(\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0 + \delta^2\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\|_0), \tag{19}$$

which and (16) complete the proof of the theorem. □

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3. Penalty finite element discretization based on two finite element space pairs

Let τ_μ denote conforming, quasi-uniform families of meshes for $\overline{\Omega}$, consisting of some affine-equivalent triangles [7] with maximum element diameter μ. We introduce two different finite element subspace pairs $\mathbf{X}_\mu$ and M_μ of $\mathbf{X}$ and M as follows:

$$\mathbf{X}_\mu = \{\mathbf{u} \in C(\overline{\Omega})^2 \cap \mathbf{X} : \mathbf{u}|_K \in P_i(K)^2, \forall K \in \tau_\mu\}, \quad i = 1, 2,$$
$$M_\mu = \{q \in M : q|_K \in P_0(K), \forall K \in \tau_\mu\},$$

where $P_i(K)$ is set of polynomials on triangles K of degree less than i. Besides, let $\rho_\mu : M \to M_\mu$ denote the L^2-orthogonal projection defined by

$$(\rho_\mu q, q_\mu) = (q, q_\mu), \quad \forall q \in M, q_\mu \in M_\mu.$$

Note that the finite element space pair $(\mathbf{X}_\mu, M_\mu)$ satisfies the following properties [12, 15].

$(\mathcal{P}_1)$. For $i = 1$, $(\mathbf{X}_\mu, M_\mu)$ does not satisfy the discrete inf-sup condition. However, it satisfies the following important relation

$$\mathrm{div}\mathbf{X}_\mu = M_\mu. \tag{20}$$

Furthermore, there exists a mapping $\mathbf{r}_\mu : H^2(\Omega)^2 \cap \mathbf{X} \to \mathbf{X}_\mu$ which and $\rho_\mu : M \to M_\mu$ satisfy

$$\|\nabla(\mathbf{u} - \mathbf{r}_\mu\mathbf{u})\|_0 + \mu^{\frac{1}{3}}\|\nabla(\mathbf{u} - \mathbf{r}_\mu\mathbf{u})\|_{L^3} \leq C\mu\|\mathbf{u}\|_2, \quad \|p - \rho_\mu p\|_0 \leq C\mu\|p\|_1, \tag{21}$$

$(\mathcal{P}_2)$. For $i = 2$, the finite element pair $(\mathbf{X}_\mu, M_\mu)$ does not satisfy (20). However, it satisfies the discrete inf-sup condition

$$\sup_{\mathbf{v}_\mu \in \mathbf{X}_\mu} \frac{(\nabla \cdot \mathbf{v}_\mu, q_\mu)}{\|\nabla \mathbf{v}_\mu\|_0} \geq \beta_0\|q_\mu\|_0, \quad \forall\, q_\mu \in M_\mu, \tag{22}$$

where β is a positive constant dependent on Ω. Furthermore, there exists a mapping $\mathbf{r}_\mu : H^2(\Omega)^2 \cap \mathbf{X} \to \mathbf{X}_\mu$ which and $\rho_\mu : M \to M_\mu$ satisfy (21) and

$$(\nabla \cdot (\mathbf{u} - \mathbf{r}_\mu\mathbf{u}), q_\mu) = 0, \quad \forall\, q_\mu \in M_\mu. \tag{23}$$

In fact, the corresponding discrete variational formulation of (11) can be defined as: solve $(\mathbf{u}_{\epsilon\mu}, p_{\epsilon\mu}) \in (\mathbf{X}_\mu, M_\mu)$ for all $(\mathbf{v}_\mu, q_\mu) \in (\mathbf{X}_\mu, M_\mu)$ such that

$$a(\mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu}, \nabla\mathbf{v}_\mu) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) - d(\mathbf{v}_\mu, p_{\epsilon\mu}) + d(\mathbf{u}_{\epsilon\mu}, q_\mu) + \frac{\epsilon}{\nu}(p_{\epsilon\mu}, q_\mu) = (\mathbf{f}, \mathbf{v}_\mu). \tag{24}$$

Then, when $i = 1$, due to the property (20), the penalty finite element scheme can be defined in the following form

$$\begin{aligned} a(\mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu}, \nabla\mathbf{v}_\mu) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + \frac{\nu}{\epsilon}(\mathrm{div}\mathbf{u}_{\epsilon\mu}, \mathrm{div}\mathbf{v}_\mu) &= (\mathbf{f}, \mathbf{v}_\mu), \\ p_{\epsilon\mu} &= -\frac{\nu}{\epsilon}\mathrm{div}\mathbf{u}_{\epsilon\mu}. \end{aligned} \tag{25}$$

When $i = 2$, the finite element space pair $(\mathbf{X}_\mu, M_\mu)$ does not satisfy the property (20). Therefore, we need to use the mapping ρ_μ to rewrite the penalty finite element scheme

$$\begin{aligned} a(\mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu}, \nabla\mathbf{v}_\mu) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + \frac{\nu}{\epsilon}(\rho_\mu\mathrm{div}\mathbf{u}_{\epsilon\mu}, \rho_\mu\mathrm{div}\mathbf{v}_\mu) &= (\mathbf{f}, \mathbf{v}_\mu), \\ p_{\epsilon\mu} &= -\frac{\nu}{\epsilon}\rho_\mu\mathrm{div}\mathbf{u}_{\epsilon\mu}. \end{aligned} \tag{26}$$

Hence, compared with the original stiffness matrix, one only needs to solve the stiffness matrix with relatively small dimension.

Next, we provide the following stability of the penalty method.

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Theorem 3.1. *Under assumption of Theorem 2.1, when* $(\mathbf{X}_\mu, M_\mu)$ *satisfies the property* $\mathcal{P}_i$, $i = 1,2$, *the solution* $(\mathbf{u}_{\epsilon\mu}, p_{\epsilon\mu}) \in (\mathbf{X}_\mu, M_\mu)$ *to the problem* (24) *satisfies*

$$\|\nabla \mathbf{u}_{\epsilon\mu}\|_0 \le \Psi(\|\mathbf{f}\|_{-1}), \quad \|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3} \le (C_S\delta)^{-1}\|\mathbf{f}\|_{-1,3}^{1/2},$$
$$\|p_{\epsilon\mu}\|_0 \le (\frac{\epsilon}{\nu}C_S\delta)^{-\frac{1}{2}}\|\mathbf{f}\|_{-1,3}^{3/4}, \quad for\ \mathcal{P}_1, \quad \|p_{\epsilon\mu}\|_0 \le \beta_0^{-1}(\Psi(\|\mathbf{f}\|_{-1}) + \nu\Psi(\|\mathbf{f}\|_{-1}) + \|\mathbf{f}\|_{-1}), \quad for\ \mathcal{P}_2.$$

Proof. It follows from (24) with $(\mathbf{v}_\mu, q_\mu) = (\mathbf{u}_{\epsilon\mu}, p_{\epsilon\mu})$ and (4) that

$$\nu\|\nabla \mathbf{u}_{\epsilon\mu}\|_0^2 + (C_S\delta)^2\|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3}^3 + \frac{\epsilon}{\nu}\|p_{\epsilon\mu}\|_0^2 \le \|\mathbf{f}\|_{-1}\|\nabla \mathbf{u}_{\epsilon\mu}\|_0 \text{ (or } \|\mathbf{f}\|_{-1,3}\|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3}). \tag{27}$$

Hence, it is easy to obtain

$$\|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3} \le (C_S\delta)^{-1}\|\mathbf{f}\|_{-1,3}^{1/2}, \tag{28}$$

and

$$\|p_{\epsilon\mu}\|_0 \le (\frac{\nu}{\epsilon}\|\mathbf{f}\|_{-1,3}\|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3})^{1/2} \le (\frac{\nu}{\epsilon})^{\frac{1}{2}}(C_S\delta)^{-\frac{1}{2}}\|\mathbf{f}\|_{-1,3}^{3/4}. \tag{29}$$

Besides, according to (27), we know that

$$\nu\|\nabla \mathbf{u}_{\epsilon\mu}\|_0^2 + (C_S\delta)^2\|\nabla \mathbf{u}_{\epsilon\mu}\|_{L^3}^3 \le \|f\|_{-1}\|\nabla \mathbf{u}_{\epsilon\mu}\|_0. \tag{30}$$

From the definition of Ψ, we can immediately get

$$\|\nabla \mathbf{u}_{\epsilon\mu}\|_0 \le \Psi(\|\mathbf{f}\|_{-1}). \tag{31}$$

Finally, utilizing the discrete inf-sup condition (22) and (24) with $q_\mu = 0$, we can deduce the bound for the pressure with $i = 2$ easily with help of (13) and the definition of Ψ. □

Now, we consider the finite element errors of the penalty system (24).

Theorem 3.2. *Assume that* $\mathbf{u}_\epsilon \in \mathbf{X} \cap W_0^{1,\infty}(\Omega)^2$ *and the properties* $\mathcal{P}_1, \mathcal{P}_2$ *of* $\mathbf{X}_\mu \times M_\mu$ *hold. If* ν *satisfies the condition* (13), *then we get*

$$\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 + \mu^{\frac{1}{3}}\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3} + \epsilon^{\frac{1}{2}}\|p_\epsilon - p_{\epsilon\mu}\|_0 \le C(\epsilon^{-\frac{1}{2}}\mu + \delta^2\mu), \quad i = 1,$$
$$\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 + \mu^{\frac{1}{3}}\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3} \le C(\mu + \delta^2\mu), \quad i = 2,$$
$$\|p_\epsilon - p_{\epsilon\mu}\|_0 \le C(\mu + \delta^2\mu + \delta\mu^{\frac{1}{3}} + \delta^3\mu^{\frac{1}{3}}), \quad i = 2.$$

Proof. Subtracting (24) from (11) with $(\mathbf{v}, q) = (\mathbf{v}_\mu, q_\mu)$, we get

$$\begin{aligned}a(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla \mathbf{u}_\epsilon|\nabla \mathbf{u}_\epsilon, \nabla \mathbf{v}_\mu) - (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon\mu}|\nabla \mathbf{u}_{\epsilon\mu}, \nabla \mathbf{v}_\mu) + b(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{u}_\epsilon, \mathbf{v}_\mu)\\ + b(\mathbf{u}_{\epsilon\mu}, \mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) - d(\mathbf{v}_\mu, p_\epsilon - p_{\epsilon\mu}) + d(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, q_\mu) + \frac{\epsilon}{\nu}(p_\epsilon - p_{\epsilon\mu}, q_\mu) = 0.\end{aligned} \tag{32}$$

Setting $(\mathbf{e}, \eta) = (\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \rho_\mu p_\epsilon - p_{\epsilon\mu})$ and choosing $(\mathbf{v}_\mu, q_\mu) = (\mathbf{e}, \eta)$ in (32), we have

$$\begin{aligned}a(\mathbf{e}, \mathbf{e}) + (C_S\delta)^2(|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon, \nabla \mathbf{e}) - (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon\mu}|\nabla \mathbf{u}_{\epsilon\mu}, \nabla \mathbf{e}) + b(\mathbf{e}, \mathbf{u}_\epsilon, \mathbf{e}) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{e}, \mathbf{e})\\ - d(\mathbf{e}, p_\epsilon - \rho_\mu p_\epsilon) + d(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_{\epsilon\mu}, \eta) + \frac{\epsilon}{\nu}(\eta, \eta) = a(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e}) + b(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{u}_\epsilon, \mathbf{e})\\ + b(\mathbf{u}_{\epsilon\mu}, \mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e}) + (C_S\delta)^2(|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon, \nabla \mathbf{e}) - (C_S\delta)^2(|\nabla \mathbf{u}_\epsilon|\nabla \mathbf{u}_\epsilon, \nabla \mathbf{e}),\end{aligned} \tag{33}$$

where we have applied the definition of the projection ρ_μ.

Besides, use Lemma 2.1 to have

$$(C_S\delta)^2(|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon, \nabla \mathbf{e}) - (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon\mu}|\nabla \mathbf{u}_{\epsilon\mu}, \nabla \mathbf{e}) \ge C(C_S\delta)^2\|\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3}^3, \tag{34}$$

as well as

$$\begin{aligned}&(C_S\delta)^2(|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon, \nabla \mathbf{e}) - (C_S\delta)^2(|\nabla \mathbf{u}_\epsilon|\nabla \mathbf{u}_\epsilon, \nabla \mathbf{e})\\ &\le |(C_S\delta)^2(|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon|\nabla (\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon), \nabla \mathbf{e})| + |(C_S\delta)^2((|\nabla \mathbf{r}_\mu \mathbf{u}_\epsilon| - |\nabla \mathbf{u}_\epsilon|)\nabla \mathbf{u}_\epsilon, \nabla \mathbf{e})|\\ &\le C\delta^2 \|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0 \|\nabla \mathbf{e}\|_0.\end{aligned} \tag{35}$$

Then, bring (34) and (35) into (33). According to (4), we obtain

$$\begin{aligned}&\nu\|\nabla \mathbf{e}\|_0^2 + C(C_S\delta)^2\|\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3}^3 + b(\mathbf{e}, \mathbf{u}_\epsilon, \mathbf{e}) + \frac{\epsilon}{\nu}\|\eta\|_0^2 - d(\mathbf{e}, p_\epsilon - \rho_\mu p_\epsilon) + d(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_{\epsilon\mu}, \eta)\\ &\le a(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e}) + b(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{u}_\epsilon, \mathbf{e}) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e}) + C\delta^2\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0\|\nabla \mathbf{e}\|_0.\end{aligned} \tag{36}$$

Due to (4), (8) and Theorem 2.1, we deduce that

$$\begin{aligned}&|b(\mathbf{e}, \mathbf{u}_\epsilon, \mathbf{e})| \le N\|\nabla \mathbf{u}_\epsilon\|_0\|\nabla \mathbf{e}\|_0^2 \le N\Psi(\|\mathbf{f}\|_{-1})\|\nabla \mathbf{e}\|_0^2,\\ &|a(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e})| \le \frac{1}{8}(\nu - N\Psi(\|\mathbf{f}\|_{-1}))\|\nabla \mathbf{e}\|_0^2 + C(\nu - N\Psi(\|\mathbf{f}\|_{-1}))^{-1}\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2,\\ &|b(\mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{u}_\epsilon, \mathbf{e}) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{r}_\mu \mathbf{u}_\epsilon - \mathbf{u}_\epsilon, \mathbf{e})| \le N(\|\nabla \mathbf{u}_\epsilon\|_0 + \|\nabla \mathbf{u}_{\epsilon\mu}\|_0)\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0\|\nabla \mathbf{e}\|_0\\ &\quad \le \frac{1}{8}(\nu - N\Psi(\|\mathbf{f}\|_{-1}))\|\nabla \mathbf{e}\|_0^2 + C(\nu - N\Psi(\|\mathbf{f}\|_{-1}))^{-1}\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2,\\ &C\delta^2\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0\|\nabla \mathbf{e}\|_0 \le \frac{1}{8}(\nu - N\Psi(\|\mathbf{f}\|_{-1}))\|\nabla \mathbf{e}\|_0^2 + C\delta^4(\nu - N\Psi(\|\mathbf{f}\|_{-1}))^{-1}\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2.\end{aligned}$$

Combining the above estimates with (36) and applying (13), we derive that

$$\begin{aligned}&(\nu - N\Psi(\|\mathbf{f}\|_{-1}))\|\nabla \mathbf{e}\|_0^2 + \frac{2\epsilon}{\nu}\|\eta\|_0^2 - 2d(\mathbf{e}, p_\epsilon - p_{\epsilon\mu}) + 2d(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_{\epsilon\mu}, \eta)\\ &\le C(\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2 + \delta^4\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2).\end{aligned} \tag{37}$$

When $i = 1$, considering (20), we obtain

$$2|d(\mathbf{e}, p_\epsilon - \rho_\mu p_\epsilon)| + 2|d(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_{\epsilon\mu}, \eta)| \le \frac{\epsilon}{\nu}\|\eta\|_0^2 + \frac{2\nu}{\epsilon}\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2,$$

which combines with (37) and (13) to get

$$\|\nabla \mathbf{e}\|_0^2 + \frac{\epsilon}{\nu}\|\eta\|_0^2 \le C(\epsilon^{-1}\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2 + \delta^4\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2).$$

Hence, based on (21) and the triangle inequality, we gain

$$\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 + \epsilon^{\frac{1}{2}}\|p_\epsilon - p_{\epsilon\mu}\|_0 \le C(\epsilon^{-\frac{1}{2}} + \delta^2)\mu,$$

and

$$\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3} \le \|\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon\|_{1,3} + C_{inv}\mu^{-\frac{1}{3}}\|\nabla(\mathbf{u}_{\epsilon\mu} - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0 \le C\mu^{-\frac{1}{3}}(\epsilon^{-\frac{1}{2}}\mu + \delta^2\mu),$$

where we have used (9).

Moreover, for $i = 2$, we use (23) to obtain

$$2|d(\mathbf{e}, p_\epsilon - \rho_\mu p_\epsilon)| + 2|d(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_{\epsilon\mu}, \eta)| \le \frac{1}{8}(\nu - N\Psi(\|f\|_{-1}))\|\nabla \mathbf{e}\|_0^2 + C(\nu - N\Psi(\|f\|_{-1}))^{-1}\|p_\epsilon - \rho_\mu p_\epsilon\|_0^2.$$

From the above inequality, (13) and (36), we have

$$\|\nabla \mathbf{e}\|_0^2 \le C(\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2 + \|p_\epsilon - \rho_\mu p_\epsilon\|_0^2 + \delta^4\|\nabla(\mathbf{u}_\epsilon - \mathbf{r}_\mu \mathbf{u}_\epsilon)\|_0^2) \le C(\mu^2 + \delta^4\mu^2), \tag{38}$$

which and the triangle inequality to further have

$$\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 \leq C(\mu + \delta^2\mu). \tag{39}$$

Next, through (9) and (39), we get

$$\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3} \leq \|\mathbf{u}_\epsilon - \mathbf{r}_\mu\mathbf{u}_\epsilon\|_{1,3} + \|\mathbf{r}_\mu\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3} \leq C\mu^{-\frac{1}{3}}(\mu + \delta^2\mu). \tag{40}$$

Moreover, taking $q_\mu = 0$ in (32), we have

$$\begin{aligned} d(\mathbf{v}_\mu, p_\epsilon - p_{\epsilon\mu}) =& a(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon, \nabla\mathbf{v}_\mu) - (C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu}, \nabla\mathbf{v}_\mu) \\ &+ b(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{u}_\epsilon, \mathbf{v}_\mu) + b(\mathbf{u}_{\epsilon\mu}, \mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}, \mathbf{v}_\mu). \end{aligned}$$

Then, by using (40) and (9), we have

$$\begin{aligned} &(C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon, \nabla\mathbf{v}_\mu) - (C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu}, \nabla\mathbf{v}_\mu) \\ \leq\ & C(C_S\delta)^2 \max\{\|\mathbf{u}_\epsilon\|_{1,3}, \|\mathbf{u}_{\epsilon\mu}\|_{1,3}\}\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_{1,3}\|\mathbf{v}_\mu\|_{1,3} \leq C(\delta\mu^{\frac{1}{3}} + \delta^3\mu^{\frac{1}{3}})\|\nabla\mathbf{v}_\mu\|_0. \end{aligned}$$

Finally, combining the above inequality with (39) and (22) yields

$$\begin{aligned} \|p_\epsilon - p_{\epsilon\mu}\|_0 &\leq \beta_0^{-1}(\nu\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 + N(\|\nabla\mathbf{u}_\epsilon\|_0 + \|\nabla\mathbf{u}_{\epsilon\mu}\|_0)\|\nabla(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu})\|_0 + C(\delta\mu^{\frac{1}{3}} + \delta^3\mu^{\frac{1}{3}})) \\ &\leq C(\mu + \delta^2\mu + \delta\mu^{\frac{1}{3}} + \delta^3\mu^{\frac{1}{3}}). \end{aligned}$$

□

Further, we list the error bounds between the solution to (10) and the finite element solution to the penalty system (24).

Theorem 3.3. *Under the assumption of Theorem 3.2, the penalized finite element solution* $(\mathbf{u}_{\epsilon\mu}, p_{\epsilon\mu})$ *has the following error estimates*

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}_{\epsilon\mu})\|_0 &\leq C(\epsilon + \epsilon\delta^2 + \epsilon\delta^4 + \epsilon^{-\frac{1}{2}}\mu + \delta^2\mu), \\ \|p - p_{\epsilon\mu}\|_0 &\leq C(\epsilon + \epsilon\delta^2 + \epsilon\delta^4 + \epsilon^{-1}\mu + \epsilon^{-\frac{1}{2}}\delta^2\mu), \quad i = 1, \\ \|\nabla(\mathbf{u} - \mathbf{u}_{\epsilon\mu})\|_0 &\leq C(\epsilon + \epsilon\delta^2 + \epsilon\delta^4 + \mu + \delta^2\mu), \\ \|p - p_{\epsilon\mu}\|_0 &\leq C(\epsilon + \epsilon\delta^2 + \epsilon\delta^4 + \mu + \delta^2\mu + \delta\mu^{\frac{1}{3}} + \delta^3\mu^{\frac{1}{3}}), \quad i = 2. \end{aligned}$$

Proof. Combining Theorem 2.2 and Theorem 3.2, we can easily obtain these results. □

Remark 3.1. *For the one-level penalty finite element method with the* $P_2 - P_0$ *element, if we assume* $\delta = O(\mu^{\frac{2}{3}})$*, then from Theorem 3.3, we have*

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\epsilon\mu})\|_0 + \|p - p_{\epsilon\mu}\|_0 \leq C(\epsilon + \mu).$$

Further, if we take $\epsilon = O(\mu)$*, then the convergence rate is* $O(\mu)$*.*

For the one-level penalty finite element method with the $P_1 - P_0$ *element, if we assume* $0 < \delta < 1$*, from Theorem 3.3, the error estimate is*

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\epsilon\mu})\|_0 \leq C(\epsilon + \epsilon^{-\frac{1}{2}}\mu), \quad \|p - p_{\epsilon\mu}\|_0 \leq C(\epsilon + \epsilon^{-1}\mu),$$

Further, if we take $\epsilon = O(\mu^{\frac{1}{2}})$*, then the convergence rate is* $O(\mu^{\frac{1}{2}})$*.*

At the last of this section, in order to show error estimate of simplified two-level penalty finite element algorithm, we next present error estimate of $\|\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon\mu}\|_0$.

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Theorem 3.4. *Under the assumptions of Theorem 3.2, we have the following error bounds:*

$$\begin{aligned}&\|\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}\|_0\le C(\epsilon^{-1}\mu^2+\epsilon^{-\frac{1}{2}}\delta\mu^{\frac{2}{3}}+\epsilon^{-\frac{1}{2}}\delta^2\mu^2+\delta^3\mu^{\frac{2}{3}}+\delta^4\mu^2),\quad i=1,\\&\|\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}\|_0\le C(\mu^2+\delta\mu^{\frac{2}{3}}+\delta^2\mu^2+\delta^3\mu^{\frac{2}{3}}+\delta^4\mu^2),\quad i=2.\end{aligned}\tag{41}$$

Proof. We will derive (41) by the standard Aubin-Nitsche duality argument.

For all $(\mathbf{v},q)\in(\mathbf{X},M)$, assume that $(\Phi,\Psi)\in(\mathbf{X},M)$ is a solution of the following dual problem [12]

$$a(\mathbf{v},\Phi)+b(\mathbf{v},\mathbf{u}_\epsilon,\Phi)+b(\mathbf{u}_\epsilon,\mathbf{v},\Phi)+d(\mathbf{v},\Psi)-d(\Phi,q)+\frac{\epsilon}{\nu}(q,\Psi)=(\mathbf{v},\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}),\tag{42}$$

and satisfies the following regularity

$$\|\Phi\|_2+\|\Psi\|_1\le C\|\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}\|_0.\tag{43}$$

According to the above regularity and the definition of projections $\mathbf{r}_\mu$ and ρ_μ, we have

$$\|\nabla(\Phi-\mathbf{r}_\mu\Phi)\|_0+\|\Psi-\rho_\mu\Psi\|_0\le C\mu\|\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}\|_0.\tag{44}$$

Set $(\mathbf{e},\eta)=(\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu},p_\epsilon-p_{\epsilon\mu})$ and take $(\mathbf{v},q)=(\mathbf{e},\eta)$ in (42).

$$\|\mathbf{e}\|_0^2=a(\mathbf{e},\Phi)+b(\mathbf{e},\mathbf{u}_\epsilon,\Phi)+b(\mathbf{u}_\epsilon,\mathbf{e},\Phi)+d(\mathbf{e},\Psi)-d(\Phi,\eta)+\frac{\epsilon}{\nu}(\eta,\Psi).\tag{45}$$

Subtracting (24) from (11) with $(\mathbf{v},q)=(\mathbf{v}_\mu,q_\mu)=(\mathbf{r}_\mu\Phi,\rho_\mu\Psi)$, we find that

$$\begin{aligned}a(\mathbf{e},\mathbf{r}_\mu\Phi)+(C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon,\nabla\mathbf{r}_\mu\Phi)-(C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu},\nabla\mathbf{r}_\mu\Phi)+b(\mathbf{e},\mathbf{u}_\epsilon,\mathbf{r}_\mu\Phi)+b(\mathbf{u}_\epsilon,\mathbf{e},\mathbf{r}_\mu\Phi)\\-b(\mathbf{e},\mathbf{e},\mathbf{r}_\mu\Phi)+d(\mathbf{e},\rho_\mu\Psi)-d(\mathbf{r}_\mu\Phi,\eta)+\frac{\epsilon}{\nu}(\eta,\rho_\mu\Psi)=0.\end{aligned}\tag{46}$$

Next, subtracting (46) from (45), we have

$$\begin{aligned}\|\mathbf{e}\|_0^2&=a(\mathbf{e},\Phi-\mathbf{r}_\mu\Phi)+((C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla\mathbf{u}_{\epsilon\mu},\nabla\mathbf{r}_\mu\Phi)-(C_S\delta)^2(|\nabla\mathbf{u}_\epsilon|\nabla\mathbf{u}_\epsilon,\nabla\mathbf{r}_\mu\Phi))+(b(\mathbf{e},\mathbf{u}_\epsilon,\Phi-\mathbf{r}_\mu\Phi)\\&\quad+b(\mathbf{u}_\epsilon,\mathbf{e},\Phi-\mathbf{r}_\mu\Phi))+(b(\mathbf{e},\mathbf{e},\Phi)-b(\mathbf{e},\mathbf{e},\Phi-\mathbf{r}_\mu\Phi))+(d(\mathbf{e},\Psi-\rho_\mu\Psi)-d(\Phi-\mathbf{r}_\mu\Phi,\eta))\\&\quad+\frac{\epsilon}{\nu}(\eta,\Psi-\rho_\mu\Psi)=:\sum_{i=1}^{6}\mathcal{I}_i.\end{aligned}\tag{47}$$

Now, we estimate each term of the right-hand side of the previous inequality. By the Cauchy-Schwarz inequality and (44), we gain

$$|\mathcal{I}_1|\le\nu\|\nabla\mathbf{e}\|_0\|\nabla(\Phi-\mathbf{r}_\mu\Phi)\|_0\le C\mu\|\nabla\mathbf{e}\|_0\|\mathbf{e}\|_0.\tag{48}$$

Then, according to (43), Lemma 2.1, Theorem 2.1 and 3.1, we deduce that

$$\begin{aligned}|\mathcal{I}_2|&\le\left|(C_S\delta)^2(|\nabla\mathbf{u}_{\epsilon\mu}|\nabla(\mathbf{u}_{\epsilon\mu}-\mathbf{u}_\epsilon),\nabla\mathbf{r}_\mu\Phi)\right|+\left|(C_S\delta)^2((|\nabla\mathbf{u}_{\epsilon\mu}|-|\nabla\mathbf{u}_\epsilon|)\nabla\mathbf{u}_\epsilon,\nabla\mathbf{r}_\mu\Phi)\right|\\&\le C\delta\|\mathbf{u}_\epsilon-\mathbf{u}_{\epsilon\mu}\|_{1,3}\|\mathbf{e}\|_0.\end{aligned}\tag{49}$$

In addition, by (4), (43), (44) and Theorem 2.1, we get

$$|\mathcal{I}_3|\le2N\|\nabla\mathbf{u}_\epsilon\|_0\|\nabla(\Phi-\mathbf{r}_\mu\Phi)\|_0\|\nabla\mathbf{e}\|_0\le C\mu\|\nabla\mathbf{e}\|_0\|\mathbf{e}\|_0,\tag{50}$$

$$|\mathcal{I}_4|\le N\|\nabla\mathbf{e}\|_0^2(\|\nabla(\Phi-\mathbf{r}_\mu\Phi)\|_0+\|\nabla\Phi\|_0)\le C\|\nabla\mathbf{e}\|_0^2\|\Phi\|_2\le C\|\nabla\mathbf{e}\|_0^2\|\mathbf{e}\|_0,\tag{51}$$

$$|\mathcal{I}_5|\le C(\|\nabla\mathbf{e}\|_0+\|\eta\|_0)(\|\nabla(\Phi-\mathbf{r}_\mu\Phi)\|_0+\|\Psi-\rho_\mu\Psi\|_0)\le C\mu(\|\nabla\mathbf{e}\|_0+\|\eta\|_0)\|\mathbf{e}\|_0,\tag{52}$$

$$|\mathcal{I}_6|\le C\mu\|\eta\|_0\|\mathbf{e}\|_0.\tag{53}$$

Finally, inserting (48)-(53) into (47), we arrive at

$$\|\mathbf{e}\|_0\le C\mu(\|\nabla\mathbf{e}\|_0+\|\eta\|_0)+C\|\nabla\mathbf{e}\|_0^2+C\delta\|\mathbf{e}\|_{1,3}.\tag{54}$$

Combining (54) and Theorem 3.2, we finish the proof of this theorem. □

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4. Simplified two-level penalty algorithm

In this section, we will consider simplified two-level penalty algorithm and its error estimates.

Let $h \ll H$, and H and h are grid scales. We will prove the error bound of $(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon^h, p_{\epsilon h} - p_\epsilon^h)$ before showing the error bound of $(\mathbf{u} - \mathbf{u}_\epsilon^h, p - p_\epsilon^h)$.

Next, we consider the simplified two-level penalty finite element algorithm.

Algorithm 4.1. *Simplified two-level penalty algorithm.*
Setp *I: Solve the penalty Smagorinsky problem on a coarse mesh. Find* $(\mathbf{u}_{\epsilon H}, p_{\epsilon H}) \in (\mathbf{X}_H, M_H)$ *such that for all* $(\mathbf{v}_H, q_H) \in (\mathbf{X}_H, M_H)$

$$a(\mathbf{u}_{\epsilon H}, \mathbf{v}_H) + (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon H}|\nabla \mathbf{u}_{\epsilon H}, \nabla \mathbf{v}_H) + b(\mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon H}, \mathbf{v}_H) - d(\mathbf{v}_H, p_{\epsilon H}) + d(\mathbf{u}_{\epsilon H}, q_H) + \frac{\epsilon}{\nu}(p_{\epsilon H}, q_H) = (f, \mathbf{v}_H). \tag{55}$$

Setp *II: Solve the penalty Stokes problem on a fine mesh. Find* $(\mathbf{u}_\epsilon^h, p_\epsilon^h) \in (\mathbf{X}_h, M_h)$ *such that for all* $(\mathbf{v}_h, q_h) \in (\mathbf{X}_h, M_h)$

$$a(\mathbf{u}_\epsilon^h, \mathbf{v}_h) + (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon H}|\nabla \mathbf{u}_{\epsilon H}, \nabla \mathbf{v}_h) + b(\mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon H}, \mathbf{v}_h) - d(\mathbf{v}_h, p_\epsilon^h) + d(\mathbf{u}_\epsilon^h, q_h) + \frac{\epsilon}{\nu}(p_\epsilon^h, q_h) = (f, \mathbf{v}_h). \tag{56}$$

For error estimate of $(\mathbf{u} - \mathbf{u}_\epsilon^h, p - p_\epsilon^h)$, we first study the convergence of $(\mathbf{u}_\epsilon^h, p_\epsilon^h)$ to $(\mathbf{u}_{\epsilon h}, p_{\epsilon h})$ in some norms. To do this, let us set $\mathbf{e}_h = \mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon^h$ and $\eta_h = p_{\epsilon h} - p_\epsilon^h$.

Theorem 4.1. *Under the assumptions of Theorem 3.2, the solution* $(\mathbf{u}_\epsilon^h, p_\epsilon^h)$ *of the problem* (55)-(56) *satisfies the following error estimates*

$$\|\nabla \mathbf{e}_h\|_0 + \epsilon^{\frac{1}{2}}\|\eta_h\|_0 \le C(\epsilon^{-1}H^2 + \epsilon^{-\frac{1}{2}}\delta H^{\frac{2}{3}}h^{-\frac{1}{3}} + \epsilon^{-\frac{1}{2}}\delta^2 H^2 + \delta^3 H^{\frac{2}{3}}h^{-\frac{1}{3}} + \delta^4 H^2), \quad i = 1,$$
$$\|\nabla \mathbf{e}_h\|_0 + \|\eta_h\|_0 \le C(H^2 + \delta h^{-\frac{1}{3}}H^{\frac{2}{3}} + \delta^2 H^2 + \delta^3 h^{-\frac{1}{3}}H^{\frac{2}{3}} + \delta^4 H^2), \quad i = 2.$$

Proof. Subtract (56) from (24) with $\mu = h$ to obtain

$$\begin{aligned} a(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon^h, \mathbf{v}_h) + (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon h}|\nabla \mathbf{u}_{\epsilon h}, \nabla \mathbf{v}_h) - (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon H}|\nabla \mathbf{u}_{\epsilon H}, \nabla \mathbf{v}_h) + b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_\epsilon, \mathbf{v}_h) \\ + b(\mathbf{u}_\epsilon, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{v}_h) + b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon, \mathbf{v}_h) + b(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{v}_h) \\ - b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{v}_h) - d(\mathbf{v}_h, p_{\epsilon h} - p_\epsilon^h) + d(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon^h, q_h) + \frac{\epsilon}{\nu}(p_{\epsilon h} - p_\epsilon^h, q_h) = 0. \end{aligned} \tag{57}$$

Next, choose $(\mathbf{v}_h, q_h) = (\mathbf{e}_h, \eta_h)$ in (57).

$$\begin{aligned} \nu\|\nabla \mathbf{e}_h\|_0^2 + \frac{\epsilon}{\nu}\|\eta_h\|_0^2 = (-b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon, \mathbf{e}_h) - b(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{e}_h)) \\ + b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{e}_h) + (-b(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{u}_\epsilon, \mathbf{e}_h) - b(\mathbf{u}_\epsilon, \mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{e}_h)) \\ + ((C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon H}|\nabla \mathbf{u}_{\epsilon H}, \nabla \mathbf{e}_h) - (C_S\delta)^2(|\nabla \mathbf{u}_{\epsilon h}|\nabla \mathbf{u}_{\epsilon h}, \nabla \mathbf{e}_h)) =: \sum_{i=1}^{4}\mathcal{S}_i. \end{aligned} \tag{58}$$

Due to (4)-(9), we get

$$|\mathcal{S}_1| \le 2N\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H})\|_0\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon)\|_0\|\nabla \mathbf{e}_h\|_0 \le \frac{\nu}{9}\|\nabla \mathbf{e}_h\|_0^2 + C\nu^{-1}\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H})\|_0^2\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_\epsilon)\|_0^2,$$
$$|\mathcal{S}_2| \le N\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H})\|_0^2\|\nabla \mathbf{e}_h\|_0 \le \frac{\nu}{9}\|\nabla \mathbf{e}_h\|_0^2 + C\nu^{-1}\|\nabla(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H})\|_0^4,$$
$$|\mathcal{S}_3| \le C(\|\nabla \mathbf{u}_\epsilon\|_{L^4}\|\mathbf{e}_h\|_{L^4} + \|\mathbf{u}_\epsilon\|_{L^\infty}\|\nabla \mathbf{e}_h\|_0)\|\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}\|_0 \le \frac{\nu}{9}\|\nabla \mathbf{e}_h\|_0^2 + C\nu^{-1}\|\mathbf{u}_\epsilon\|_2^2\|\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}\|_0^2.$$

Besides, owing to (9) and Lemma 2.1, we have

$$
\begin{aligned}
|\mathcal{S}_4| &\le C(C_S\delta)^2(\|\mathbf{u}_{\epsilon h}\|_{1,3}+\|\mathbf{u}_{\epsilon H}\|_{1,3})\|\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}\|_{1,3}\|\mathbf{e}_h\|_{1,3}\\
&\le C\delta\|\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon}\|_{1,3}\|\mathbf{e}_h\|_{1,3}+C\delta\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}\|\mathbf{e}_h\|_{1,3}\\
&\le C\delta\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}\|\mathbf{e}_h\|_{1,3}\le C\delta h^{-\frac{1}{3}}\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}\|\nabla\mathbf{e}_h\|_0\\
&\le \frac{\nu}{9}\|\nabla\mathbf{e}_h\|_0^2+C\nu^{-1}\delta^2h^{-\frac{2}{3}}\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}^2.
\end{aligned}
$$

Then, inserting above estimates on $\mathcal{S}_i$ into (58) yields

$$
\begin{aligned}
\nu\|\nabla\mathbf{e}_h\|_0^2+\frac{\epsilon}{\nu}\|\eta_h\|_0^2 &\le C(\|\nabla(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H})\|_0^4+\|\nabla(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon})\|_0^4)+C\nu^{-1}\|\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}\|_0^2\\
&\quad +C\nu^{-1}\delta^2h^{-\frac{2}{3}}\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}^2.
\end{aligned}
\tag{59}
$$

Moreover, applying Theorem 3.2 and 3.4 to (59) yields

$$
\begin{aligned}
\|\nabla\mathbf{e}_h\|_0+\epsilon^{\frac{1}{2}}\|\eta_h\|_0 &\le C(\epsilon^{-1}H^2+\epsilon^{-\frac{1}{2}}\delta H^{\frac{2}{3}}h^{-\frac{1}{3}}+\epsilon^{-\frac{1}{2}}\delta^2H^2+\delta^3H^{\frac{2}{3}}h^{-\frac{1}{3}}+\delta^4H^2),\quad i=1,\\
\|\nabla\mathbf{e}_h\|_0 &\le C(H^2+\delta h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^2H^2+\delta^3h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^4H^2),\quad i=2.
\end{aligned}
\tag{60}
$$

Besides, for $i=2$, by using the discrete inf-sup condition and (57) with $q_h=0$, we have

$$
\begin{aligned}
\|\eta_h\|_0 &\le C\nu\|\nabla\mathbf{e}_h\|_0+\|\mathbf{u}_\epsilon\|_2\|\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}\|_0+\|\nabla(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H})\|_0^2+\|\nabla(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon})\|_0\|\nabla(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H})\|_0\\
&\quad +C\delta h^{-\frac{1}{3}}\|\mathbf{u}_{\epsilon}-\mathbf{u}_{\epsilon H}\|_{1,3}.
\end{aligned}
\tag{61}
$$

Using Theorem 3.2, 3.4 and (60) for (61), we get

$$
\|\eta_h\|_0\le C(H^2+\delta h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^2H^2+\delta^3h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^4H^2).
\tag{62}
$$

The proof ends. □

Now, we list the error bounds between the solution to (10) and Algorithm 4.1.

Theorem 4.2. *Under the assumptions of Theorem 3.2, the penalized finite element solution $(\mathbf{u}_\epsilon^h, p_\epsilon^h)$ satisfies the error estimates*

$$
\begin{aligned}
\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon^h)\|_0 &\le C\epsilon(1+\delta^2+\delta^4)+C\epsilon^{-\frac{1}{2}}(h+\epsilon^{\frac{1}{2}}h\delta^2+\epsilon^{-\frac{1}{2}}H^2+\delta h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^2H^2\\
&\quad +\epsilon^{\frac{1}{2}}\delta^3H^{\frac{2}{3}}h^{-\frac{1}{3}}+\epsilon^{\frac{1}{2}}\delta^4H^2),\\
\|p-p_\epsilon^h\|_0 &\le C\epsilon(1+\delta^2+\delta^4)+C\epsilon^{-1}(h+\epsilon^{\frac{1}{2}}h\delta^2+\epsilon^{-\frac{1}{2}}H^2+\delta h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^2H^2\\
&\quad +\epsilon^{\frac{1}{2}}\delta^3H^{\frac{2}{3}}h^{-\frac{1}{3}}+\epsilon^{\frac{1}{2}}\delta^4H^2),\quad i=1,\\
\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon^h)\|_0+\|p-p_\epsilon^h\|_0 &\le C(\epsilon+\epsilon\delta^2+\epsilon\delta^4+h+\delta^2h+\delta h^{\frac{1}{3}}+\delta^3h^{\frac{1}{3}}+H^2+\delta h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^2H^2\\
&\quad +\delta^3h^{-\frac{1}{3}}H^{\frac{2}{3}}+\delta^4H^2),\quad i=2.
\end{aligned}
$$

Proof. Combining Theorem 2.2, 3.2 and 4.1, we can easily obtain the desired results. □

Remark 4.1. *For Algorithm 4.1 with the P_2-P_0 element, if we assume $\delta=O(h)$, from Theorem 4.2, then we have*

$$
\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon^h)\|_0+\|p-p_\epsilon^h\|_0\le C(\epsilon+h+H^2+h^{\frac{2}{3}}H^{\frac{2}{3}}+h^2H^2+h^{\frac{8}{3}}H^{\frac{2}{3}}+h^4H^2).
$$

Further, if we take $\epsilon=O(h)$ and $h=O(H^2)$ then the convergence rate is $O(h)$.

For Algorithm 4.1 with the P_1-P_0 element, if we assume $\delta=O(h^{\frac{11}{12}})$, from Theorem 4.2, then we have

$$
\begin{aligned}
\|\nabla(\mathbf{u}-\mathbf{u}_\epsilon^h)\|_0 &\le C\epsilon+C\epsilon^{-\frac{1}{2}}(h+\epsilon^{-\frac{1}{2}}H^2+h^{\frac{7}{12}}H^{\frac{2}{3}}+h^{\frac{11}{6}}H^2+\epsilon^{\frac{1}{2}}h^{\frac{29}{12}}H^{\frac{2}{3}}+\epsilon^{\frac{1}{2}}h^{\frac{11}{3}}H^2),\\
\|p-p_\epsilon^h\|_0 &\le C\epsilon+C\epsilon^{-1}(h+\epsilon^{-\frac{1}{2}}H^2+h^{\frac{7}{12}}H^{\frac{2}{3}}+h^{\frac{11}{6}}H^2+\epsilon^{\frac{1}{2}}h^{\frac{29}{12}}H^{\frac{2}{3}}+\epsilon^{\frac{1}{2}}h^{\frac{11}{3}}H^2).
\end{aligned}
$$

Further, if we take $\epsilon=O(h^{\frac{1}{2}})$ and $H^2=O(\epsilon^{\frac{1}{2}}h)$ then the convergence rate is $O(h^{\frac{1}{2}})$.

5. Numerical experiments

In this section, we give some numerical results to show the effectiveness of Algorithm 4.1. On one hand, the goal of the first experiment is to illustrate the performance of Algorithm 4.1 compared with the one-level penalty finite element method (24). On other hand, the second numerical example shows the advantages of using the penalty finite element method based on two finite element pairs.

From Remark 4.1 and 3.1, when using the $P_1 - P_0$ finite element pair, the penalty parameter ϵ is selected as $\epsilon = O(h^{\frac{1}{2}})$, and when using the $P_2 - P_0$ finite element pair, the penalty parameter ϵ is selected as $\epsilon = O(h)$. In addition, we choose the Smagorinsky constant $C_S = 0.17$ and iterative tolerance 1.0E-6 are used in all numerical implementations.

5.1. Experiment one

In the first experiment, the computational domain $\Omega = [0, 1]^2$. The exact solution for the velocity $\mathbf{u} = (u_1, u_2)$ and the pressure p is given as follows:

$$u_1(x, y) = 10x^2(x-1)^2 y(y-1)(2y-1), \quad u_2(x, y) = -10x(x-1)(2x-1)y^2(y-1)^2,$$
$$p(x, y) = 10(2x-1)(2y-1),$$

and the forcing term $\mathbf{f} = (f_1(x, y), f_2(x, y))$ is determined by the original problem (1). Here we consider $\nu = 1$.

In this test, we compared the simulation time of the one-level penalty finite element method (24) and Algorithm 4.1 based on two finite element pairs, $P_2 - P_0$ and $P_1 - P_0$, respectively.

When we use the $P_1 - P_0$ element, we choose $\delta = h^{\frac{2}{3}}$ for the one-level penalty finite element method, and $\delta = h^{\frac{11}{12}}$ and $h = H^{\frac{8}{5}}$ for Algorithm 4.1. Table 1 and 2 give the numerical results of the relative errors of the velocity and pressure, CPU time and convergence order of both methods at different mesh sizes. From these tables, we can see that all methods work well and keep the convergence rates just like the theoretical analysis. The comparison shows that the relative errors of the velocity and pressure of the one-level penalty method and Algorithm 4.1 are almost the same, but Algorithm 4.1 spends less time than the one-level penalty method.

Table 1 Numerical results of the one-level penalty finite element method with the $P_1 - P_0$ element.

$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	Rate	CPU time
1/9	7.93283E−1	—	4.37640E−1	—	0.172
1/28	4.31730E−1	0.54	2.58672E−1	0.46	1.078
1/53	3.06542E−1	0.54	1.92452E−1	0.46	4.721
1/84	2.42055E−1	0.51	1.53663E−1	0.49	11.081
1/121	2.01596E−1	0.50	1.27918E−1	0.50	23.069
1/162	1.74426E−1	0.50	1.10276E−1	0.51	46.202

When we use the $P_2 - P_0$ element, we choose $\delta = h^{\frac{2}{3}}$ for the one-level penalty finite element method, and $\delta = h$ and $h = H^2$ for Algorithm 4.1. Table 3 and 4 give the relative errors, CPU time and convergence order of both methods at different grid scales. From these tables, we can find that the one-level penalty method and Algorithm 4.1 work well and keep the convergence rates just like the theoretical analysis. As expected, Algorithm 4.1 costs less time than the one-level penalty method to achieve almost the same accuracy.

5.2. Experiment two

To show the benefits of the penalty finite element method, we calculated the stationary version of the vortex decay problem of Chorin (see [6]). The computational domain $\Omega = [0, 1]^2$. The exact solution for the velocity $\mathbf{u} = (u_1, u_2)$ and the pressure p is given as follows:

$$u_1(x, y) = -\cos(3\pi x)\sin(3\pi y), \quad u_2(x, y) = \sin(3\pi x)\cos(3\pi y),$$
$$p(x, y) = -1/4(\cos(6\pi x) + \cos(6\pi y)),$$

Table 2 Numerical results of Algorithm 4.1 with the $P_1 - P_0$ element.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_\epsilon^h\|_0}{\|p\|_0}$	Rate	CPU time
1/4	1/9	7.93456E−1	—	4.37571E−1	—	0.094
1/8	1/28	4.31716E−1	0.54	2.58644E−1	0.46	0.593
1/12	1/53	3.06561E−1	0.54	1.92435E−1	0.46	1.984
1/16	1/84	2.42067E−1	0.51	1.53651E−1	0.49	4.719
1/20	1/121	2.01604E−1	0.50	1.27907E−1	0.50	9.546
1/24	1/162	1.74432E−1	0.50	1.10267E−1	0.51	17.672

Table 3 Numerical results of the one-level penalty finite element method with the $P_2 - P_0$ element.

$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	Rate	CPU time
1/16	7.86844E−1	—	8.11220E−2	—	0.657
1/36	3.66781E−1	0.94	3.50431E−2	1.04	3.016
1/64	2.09537E−1	0.97	1.94441E−2	1.02	9.674
1/100	1.35032E−1	0.98	1.23560E−2	1.02	26.347
1/144	9.41160E−2	0.99	8.54601E−3	1.01	51.111
1/196	6.92971E−2	0.99	6.26311E−3	1.01	110.992

Table 4 Numerical results of Algorithm 4.1 with the $P_2 - P_0$ element.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_\epsilon^h\|_0}{\|p\|_0}$	Rate	CPU time
1/4	1/16	7.87204E−1	—	8.11011E−2	—	0.313
1/6	1/36	3.66828E−1	0.94	3.50401E−2	1.03	1.361
1/8	1/64	2.09549E−1	0.97	1.94441E−2	1.02	4.016
1/10	1/100	1.36346E−1	0.96	1.24690E−2	1.00	9.875
1/12	1/144	9.41181E−2	1.02	8.54601E−3	1.04	20.501
1/14	1/196	6.92980E−2	0.99	6.26301E−3	1.01	42.712

and the right-hand side of (1), $\mathbf{f} = (f_1(x,y), f_2(x,y))$, is determined by the original problem (1).

In Table 5, we present the relative errors of the velocity and pressure at different values of viscosity $\nu = 0.1$, 0.07 and 0.04 by eight methods. Methods 1-8 are the one-level penalty finite element method based on $P_1 - P_0$ element, the two-level penalty finite element method based on $P_1 - P_0$ element, the one-level finite element method based on $P_1 - P_0$ element, the two-level finite element method based on $P_1 - P_0$ element, the one-level penalty finite element method based on $P_2 - P_0$ element, the two-level penalty finite element method based on $P_2 - P_0$ element, the one-level finite element method based on $P_2 - P_0$ element and the two-level finite element method based on $P_2 - P_0$ element, respectively. Here, if the $P_1 - P_0$ element is used, then we take $h = 1/64$, $\delta = h^{\frac{2}{3}}$ as the one-level penalty finite element method, and $\delta = h^{\frac{11}{12}}$, $h = H^{\frac{8}{5}}$ as the two-level penalty finite element method. If the $P_2 - P_0$ element is used, we take $h = 1/64$, $\delta = h^{\frac{2}{3}}$ as the one-level penalty finite element method, and $\delta = h$, $h = H^2$ as the two-level penalty finite element method. As Experiment one, for the smaller value of viscosity, the two-level methods also spend less time than one-level method under nearly the same relative error. Besides, the method 2 (Algorithm 4.1 with the $P_1 - P_0$ element) and method 6 (Algorithm 4.1 with the $P_2 - P_0$ element) are more efficient than the another methods.

For the $P_1 - P_0$ element, we choose $\delta = h^{\frac{2}{3}}$ for the one-level penalty finite element method. Then, based on the $P_1 - P_0$ element, Table 6 and 7 show the relative error of the velocity and pressure with $\nu = 0.1$ and the convergence order of the one-level penalty finite element method and the one-level finite element

Table 5 Numerical results of the proposed method under different viscosities.

Methods	$\nu = 0.1$		$\nu = 0.07$		$\nu = 0.04$		CPU time
	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	($\nu = 0.04$)
Method 1	1.034E−1	6.240E−1	1.072E−1	6.106E−1	1.537E−1	4.447E−1	132.496
Method 2	9.082E−2	7.813E−1	9.526E−2	8.042E−1	1.330E−1	6.900E−1	4.201
Method 3	1.182E+0	5.885E+7	1.182E+0	5.885E+7	1.182E+0	5.885E+7	12.394
Method 4	1.182E+0	5.885E+7	1.182E+0	5.885E+7	1.182E+0	5.885E+7	6.642
Method 5	2.252E−2	7.975E−2	3.198E−2	8.461E−2	5.535E−2	1.010E−1	40.153
Method 6	2.197E−2	1.005E−1	3.526E−2	1.232E−1	8.011E−2	2.122E−1	8.146
Method 7	2.256E−2	8.026E−2	3.206E−2	8.545E−2	5.561E−2	1.030E−1	92.842
Method 8	2.194E−2	9.953E−2	3.520E−2	1.222E−1	7.994E−2	2.109E−1	11.629

method, respectively. From Table 6, as the theory predicts, the optimal convergence rates for the one-level penalty finite element method the new method are obtained. However, we notice that the relative errors of the velocity and pressure are not good from Table 7, which is not surprising since the lowest order pair does not satisfying the inf-sup condition.

Table 6 One-level penalty finite element method based on the $P_1 - P_0$ element.

$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_{\epsilon h}\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_{\epsilon h}\|_0}{\|p\|_0}$	Rate
1/20	2.54347E−1	—	9.65289E−1	—
1/30	1.83957E−1	0.80	8.11466E−1	0.43
1/40	1.48703E−1	0.74	7.16264E−1	0.43
1/50	1.27523E−1	0.69	6.50283E−1	0.43
1/60	1.13328E−1	0.65	6.01077E−1	0.43
1/70	1.03093E−1	0.61	5.62505E−1	0.43
1/80	9.53181E−2	0.59	5.31169E−1	0.43

Table 7 One-level finite element method based on the $P_1 - P_0$ element.

$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate
1/20	1.17854	—	1.87E+8	—
1/30	1.18090	—	1.25E+8	—
1/40	1.18174	—	9.40E+7	—
1/50	1.18212	—	7.53E+7	—
1/60	1.18234	—	6.28E+7	—
1/70	1.18246	—	5.38E+7	—
1/80	1.18254	—	4.71E+7	—

For the $P_2 - P_0$ element, we choose $\delta = h$ and $h = H^2$ with $\nu = 0.1$ for the two-level penalty finite element method and the two-level finite element method. Table 8 shows the numerical results for the two-level penalty finite element method, and Table 9 lists the numerical results for the two-level finite element method. By comparing Table 8 and 9, we find that the two-level penalty finite element method costs less computational time than two-level finite element method to get almost the same error. In fact, compared with the original stiffness matrix, for the penalty finite element method, one only needs to solve the stiffness matrix with relatively small dimension. Hence, as expected, the penalty finite element method spends less time than the finite element method under nearly the same relative error.

Table 8 Two-level penalty finite element method based on the $P_2 - P_0$ element.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p_\epsilon^h\|_0}{\|p\|_0}$	Rate	CPU time
1/3	1/9	1.34325E−1	—	8.50859E−1	—	0.125
1/5	1/25	4.75191E−2	1.02	2.80241E−1	1.09	0.587
1/7	1/49	2.75660E−2	0.81	1.30059E−1	1.14	2.047
1/9	1/81	1.86040E−2	0.78	8.52701E−2	0.83	5.220
1/11	1/121	1.24131E−2	1.01	5.47641E−2	1.12	11.253

Table 9 Two-level finite element method based on the $P_2 - P_0$ element.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_1}{\|\mathbf{u}\|_1}$	Rate	$\frac{\|p-p^h\|_0}{\|p\|_0}$	Rate	CPU time
1/3	1/9	1.34391E−1	—	8.51085E−1	—	0.125
1/5	1/25	4.75220E−2	1.02	2.78004E−1	1.10	0.672
1/7	1/49	2.75441E−2	0.81	1.28687E−1	1.14	3.095
1/9	1/81	1.85771E−2	0.78	8.51205E−2	0.82	13.161
1/11	1/121	1.23910E−2	1.01	5.42953E−2	1.12	61.581

6. Conclusion

In this paper, we establish simplified two-level penalty finite element algorithm for the Smagorinsky model, based on the inf-sup stable finite element pair $P_2 - P_0$ element and inf-sup unstable one $P_1 - P_0$ element. We prove theoretically that the relationship between coarse grid H and fine grid h is $H^2 = O(\epsilon^{\frac{1}{2}}h)$ by selecting appropriate δ and ϵ when the $P_1 - P_0$ element are used; $h = O(H^2)$ when the $P_2 - P_0$ element is selected. Besides, the numerical experiments are showed to clarify the theoretical convergence order, and the effectiveness of the penalty finite element method and simplified two-level penalty finite element algorithm.

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