HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR SEMI HARMONICALLY P-FUNCTIONS

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Abstract. In this paper, the authors introduce the concept of semi harmonically P-functions and establish some Hermite-Hadamard type integral inequalities for these classes of functions. Also, the authors compare the results obtained Hölder and Hölder-İşcan inequalities and show that the result obtained with Hölder-İşcan inequalities give better approach than the others. Some applications to special means of real numbers are also given.

1. Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [2, 3, 5, 10, 12, 13, 14, 18]).

Definition 1 ([6]). *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2.1) is reversed, then f is said to be harmonically concave.

Theorem 1 ([6]). *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

In [8], İşcan gave the Hölder-İşcan integral inequality, which gives a better approximation than the Hölder integral inequality as follows:

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Theorem 2 (Hölder-İşcan integral inequality [8]). *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned} &\int_a^b |f(x)g(x)|\,dx \\ \leq\; & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{1.3}$$

2. Main Results for semi harmonically P-functions

In recent years, many function classes such as P-function, harmonic convex functions, harmonic P-function, harmonically quasi-convex function, harmonically s-convex function in the first sense, harmonically s-convex function in the second sense and HH-convex function, etc. have been studied by many researchers, and integral inequalities belonging to these function classes have been studied in the literature (see [1, 2, 4, 7, 9, 11, 15, 16, 17, 19]).

The main purpose of this study is to introduce the concept of semi harmonically P-functions and establish some results connected with the right-hand side of new inequalities similar to the inequality (1.1) for these classes of functions. The relations of this new definition we have given with other convexity types are examined below. Examples suitable for the new definition are given. Some applications to special means of positive real numbers are also given.

Definition 2. *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be semi harmonically P-function(or semi harmonically P-convex function), if*

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq[ty+(1-t)x]\,[f(x)+f(y)] \tag{2.1}$$

for all $x,y\in I$ and $t\in[0,1]$.

We note that if the function $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is semi harmonically P-function then

$$f(x)=f\left(\frac{xx}{tx+(1-t)x}\right)\leq(tx+(1-t)x)\,[f(x)+f(x)]=2xf(x)$$

for all $x\in I$, i.e. $(2x-1)f(x)\geq 0$ for all $x\in I$. In this case, we can say that either "$x\geq\frac{1}{2}$ and $f(x)\geq 0$" or "$x\leq\frac{1}{2}$ and $f(x)\leq 0$".

Example 1. *The function $f:[1,\infty)\to\mathbb{R}$, $f(x)=x$ is a semi harmonically P-function.*

Example 2. *The function $f:(-\infty,0)\to\mathbb{R}$, $f(x)=x$ is a semi harmonically P-function.*

Example 3. *The function $f:[1,\infty)\to\mathbb{R}$, $f(x)=x^r, r\in\mathbb{R}$, is a semi harmonically P-function.*

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Example 4. *The function* $f:[1,\infty)\to\mathbb{R}$, $f(x)=\ln x$ *is a semi harmonically* P*-function.*

Example 5. *For every* $c\in\mathbb{R}\,(c\geq 0)$, *the function* $f:\left[\frac{1}{2},\infty\right)\subset\mathbb{R}\to\mathbb{R}$, $f(x)=c$ *is a semi harmonically* P*-function.*

Remark 1. *If* $f:[1,\infty)\to[0,\infty)$ *is a harmonically convex function, then the function* f *is also semi harmonically* P*-function. Since,* $t\leq tb+(1-t)a$, $1-t\leq tb+(1-t)a$ *for every* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$f\left(\frac{ab}{tb+(1-t)a}\right)\leq tf(a)+(1-t)f(b)\leq(tb+(1-t)a)\left[f(a)+f(b)\right].$$

Remark 2. *If* $f:[1,\infty)\to[0,\infty)$ *is a harmonically* P*-function, then* f *is also a semi harmonically* P*-function. Since,* $1\leq tb+(1-t)a$ *for every* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$\begin{aligned}f\left(\frac{ab}{tb+(1-t)a}\right)&\leq f(a)+f(b)\leq(tb+(1-t)a)\,f(a)+(tb+(1-t)a)\,f(b)\\&\leq(tb+(1-t)a)\left[f(a)+f(b)\right].\end{aligned}$$

Remark 3. *If* $f:[1,\infty)\to[0,\infty)$ *is a harmonically quasi convex function, then the function* f *is also a semi harmonically* P*-function. Since,* $f(a)\leq(tb+(1-t)a)\,f(a)$, $f(b)\leq(tb+(1-t)a)\,f(b)$ *for* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$\begin{aligned}f\left(\frac{ab}{tb+(1-t)a}\right)&\leq\max\left\{f(a),f(b)\right\}\leq\max\left\{(tb+(1-t)a)\,f(a),(tb+(1-t)a)\,f(b)\right\}\\&\leq(tb+(1-t)a)\left[f(a)+f(b)\right].\end{aligned}$$

Remark 4. *Let* $f:[1,\infty)\to[0,\infty)$ *be a nonnegative and* $s\in(0,1]$. *If* f *is a harmonically* s*-convex function in the first sense, then the function* f *is also a semi harmonically* P*-function. Since,* $t^{s}\leq 1\leq tb+(1-t)a$ *and* $1-t^{s}\leq 1\leq tb+(1-t)a$ *for every* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$\begin{aligned}f\left(\frac{ab}{tb+(1-t)a}\right)&\leq t^{s}f(a)+(1-t^{s})f(b)\leq(tb+(1-t)a)\,f(b)+(tb+(1-t)a)\,f(a)\\&=(tb+(1-t)a)\left[f(a)+f(b)\right].\end{aligned}$$

Remark 5. *Let* $f:[1,\infty)\to[0,\infty)$ *be a nonnegative and* $s\in(0,1]$. *If* f *is a harmonically* s*-convex function in the second sense, then* f *is also a semi harmonically* P*-function. Since,* $t^{s}\leq 1\leq tb+(1-t)a$ *and* $(1-t)^{s}\leq 1\leq tb+(1-t)a$ *for every* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$\begin{aligned}f\left(\frac{ab}{tb+(1-t)a}\right)&\leq t^{s}f(a)+(1-t)^{s}f(b)\leq(tb+(1-t)a)\,f(a)+(tb+(1-t)a)\,f(b)\\&=(tb+(1-t)a)\left[f(a)+f(b)\right].\end{aligned}$$

Remark 6. *If* $f:[1,\infty)\to(0,\infty)$ *is a HH convex function, then the function* f *is also a semi harmonically* P*-function. Since,* $t\leq tb+(1-t)a$, $1-t\leq tb+(1-t)a$ *for* $a,b\in[1,\infty)$ *and* $t\in[0,1]$, *we can write*

$$f\left(\frac{ab}{tb+(1-t)a}\right)\leq\frac{f(a)f(b)}{tf(b)+(1-t)f(a)}\leq tf(a)+(1-t)f(b)\leq(tb+(1-t)a)\left[f(a)+f(b)\right].$$

Theorem 3. *Let $f,g : I \subset \left[\frac{1}{2},\infty\right) \rightarrow \mathbb{R}$. If the functions f and g are semi harmonically P-functions, then $f+g$ is a semi harmonically P-function, for $c \in \mathbb{R}$ $(c \geq 0)$ cf is a semi harmonically P-function.*

Proof. Let f,g be semi harmonically P-functions, then

$$\begin{aligned}
& (f+g)\left(\frac{ab}{tb+(1-t)a}\right) \\
= & f\left(\frac{ab}{tb+(1-t)a}\right)+g\left(\frac{ab}{tb+(1-t)a}\right) \\
\leq & (tb+(1-t)a)\left[f(a)+f(b)\right]+(tb+(1-t)a)\left[g(a)+g(b)\right] \\
= & (tb+(1-t)a)\left[f(a)+g(a)\right]+(tb+(1-t)a)\left[f(b)+g(b)\right] \\
= & (tb+(1-t)a)\left(f+g\right)(a)+(tb+(1-t)a)\left(f+g\right)(b),
\end{aligned}$$

for every $a,b \in I$ and $t \in [0,1]$.

Let f be semi harmonically P-function and $c \in \mathbb{R}$ *$(c \geq 0)$, then*

$$\begin{aligned}
(cf)\left(\frac{ab}{tb+(1-t)a}\right) & \leq c\,(tb+(1-t)a)\left[f(a)+f(b)\right] \\
& = (tb+(1-t)a)\left[cf(a)+cf(b)\right] \\
& = (tb+(1-t)a)\left[(cf)(a)+(cf)(b)\right],
\end{aligned}$$

for every $a,b \in I$ and $t \in [0,1]$. This completes the proof of theorem. □

Theorem 4. *Let $f_\alpha : I \subset \left[\frac{1}{2},\infty\right) \rightarrow \mathbb{R}$ be an arbitrary family of semi harmonically P-functions and let $f(x)=\sup_\alpha f_\alpha(x)$. If $J=\{u \in I : f(u)<\infty\}$ is nonempty, then J is an interval and the function f is a semi harmonically P-function on interval J.*

Proof. Let $t \in [0,1]$ and $a,b \in J$ be arbitrary. Then

$$\begin{aligned}
f\left(\frac{ab}{tb+(1-t)a}\right) & = \sup_\alpha f_\alpha\left(\frac{ab}{tb+(1-t)a}\right) \\
& \leq \sup_\alpha\left[(tb+(1-t)a)\,f_\alpha(a)+(tb+(1-t)a)\,f_\alpha(b)\right] \\
& \leq (tb+(1-t)a)\sup_\alpha f_\alpha(a)+(tb+(1-t)a)\sup_\alpha f_\alpha(b) \\
& = (tb+(1-t)a)\,f(a)+(tb+(1-t)a)\,f(b) \\
& = (tb+(1-t)a)\left[f(a)+f(b)\right]<\infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a semi harmonically P-function on J. This completes the proof of theorem. □

The following result of the Hermite-Hadamard type integral inequality holds for the semi harmonically P-functions.

Theorem 5. *Let $f : I \subset \left[\frac{1}{2},\infty\right) \rightarrow \mathbb{R}$ be a semi harmonically P-function and $a,b \in I$ with $a<b$. If the function $f \in L[a,b]$, then the following inequalities hold*

$$\frac{1}{a+b}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{a+b}{2}\left[f(a)+f(b)\right]. \tag{2.2}$$

Proof. Since $f : I \to \mathbb{R}$ is a semi harmonically P-function, we get, for all $x, y \in I$ (with $t = \frac{1}{2}$ in the inequality (2.1))

$$f\left(\frac{2xy}{x+y}\right) \le \frac{a+b}{2}\left[f(y) + f(x)\right].$$

Choosing $x = \frac{ab}{ta+(1-t)b}$, $y = \frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2ab}{a+b}\right) \le \frac{a+b}{2}\left[f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)\right].$$

Further, integrating for $t \in [0,1]$, we have

$$f\left(\frac{2ab}{a+b}\right) \le \frac{a+b}{2}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt + \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right], \tag{2.3}$$

where

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt = \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx.$$

So, we obtain

$$\frac{1}{a+b} f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

The proof of the second inequality follows by using (2.1) with $x = a$ and $y = b$ and integrating with respect to t over $[0,1]$.

$$\begin{aligned} \int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt &\le \int_0^1 [tb+(1-t)a]\,[f(a)+f(b)]\,dt \\ \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &\le \frac{a+b}{2}\left[f(a)+f(b)\right]. \end{aligned}$$

This completes the proof of the theorem. □

3. Some new integral inequalities for the semi harmonically P-functions

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are semi harmonically P-functions, we need the following lemma.

Lemma 1 ([6])**.** *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt. \tag{3.1}$$

Theorem 6. *Let $f : I \subset \left[\frac{1}{2}, \infty\right) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If the function $|f'|$ is a semi harmonically P-function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab}{b-a}A(a,b)A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\left[1 - A\left(\ln a, \ln b\right)\right], \tag{3.2}$$

where $A(a,b) = \frac{a+b}{2}$ is arithmetic mean.

Proof. Using Lemma 1 and the inequality

$$\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right| \leq (tb+(1-t)a)\left[|f'(a)|+|f'(b)|\right],$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
=\ & \frac{ab\,(b-a)}{2}\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}(tb+(1-t)a)\left[|f'(a)|+|f'(b)|\right]dt \\
=\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|+|f'(b)|\right]\int_0^1\frac{|1-2t|}{(tb+(1-t)a)}dt \\
=\ & \frac{ab}{b-a}A(a,b)A\left(|f'(a)|,|f'(b)|\right)\left[1-A\left(\ln a,\ln b\right)\right]
\end{aligned}$$

where

$$\int_0^1\frac{|1-2t|}{(tb+(1-t)a)}dt=\frac{A(a,b)}{(b-a)^2}\left[1-A\left(\ln a,\ln b\right)\right]$$

and A is the arithmetic mean. This completes the proof of theorem. □

Theorem 7. *Let $f: I\subset\left[\frac{1}{2},\infty\right)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^q$ is a semi harmonically P-function on interval $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \leq \frac{2^{\frac{1}{q}-1}abA^{\frac{1}{q}}(a,b)}{(b-a)^{\frac{2}{q}-1}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right) \times\left(\left[1-A\left(\ln a,\ln b\right)\right]\right)^{\frac{1}{q}}\left(\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right)^{1-\frac{1}{q}}. \tag{3.3}$$

Proof. From Lemma 1 and using the power-mean inequality, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is semi harmonically P-function on interval $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_0^1\frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab(b-a)}{2}\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|(tb+(1-t)a)\left[|f'(a)|^q+|f'(b)|^q\right]}{(tb+(1-t)a)^2}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right)^{1-\frac{1}{q}}\left(\frac{A(a,b)}{(b-a)^2}\left[1-A(\ln a,\ln b)\right]\right)^{\frac{1}{q}} \\
=\ & \frac{2^{\frac{1}{q}-1}abA^{\frac{1}{q}}(a,b)}{(b-a)^{\frac{2}{q}-1}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)\left([1-A(\ln a,\ln b)]\right)^{\frac{1}{q}}\left(\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right)^{1-\frac{1}{q}}.
\end{aligned}$$

It is easily check that

$$\begin{aligned}
\int_0^1\frac{|1-2t|}{(tb+(1-t)a)}dt &= \frac{A(a,b)}{(b-a)^2}\left[1-A(\ln a,\ln b)\right], \\
\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}dt &= \frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right).
\end{aligned}$$

This completes the proof of the Theorem. □

Corollary 1. *Under the assumption of Theorem 7 with $q=1$, we get the conclusion of Theorem 6.*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab}{b-a}A(a,b)A\left(|f'(a)|,|f'(b)|\right)\left[1-A(\ln a,\ln b)\right].$$

Theorem 8. *Let $f:I\subset\left[\frac{1}{2},\infty\right)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^q$ is semi harmonically P-function on interval $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

(3.4)
$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}L_{1-2q}^{\frac{1-2q}{q}}(a,b),$$

where $L_p(a,b):=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $p\in\mathbb{R}\setminus\{-1,0\}$, is p-Logarithmic mean.

Proof. From Lemma 1, Hölder's integral inequality and the semi harmonically P-convexity of the function $|f'|^q$ on interval $[a,b]$,we obtain,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\frac{[tb+(1-t)a]\left[|f'(a)|^q+|f'(b)|^q\right]}{(tb+(1-t)a)^{2q}}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q-1}}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{b^{2-2q}-a^{2-2q}}{2\,(1-q)\,(b-a)}\right)^{\frac{1}{q}} \\
=\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}L_{1-2q}^{\frac{1-2q}{q}}(a,b)\,,
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
\int_0^1|1-2t|^p\,dt &= \frac{1}{p+1}, \\
\int_0^1\frac{1}{(tb+(1-t)a)^{2q-1}}dt &= \frac{b^{2-2q}-a^{2-2q}}{2\,(1-q)\,(b-a)}=L_{1-2q}^{\frac{1-2q}{q}}(a,b)\,.
\end{aligned}$$

This completes the proof of the Theorem. □

Now, we will prove the Theorem 7 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 7.

Theorem 9. *Let $f:I\subset\left[\frac{1}{2},\infty\right)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^q$ is semi harmonically P-function on interval $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

(3.5)

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\times\left[\begin{array}{c}\left(bL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)-L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)\right)^{\frac{1}{q}} \\ +\left(L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)-aL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q\leq(tb+(1-t)a)\left[|f'(a)|^q+|f'(b)|^q\right]$$

which is the semi harmonically P-function of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1-t}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{ab\,(b-a)}{2}\left(\int_0^1 t\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{t}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\left[|f'(a)|^q+|f'(b)|^q\right]\int_0^1 \frac{(1-t)}{(tb+(1-t)a)^{2q-1}}dt\right)^{\frac{1}{q}} \\
&+\frac{ab\,(b-a)}{2}\left(\int_0^1 t\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\left[|f'(a)|^q+|f'(b)|^q\right]\int_0^1 \frac{t}{(tb+(1-t)a)^{2q-1}}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(bL_{1-2q}^{\frac{1-2q}{q}}(a,b)-L_{2-2q}^{\frac{2-2q}{q}}(a,b)\right)^{\frac{1}{q}} \\
&+\frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(L_{2-2q}^{\frac{2-2q}{q}}(a,b)-aL_{1-2q}^{\frac{1-2q}{q}}(a,b)\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-t)\,|1-2t|^p\,dt &= \int_0^1 t\,|1-2t|^p\,dt=\frac{1}{2(p+1)}, \\
\int_0^1 \frac{(1-t)\,dt}{(tb+(1-t)a)^{2q-1}} &= \frac{2a^2b^{2q}\,(b-a)\,q+a^{2q}b^3+a^2b^{2q}\,(2a-3b)}{2a^{2q}b^{2q}(b-a)^2(q-1)(3q-1)}, \\
\int_0^1 \frac{tdt}{(tb+(1-t)a)^{2q-1}} &= -\frac{2b^2a^{2q}\,(b-a)\,q-a^3b^{2q}-b^2a^{2q}\,(2b-3a)}{2a^{2q}b^{2q}(b-a)^2(q-1)(3q-1)}.
\end{aligned}$$

This completes the proof of theorem. □

Remark 7. *The inequality (3.5) gives better results than the inequality (3.4). Let us show that*

$$\begin{aligned}
&\frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\times\left[\begin{array}{c}\left(bL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)-L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)\right)^{\frac{1}{q}} \\ +\left(L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)-aL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)\right)^{\frac{1}{q}}\end{array}\right] \\
\leq\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}L_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)
\end{aligned}$$

Using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$ by simple calculation we obtain

$$\begin{aligned}
&\frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\begin{array}{c}\left(bL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)-L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)\right)^{\frac{1}{q}}\\+\left(L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)-aL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)\right)^{\frac{1}{q}}\end{array}\right]\\
\leq\;& 2\frac{ab\,(b-a)}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{b-a}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\begin{array}{c}\frac{1}{2}\left(bL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)-L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)\right)\\+\frac{1}{2}\left(L_{2-2q}^{\frac{2-2q}{q}}\left(a^{2-2q},b^{2-2q}\right)-aL_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)\right)\end{array}\right]^{\frac{1}{q}}\\
=\;&\frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}L_{1-2q}^{\frac{1-2q}{q}}\left(a^{1-2q},b^{1-2q}\right)
\end{aligned}$$

which is the required.

Theorem 10. *Let $f:I\subset\left[\frac{1}{2},\infty\right)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$, $q\geq 1$ and assume that $f'\in L[a,b]$. If $|f'|^q$ is a semi harmonically P-function on the interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| &\leq \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\\
&\times\left[\left(-\frac{2(3b+a)\ln(b-a)+(3b+a)\ln(ab)}{(b-a)^3}+\frac{b^2-3a^2}{a(b-a)^3}\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\frac{2b(b+2a)}{(b-a)^3}-\frac{b(a+b)\ln(ab)}{(b-a)^3}\right)^{\frac{1}{q}}\\
&+\left(\frac{2(b+3a)\ln(b-a)-(b+3a)\ln(ab)}{(b-a)^3}+\frac{(b-a)^2-2ab}{b(b-a)^3}\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\frac{a(a+b)\ln(ab)}{(b-a)^3}-\frac{2a(2b+a)}{(b-a)^3}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{3.6}$$

Proof. Assume first that $q>1$. Using the Lemma 1, improved power-mean inequality and the property of the semi harmonically P-function of the function $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)^2}\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)^2}\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)}dt\right)^{\frac{1}{q}} \\
&+\frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\left(\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)}dt\right)^{\frac{1}{q}} \\
=\ & \frac{ab\,(b-a)}{2}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}} \\
&\times\left[\left(-\frac{2(3b+a)\ln(b-a)+(3b+a)\ln(ab)}{(b-a)^3}+\frac{b^2-3a^2}{a(b-a)^3}\right)^{1-\frac{1}{q}}\left(-\frac{b(a+b)\ln(ab)}{(b-a)^3}+\frac{2b(b+2a)}{(b-a)^3}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2(b+3a)\ln(b-a)-(b+3a)\ln(ab)}{(b-a)^3}+\frac{(b-a)^2-2ab}{b(b-a)^3}\right)^{1-\frac{1}{q}}\left(\frac{a(a+b)\ln(ab)}{(b-a)^3}-\frac{2a(2b+a)}{(b-a)^3}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)^2}dt=-\frac{2(3b+a)\ln(b-a)+(3b+a)\ln(ab)}{(b-a)^3}+\frac{b^2-3a^2}{a(b-a)^3},$$

$$\begin{aligned}
\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)^2}dt &= \frac{2(b+3a)\ln(b-a)-(b+3a)\ln(ab)}{(b-a)^3}+\frac{(b-a)^2-2ab}{b(b-a)^3}, \\
\int_0^1 \frac{(1-t)\,|1-2t|}{(tb+(1-t)a)}dt &= -\frac{b(a+b)\ln(ab)}{(b-a)^3}+\frac{2b(b+2a)}{(b-a)^3}, \\
\int_0^1 \frac{t\,|1-2t|}{(tb+(1-t)a)}dt &= \frac{a(a+b)\ln(ab)}{(b-a)^3}-\frac{2a(2b+a)}{(b-a)^3}.
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 2. *Under the assumption of Theorem 7 with $q=1$, we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab}{2}\left[|f'(a)|+|f'(b)|\right]\left[\frac{2(a+b)-(a+b)\ln(ab)}{b-a}\right]$$

4. Some applications for special means

Let us recall the following special means of two nonnegative number a, b with $b > a$:

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

(2) The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

(3) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}.$$

(4) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

(5) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

(6) The Identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $1 \leq a < b$ and $r \in \mathbb{R}\diagdown\{1,2\}$ Then we have the following inequality*

$$\frac{H^r}{G^2} \leq G^2 L_{r-2}^{r-2} \leq 2A.A(a^r, b^r)$$

Proof. The assertion follows from the inequality (2.2) in Theorem 5, for $f : [1,\infty) \to \mathbb{R}$, $f(x) = x^r$. □

Proposition 2. *Let $1 \leq a < b$. Then we have the following inequality*

$$\frac{H}{G^2} \leq G^2 L_{r-2}^{r-2} \leq 2A^2.$$

Proof. The assertion follows from the inequality (2.2) in Theorem 5, for $f : [1,\infty) \to \mathbb{R}$, $f(x) = x$. □

Proposition 3. *Let $1 \leq a < b$ and $p \in (-1,\infty)\setminus\{0\}$. Then we have the following inequality*

$$\frac{H^2}{G^2} \leq G^2 \leq 2A.A(a^2, b^2).$$

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Proof. The assertion follows from the inequality (2.2) in Theorem 5, for $f:[1,\infty)\to\mathbb{R},\ f(x)=x^2$. $\square$

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