On entire solutions of different variants of Fermat-type partial delay differential equations in several complex variables

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Abstract. In this paper, we have introduced different variants of partial delay differential operators in $\mathbb{C}^n$ which conveniently accommodate all the existing operators in the literature under one umbrella. Manipulating the operators, we extend a number of results related to the various forms of finite order transcendental entire solutions of several Fermat-type partial delay differential equations upto $\mathbb{C}^3$. Our results are the improvements of some recent results like [Mediterr. J. Math., 15 (2018), 1–14], [Anal. Math., 48 (2022), 199–226], [Electron. J. Differ. Equ., 2021(18) (2021), 1–11] and [Sib. Electron. Math. Report, 18(1) (2021), 479–494]. In addition, some examples relevant to the content of the paper have been exhibited.

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1. Introduction

It is well known that for $m \geqslant 3$ the Fermat equation $x^m + y^m = 1$ does not admit nontrivial solutions in rational numbers but it does so for $m = 2$ ([33, 34]).

The most basic Fermat-type functional equations are the circle functional equation $f^2 + g^2 = 1$, and the cubic equation $f^3 + g^3 = 1$. Obviously, generalizing the powers of these functions, a series of Fermat-type functional equations can be produced. The main theme to study these equations are to

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seek proper types of solutions of the concerned equations. Using Nevanlinna theory [11] as a tool, for the Fermat-type functional equation

$$f^m(z) + g^m(z) = 1, \tag{1.1}$$

Montel [25] and Gross [7] established some remarkable results about the existence of entire and meromorphic solutions of (1.1). In the mean time, in 1939, Iyer [13] proved that when $m = 2$, the entire solutions of equation (1.1) are $f(z) = \cos(\alpha(z))$, $g(z) = \sin(\alpha(z))$, where $\alpha(z)$ is an entire function. As the time progressed researchers perceived that Nevanlinna theory of meromorphic functions (see [11]) could be rendered as a theoretical tool to investigate the existence as well as to determine the form of transcendental entire or meromorphic solutions of different Fermat-types differential equations in the complex field. For several research questions on Fermat-type equations, we refer to the article of Gundersen [8]. For extensive research work, we refer the readers to go through the articles (see [16, 32, 38, 39]) and references therein.

In 2004, Yang and Li [39] studied the existence of meromorphic solutions to the equation

$$f^2(z) + \Big(a_n f^{(n)}(z) + a_{n+1} f^{(n+1)}(z)\Big)^2 = 1, \tag{1.2}$$

a_n and a_{n+1} are non-zero constants, and proved that (1.2) has no transcendental meromorphic solutions.

The establishment of the difference analogues of Nevanlinna theories in 2006, specially the difference analogue lemma of the logarithmic derivative lemma (see [5, 9, 10]) expedite the research activities regarding the existence and the form of the entire or meromorphic solutions of Fermat-type difference and differential-difference equations (see [21–23]).

In 2012, Liu *et al.* [22] first inspected finite order transcendental entire solutions of the Fermat-type shift equation $f(z)^2 + f(z+c)^2 = 1$ and proved that the form of the solution is $f(z) = \sin(Az + B)$, where $A = (4k+1)\pi/2c$, $B \in \mathbb{C}$, $k \in \mathbb{Z}$. As far as the development of finding the solutions of Fermat type shift equations are concerned, this result can be considered as a path breaking achievement. After that, citing this article, a plethora of research works were appeared in the literature.

In 2021, Banerjee and Biswas [1] generalized the result as follows:

Theorem A. [1] The non-linear c-shift equation $f(z)^2 + L_c^2(z, f) = 1$, where $L_c(z, f) = \sum_{j=0}^{n} a_j f(z + jc)$ has finite order transcendental entire solution of the form

$$f(z) = \frac{e^{az+b} + e^{-az-b}}{2}$$

satisfying the conditions $a_0 + \sum_{j=1}^{n} a_j e^{jc} = -i$ and $a_0 + \sum_{j=1}^{n} a_j e^{-jc} = i$. Here $e^{ac} \neq \pm 1$. Also if $n = 1$, $a_0 \neq \pm a_1$ is required.

The advent of partial differential equations in the realm of Fermat types equations further enriched the field (see [6,27]). Before we discuss in detail, we recall here the definition of it.

Definition 1.1. [24,40] An equation is called a differential-difference equation, if it includes derivatives, shifts, or differences of $f(z)$, which can be called DDE, in short. An equation is called a complex partial differential-difference equation, if it includes partial derivatives, shifts, or differences of $f(z)$, which can be called PDDE, in short.

Note 1.1. For the sake of convenience, we will write partial shift differential equation and partial difference differential equation according as the presence of shift or difference operator in the concern equations.

As far as the authors knowledge is concerned, in 1999, Saleeby, first started an investigation into the existence and form of entire and meromorphic solutions of Fermat-type partial differential equations (see [29,30]). Most noticeably, in 1995, Khavinson [14], proved that any entire solution of the partial differential equation

$$\left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 = 1$$

must be linear, i.e., $u(z_1, z_2) = az_1 + bz_2 + c$, where $a, b, c \in \mathbb{C}$, and $a^2 + b^2 = 1$. Inspired by this result, Li [18–20] investigated on the partial differential equations with more general forms such as $u_{z_1}^2 + u_{z_2}^2 = p$, $u_{z_1}^2 + u_{z_2}^2 = e^q$, etc, where p, q are polynomials in $\mathbb{C}^2$. With the help of the difference Nevanlinna theory for several complex variables (see [2,3,15]), Xu and Cao [35], Xu and his co-authors [37,40] obtained some interesting results on the characterizations of entire and meromorphic solutions for some Fermat-type difference equations and systems of difference equations which are the extensions from one complex variable to several several complex variables.

From now on, we denote by $z + w = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$ for any $z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$. By jc we mean $(jc_1, jc_2, \ldots, jc_n)$ for any $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$ and $j \in \mathbb{N}$. The shift of $f(z)$ is defined by $f(z + c)$, whereas the difference of $f(z)$ is defined by $\Delta_c f(z) = f(z + c) - f(z)$ (see [15]).

Xu and Cao [35] extended Theorem 1.1 of [22] to the case of several complex variables as follows.

Theorem B. [35] Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}$. Then any non-constant entire solution $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ with finite order of the Fermat-type shift equation $f(z)^2 + f(z + c)^2 = 1$ has the form of $f(z) = \cos(L(z) + B)$, where L is a linear function of the form $L(z) = a_1 z_1 + \cdots + a_n z_n$ on $\mathbb{C}^n$ such that $L(c) = -\pi/2 - 2k\pi$ $(k \in \mathbb{Z})$, and B is a constant on $\mathbb{C}$.

However, Zheng and Xu [40] proved that the difference analogue of *Theorem B* cease to hold in $\mathbb{C}^2$. In fact, Zheng and Xu's [40] result is as follows:

Theorem C. [40] Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Then the equation

$$f(z)^2 + [f(z+c) - f(z)]^2 = 1$$

has no transcendental entire solutions with finite order.

Let us now define three operators L_i, $i = 1, 2, 3$ as follows:

(i) $L_1(f) := \sum_{j=1}^{k} a_j f(z + c_j)$,

(ii) $L_2(f) := \sum_{m=1}^{s} b_m \frac{\partial^m f(z + d_m)}{\partial z_1^m}$ and

(iii) $L_3(f) = \sum_{l=1}^{t} e_l \frac{\partial^l f(z)}{\partial z_1^l}$, where z, c_j, d_m are all constants in $\mathbb{C}^n$; a_j, b_m, e_l are constants in $\mathbb{C}$, $j = 1, 2, \ldots, k$, $m = 1, 2, \ldots, s$, $l = 1, 2, \ldots, t$.

We call L_1, L_2 and L_3 as linear shift operator, linear shift partial differential operator and linear partial differential operator, respectively. We also call $\tilde{L} := L_1 + L_2 + L_3$ as partial delay differential operator and $\tilde{L_p} := L_1 + L_2$ as proper partial delay differential operator.

In particular, when $c_j = jc$ and $d_m = md$; $c, d \in \mathbb{C}^n$, $j = 1, 2, \ldots, k$, $m = 1, 2, \ldots, s$, then L_1 and L_2 are called the reduced linear shift and reduced linear shift partial differential operators, and they are denoted by $L_1^r(f)$ and $L_2^r(f)$, respectively. We also call $\tilde{L_p}^r := L_1^r + L_2^r$ as proper reduced partial delay differential operator.

In view of the above definitions, it will be interesting to further investigate Theorems A–C in $\mathbb{C}^3$ under the aegis of the following Fermat-type equation:

$$f(z)^2 + [a_0 f(z) + \tilde{L_p}^r(f)]^2 = 1. \tag{1.3}$$

So, we have our first result as follows:

Theorem 1.2. *Let $c = (c_1, c_2, c_3), d = (d_1, d_2, c_3)$ be two non-zero constants in $\mathbb{C}^3$; a_j, b_m are constants in $\mathbb{C}$ with at least one of a_j or b_m are non-zero, $j = 1, 2 \ldots, k$, $m = 1, 2 \ldots, s$, where n, k, s be positive integers. Then any finite order transcendental entire solutions of* (1.3) *must be one of the following three types:*

(i) *If $L_1^r(f) \not\equiv 0$ and $L_2^r(f) \equiv 0$, then $f(z) = -i \sinh(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi)$, where $L(z) = \sum_{\mu=1}^{3} \alpha_\mu z_\mu$, $H_1(s_1)$ is a polynomial in $s_1 := d_{11}z_1 + d_{12}z_2$, $H_2(s_2)$ is a polynomial in $s_2 := d_{22}z_2 + d_{23}z_3$, $H_3(s_3)$ is a polynomial in $s_3 := d_{31}z_1 + d_{33}z_3$ and $H_4(s_4)$ is a polynomial in $s_4 := d_{41}z_1 + d_{42}z_2 + d_{43}z_3$ with $d_{11}c_1 + d_{12}c_2 = 0$, $d_{22}c_2 + d_{23}c_3 = 0$, $d_{31}c_1 + d_{33}c_3 = 0$ and $d_{41}c_1 + d_{42}c_2 + d_{43}c_3 = 0$, ξ, α_μ, d_{ij} are all constants in $\mathbb{C}$, and $L(z)$ satisfy relations*

$$\begin{cases} a_0 + \sum_{j=1}^{k} a_j e^{jL(c)} = i; \\ a_0 + \sum_{j=1}^{k} a_j e^{-jL(c)} = -i. \end{cases} \tag{1.4}$$

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(ii) *If $L_1^r(f) \equiv 0$ and $L_2^r(f) \not\equiv 0$, then $f(z) = -i\sinh(L(z)+\sum_{j=1}^4 H_j(s_j)+\xi)$, where $L(z)$ and $H_j(s_j)$ are defined as in (i), $j = 1,2,3,4$, and $L(z)$ satisfies the relation*

$$\begin{cases} a_0 + \sum_{m=1}^{s} b_m \alpha^m e^{mL(d)} = i; \\ a_0 + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(d)} = -i, \end{cases}$$

where α can be found from the relation

$$\alpha_1 + d_{11}H_1'(s_1) + d_{31}H_3'(s_3) + d_{41}H_4'(s_4) = \alpha. \tag{1.5}$$

In particular, if $d_{11} \neq 0$, then $H_1(s_1)$ is linear in s_1. If $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

(iii) *If $L_1^r(f) \not\equiv 0$, $L_2^r(f) \not\equiv 0$ with $c = d$, then $f(z) = -i\sinh(L(z) + \sum_{j=1}^4 H_j(s_j) + \xi)$, where $L(z)$ and $H_j(s_j)$ are defined as in (i), $j = 1,2,3,4$, and $L(z)$ satisfy relations*

$$\begin{cases} a_0 + \sum_{j=1}^{k} a_j e^{-jL(c)} + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(c)} = -i, \\ a_0 + \sum_{j=1}^{k} a_j e^{jL(c)} + \sum_{m=1}^{s} b_m \alpha^m e^{mL(c)} = i, \end{cases}$$

where α satisfies the relation (1.5).

In particular, if $d_{11} \neq 0$, then $H_1(s_1)$ is linear in s_1. If $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

Remark 1.3. Following the same procedure adopted to prove Theorem 1.2 (i), one can easily verify that in $\mathbb{C}^2$, the form of transcendental entire solutions of

$$f(z)^2 + (a_0 f(z) + L_1^r(f))^2 = 1$$

will be $f(z) = -i\sinh(L(z) + H(s_1) + \xi)$, where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $H(s_1)$ is a polynomial in $s_1 := d_1 z_1 + d_2 z_2$ with $d_1 c_1 + d_2 c_2 = 0$, and $L(z)$ satisfies the relation (1.4) in $\mathbb{C}^2$, which is clearly the generalization of Theorems B and C.

Remark 1.4. In view of (i) of Theorem 1.2, we can easily see that Theorem 1.2 is an improvement of Theorem A to a large extent.

Remark 1.5. Let $a_0 = -1$, $L_1^r(f) = f(z+c)$ and $L_2^r(f) \equiv 0$. If f be a finite order transcendental entire solution of (1.3), then in view of the conclusion (i) of Theorem 1.2, we must have $e^{L(c)} = 1+i$ and $e^{-L(c)} = 1-i$, which is not possible. Therefore, in this case (1.3) has no solution. Hence from Theorem 1.2, we can easily deduce Theorem C.

Remark 1.6. Though in the statements of Theorems A and B, no restrictions other than finite were imposed on the order of the solutions, we see that in the conclusion of both the theorems, the concerned solutions are of order 1. But, in Theorem 1.2, we see that the form of the solutions may be of any integer order. In particular, if $H_j(s_j)$ be constant for all $j = 1, 2, 3, 4$, then we obtain exactly the form of the solutions as exhibited in Theorems A and B.

Next, we exhibit some examples in support of Theorem 1.2.

Example 1.7. Let $L(z) = 2z_1 - 3z_2$, $c = \left(\frac{3\pi i}{20}, -\frac{3\pi i}{20}\right) \in \mathbb{C}^2$ and n be any positive integer. Then, in view of (i) of Theorem 1.2, it can be shown that $f(z) = -i \sinh(2z_1 + 3z_2 + (z_1 + z_2)^n + 1)$ is a solution of

$$f(z)^2 + [f(z) + \sqrt{2} f(z + c)]^2 = 1.$$

Example 1.8. Let $a_0 = \alpha_1 = 1$, $b_2 = \sqrt{2}$. Let $c = (c_1, 0) \in \mathbb{C}^2 \setminus \{0\}$ such that $L(c) = \left(2m\pi + \frac{3\pi}{4}\right) i$. Then, in view of conclusion (ii) of Theorem 1.2, we can easily see that $f(z) = -i \sinh(z_1 + 2z_2 + z_2^n + 1)$ is a solution of (1.3), where n is any positive integer.

Example 1.9. Let $a_0 = b_1 = b_3 = \alpha_1 = 1$, $b_2 = 2$. Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$ such that $L(c) = \left(2m\pi + \frac{3\pi}{4}\right) i$. Then, in view of (ii) of Theorem 1.2, we can easily see that $f(z) = -i \sinh(z_1 + 2z_2 + 1)$ is a solution of

$$f(z)^2 + \left[f(z) + \frac{\partial f(z + c)}{\partial z_1} + 2\frac{\partial^2 f(z + c)}{\partial z_1^2} + \frac{\partial^3 f(z + c)}{\partial z_1^3}\right]^2 = 1.$$

From Theorem 1.2, we easily obtain the following Corollaries.

Corollary 1.10. *Let $a_1 \neq 0$, $b_1 \neq 0$ and $\alpha_\lambda \neq \pm a_1/b_1$. Then the finite order transcendental entire solutions of the equation*

$$f(z)^2 + \left[a_1 f(z + c) + b_1 \frac{\partial f(z + c)}{\partial z_1}\right]^2 = 1$$

must be of the form $f(z) = -i \sinh\left(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi\right)$, where $L(z)$ and $H_j(s_j)$ are defined as in (iii) of Theorem 1.2 such that

$$e^{L(c)} = \frac{i}{a_1 + b_1\alpha} \quad \text{and} \quad a_1^2 = 1 + b_1^2\alpha^2,$$

and α satisfies the relation (1.5).

Corollary 1.11. *Let a_1, b_1 be two non-zero constants in $\mathbb{C}$. Then any finite order transcendental entire solutions of the equation*

$$f(z)^2 + \left[a_1 \Delta_c f(z) + b_1 \frac{\partial f(z + c)}{\partial z_1}\right]^2 = 1$$

must be of the form $f(z) = -i\sinh\left(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi\right)$, *where* $L(z)$ *and* $H_j(s_j)$ *are defined as in (iii) of Theorem* 1.2 *such that*

$$e^{L(c)} = \frac{a_1 + i}{a_1 + b_1\alpha} \quad \textit{and} \quad b_1^2\alpha^2 + 1 = 0,$$

and α *satisfies the relation* (1.5).

Corollary 1.12. *Let* b_1, b_2 *and* α_1 *be non-zero constants in* $\mathbb{C}$ *such that* $\alpha_1 \neq \pm b_1/b_2$. *Then, the finite order transcendental entire solutions of the equation*

$$f(z)^2 + \left[b_1\frac{\partial f(z+c)}{\partial z_1} + b_2\frac{\partial^2 f(z+c)}{\partial z_1^2}\right]^2 = 1$$

must be of the form $f(z) = -i\sinh\left(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi\right)$, *where* $L(z)$ *and* $H_j(s_j)$ *are defined as in (ii) of Theorem* 1.2 *such that*

$$e^{L(c)} = \frac{i}{(b_1 + b_2\alpha)\alpha} \quad \textit{and} \quad (b_2^2\alpha^2 - b_1^2)\alpha^2 = 1,$$

and α *satisfies the relation* (1.5).

In 2020, Xu and Cao [36] obtained the following result.

Theorem D. [36] Any transcendental entire solution with finite order of the of the Fermat-type partial differential equation

$$f^2(z_1, z_2) + \left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 = 1 \tag{1.6}$$

has the form of $f(z_1, z_2) = \sin(z_1 + g(z_2))$, where $g(z_2)$ is a polynomial in one variable z_2.

In connection to *Theorem D*, let us now demonstrate the following result of Chen and Xu [4].

Theorem E. [4] Let b_1 and b_2 be two nonzero constants in $\mathbb{C}$. Then

$$f^2(z_1, z_2) + \left[b_1\frac{\partial f}{\partial z_1} + b_2\frac{\partial^2 f}{\partial z_1^2}\right]^2 = 1 \tag{1.7}$$

has no transcendental entire solutions with finite order in $\mathbb{C}^2$.

In this paper, we are interested to investigate the existence and forms of finite order transcendental entire solutions of more general equation than (1.6) and (1.7). For this we consider the following partial differential-difference equation

$$f(z)^2 + L_3(f)^2 = 1, \tag{1.8}$$

and obtained the result as follows.

Theorem 1.13. *The finite order transcendental entire solutions of* (1.8) *must be of the form*

$$f(z)=-i\sinh(\alpha_\lambda z_\lambda+\Phi(z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n)),$$

where α_λ *be a non-zero constant in* $\mathbb{C}$ *and* $\Phi(z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n)$ *is a polynomial in* $z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n$ *such that*

$$\sum_{l=1}^{t}e_l\alpha_\lambda^l=i \ \text{ and } \ \sum_{l=1}^{t}(-1)^l e_l\alpha_\lambda^l=-i.$$

Remark 1.14. From Theorem 1.13, it can be easily obtain Theorems D and E by taking $L_3(f)=e_1\frac{\partial f}{\partial z_1}+e_2\frac{\partial^2 f}{\partial z_1^2}$, e_1,e_2 are non-zero constants in $\mathbb{C}$. Therefore, Theorem 1.13 is more general than Theorems D and E.

Example 1.15. In view of Theorem 1.13, it is easily seen that

$$f(z)=-i\sinh(z_1+z_2^{10}+z_2^5+z_2+1)$$

is a solution of

$$f(z)^2+\left[\frac{\partial f}{\partial z_1}+\frac{\partial^2 f}{\partial z_1^2}+(i-1)\frac{\partial^3 f}{\partial z_1^3}-\frac{\partial^4 f}{\partial z_1^4}\right]^2=1.$$

Next, we would like to mention a result due to Xu and Cao [35], in which the authors have investigated the solutions of the following Fermat-type partial differential-difference equation

$$f^2(z_1+c_1,z_2+c_2)+\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2=1. \tag{1.9}$$

Xu and Cao [35] proved the following theorem.

Theorem F. [35] Let $c=(c_1,c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solution with finite order of the Fermat-type partial differential-difference equation (2.14) has the form $f(z_1,z_2)=\sin(Az_1+Bz_2+H(z_2))$, Where A,B are constant on $\mathbb{C}$ satisfying $A^2=1$ and $Ae^{i(Ac_1+Bc_2)}=1$, and $H(z_2)$ is a polynomial in one variable z_2 such that $H(z_2)\equiv H(z_2+c_2)$. In the special case whenever $c_2\neq 0$, we have $f(z_1,z_2)=\sin(Az_1+Bz_2+Constant)$.

In this paper, for further extension and generalization of Theorem F, we consider the following partial differential-difference equation

$$f(z+c)^2+(L_2^r(f)+L_3(f))^2=1 \tag{1.10}$$

in $\mathbb{C}^3$ which is more general setting than (1.9), and obtained the result as follows.

Theorem 1.16. *Let* $c=(c_1,c_2,c_3)\in\mathbb{C}^3\setminus\{0\}$ *and* $d=mc$, $m\in\mathbb{N}$. *Then, any finite order transcendental entire solutions of* (1.10) *must be of the form*

$$f(z)=-i\sinh\left(L(z)-L(c)+\sum_{j=1}^{4}H_j(s_j)+\xi\right),$$

where $L(z)$ and $H_j(s_j)$ are defined as in (i), $j = 1, 2, 3, 4$, *and $L(z)$ satisfy the relations*

$$\begin{cases} \displaystyle\sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(c)+L(c)} + \sum_{l=1}^{t} (-1)^l e_l \alpha^l e^{L(c)} = -i, \\ \displaystyle\sum_{m=1}^{s} b_m \alpha^m e^{mL(c)-L(c)} + \sum_{l=1}^{t} e_l \alpha^l e^{-L(c)} = i, \end{cases}$$

where α satisfies the relation (1.5).

In particular, if $d_{11} \neq 0$, then $H_1(s_1)$ is linear in s_1. If $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

Now, we exhibit some examples in order to show the existence of solutions in Theorem 1.16.

Example 1.17. Let $\alpha_1 = \beta_1 = 1, \beta_2 = -1$ and $c = (\pi i, \pi i) \in \mathbb{C}^2$. Then in view of Theorem 1.16, it is clear that $f(z) = -i \sinh(z_1 + z_2 + 3 - 2\pi i)$ is a solution of

$$f(z+c)^2 + \left[\frac{\partial \Delta_c f(z)}{\partial z_1} - i\frac{\partial f(z+c)}{\partial z_1}\right]^2 = 1.$$

Example 1.18. Let $\alpha_1 = 1$ and $c = \left(\frac{1}{3}\log(\sqrt{2}+1), \frac{1}{3}\log(\sqrt{2}+1)\right) \in \mathbb{C}^2$. Then we can easily verified that $f(z) = -i\sinh(z_1 + z_2 - \log(\sqrt{2}+1) + 1)$ is a solution of

$$f(z+c)^2 + \left[\frac{\partial f}{\partial z_1} + i\frac{\partial^2 f}{\partial z_1^2} + (\sqrt{2}i - 1)\frac{\partial^3 f}{\partial_1^3}\right]^2 = 1.$$

Example 1.19. Let $\alpha_1 = e_1 = 1$, $e_3 = -1$, $e_5 = i$ and $c = (2\pi, 0) \in \mathbb{C}^2$. Then for any positive integer n, $f(z) = -i\sinh(z_1 + z_2 + z_2^n + z_2^{n-1} + 1)$ is a solution of

$$f(z+c)^2 + \left[\frac{\partial f}{\partial z_1} - \frac{\partial^3 f}{\partial z_1^3} + i\frac{\partial^5 f}{\partial_1^5}\right]^2 = 1.$$

From the above Theorem 1.16, we can easily deduce the following Corollary.

Corollary 1.20. *Let $L_2(f) \equiv 0$ and $L_3(f) = b_1\frac{\partial f}{\partial z_1} + b_2\frac{\partial^2 f}{\partial z_1^2}$, where b_1, b_2 are non-zero constants in $\mathbb{C}$. Then any transcendental entire solutions of* (1.10) *has the form*

$$f(z) = -i\sinh\left(L(z) - L(c) + \sum_{j=1}^{4} H_j(s_j) + \xi\right),$$

where $L(z)$ and $H_j(s_1)$ are defined as in Theorem 1.16 *and and $L(z)$ satisfies the relation*

$$e^{L(c)} = -i(b_1 + b_2\alpha)\alpha, \;\; \text{and} \;\; (b_2^2\alpha^2 - b_1^2)\alpha^2 = 1,$$

and α satisfies the relation (1.5).

Remark 1.21. In Corollary 1.20, the surprising fact one can observe is that if $f(z)$ in equation (1.7) is replaced by its shift $f(z+c)$, then solution must exists, and the precise form of the solutions can be obtained.

2. Proofs of the main results

Before we starting the proof of the main results, we present here some necessary lemmas which will play key role to prove the main results of this paper.

Lemma 2.1. *[17, 28, 31] For an entire function F on $\mathbb{C}^n$, $F(0) \not\equiv 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Lemma 2.2. *[26] If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(i) *the internal function h is a polynomial and the external function g is of finite order; or else*
(ii) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 2.3. *[12] Suppose that $a_0(z), a_1(z), \ldots, a_n(z)$ $(n \geqslant 1)$ are meromorphic functions on $\mathbb{C}^m$ and $g_0(z), g_1(z), \ldots, g_n(z)$ are entire functions on $\mathbb{C}^m$ such that $g_j(z) - g_k(z)$ are not constants for $0 \leqslant j < k \leqslant n$. If $\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$ and $\|\ T(r, a_j) = o(T(r))$, $j = 0, 1, \ldots, n$, where $T(r) = min_{0\leqslant j<k\leqslant n} T\left(r, e^{g_j(z)-g_k(z)}\right)$, then $a_j(z) \equiv 0$ $(j = 0, 1, \ldots, n)$.*

Proof of Theorem 1.2. First we rewrite (1.3) as the following.

$$(a_0 f(z) + \tilde{L}_p^r(f) + if(z))(a_0 f(z) + \tilde{L}_p^r(f) - if(z)) = 1.$$

Since $f(z)$ is finite order transcendental entire, in view of Lemma 2.2, we get a non-constant polynomial $p(z)$ in $\mathbb{C}^n$ such that

$$a_0 f(z) + \tilde{L}_p^r(f) + if(z) = e^{p(z)}, \;\; a_0 f(z) + \tilde{L}_p^r(f) - if(z) = e^{-p(z)},$$

which yield that

$$a_0 f(z) + \tilde{L}_p^r(f) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \;\; f(z) = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{2.1}$$

Let us consider three possible cases in the following.

Case 1. Let $L_1^r(f) \not\equiv 0$ and $L_2^r(f) \equiv 0$. Then (2.1) leads to

$$\begin{aligned}&\sum_{j=1}^{k} a_j e^{p(z+jc)+p(z)} - \sum_{j=1}^{k} a_j e^{p(z)-p(z+jc)} + (a_0 - i)e^{2p(z)}\\ &= a_0 + i.\end{aligned} \tag{2.2}$$

Now, we consider the following two subcases.

Subcase 1.1. Let $p(z+d)-p(z)=\eta_1$, a constant in $\mathbb{C}$. As $p(z)$ is a polynomial, this leads to

$$p(z)=L(z)+H_1(s_1)+H_2(s_2)+H_3(s_3)+H_4(s_4)+\xi, \tag{2.3}$$

where $L(z)=\sum_{\mu=1}^{3}\alpha_\mu z_\mu$, $H_1(s_1)$ is a polynomial in $s_1:=d_{11}z_1+d_{12}z_2$, $H_2(s_2)$ is a polynomial in $s_2:=d_{22}z_2+d_{23}z_3$, $H_3(s_3)$ is a polynomial in $s_3:=d_{31}z_1+d_{33}z_3$ and $H_4(s_4)$ is a polynomial in $s_4:=d_{41}z_1+d_{42}z_2+d_{43}z_3$ with $d_{11}c_1+d_{12}c_2=0$, $d_{22}c_2+d_{23}c_3=0$, $d_{31}c_1+d_{33}c_3=0$ and $d_{41}c_1+d_{42}c_2+d_{43}c_3=0$, α_μ, d_{ij} are all constants in $\mathbb{C}$. Thus, $p(z+jd)-p(z)=L(jd)$ for all $j\in\mathbb{N}$.

Hence, (2.2) reduces to

$$\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}+(a_0-i)e^{2p(z)}=a_0+i+\sum_{j=1}^{k}a_je^{-jL(c)}. \tag{2.4}$$

We claim that

$$a_0+i+\sum_{j=1}^{k}a_je^{-jL(c)}=0. \tag{2.5}$$

Otherwise, from (2.4), we get

$$\left[a_0+i+\sum_{j=1}^{k}a_je^{-jL(c)}\right]e^{-2p(z)}=\sum_{j=1}^{k}a_je^{jL(c)}+(a_0-i).$$

This implies that $p(z)$ must be constant, which is a contradiction. Hence, our claim is established.

Therefore, in view of (2.4) and (2.5), we obtain that

$$a_0-i+\sum_{j=1}^{k}a_je^{jL(c)}=0. \tag{2.6}$$

Thus, in view of (2.1), we obtain

$$f(z)=-i\sinh(L(z)+H_1(s_1)+H_2(s_2)+H_3(s_3)+H_4(s_4)+\xi),$$

where $L(z)$ satisfies the relation (2.5) and (2.6).

Subcase 1.2. Let $p(z+c)-p(z)$ be non-constant. Then we can easily verify that $p(z+jc)-p(z)$ are all non-constants, $j=1,2,\ldots,k$.

Now, in view of Lemma 2.3, we must conclude from (2.2) that $a_0=\pm i$ and $a_j=0$ for all $j=1,2,\ldots,k$. But this is a contradiction to the assumption that at-least one $a_j\neq 0$.

Case 2. Let $L_1^r(f)\equiv 0$ and $L_2^r(f)\not\equiv 0$.

Differentiating second equation of (2.1) j times partially with respect to z_λ, we obtain

$$\frac{\partial^j f(z)}{\partial z_1^j}=\frac{h_1(z)e^{p(z)}-h_2(z)e^{-p(z)}}{2i}, \tag{2.7}$$

where

$$\begin{cases} h_1(z) = \left(\frac{\partial p}{\partial z_1}\right)^j + H_{1j}\left(\frac{\partial^j p}{\partial z_1^j}, \ldots, \frac{\partial p}{\partial z_1}\right), \\ h_2(z) = (-1)^j\left(\frac{\partial p}{\partial z_1}\right)^j + H_{2j}\left(\frac{\partial^j p}{\partial z_1^j}, \ldots, \frac{\partial p}{\partial z_1}\right), \end{cases} \tag{2.8}$$

H_{1j} and H_{2j} are partial differential polynomials of $p(z)$ of degree less than j, $j = 1, 2, \ldots, s$.

Then, it follows from (2.1) that

$$\sum_{m=1}^{s} b_m h_1(z+md)e^{p(z+md)+p(z)} - \sum_{m=1}^{s} b_m h_2(z+md)e^{p(z)-p(z+md)}$$
$$+(a_0 - i)e^{2p(z)} = a_0 + i. \tag{2.9}$$

Now, we consider the following two possible subcases.

Subcase 2.1. Let $p(z+d) - p(z) = \eta_1$, a constant in $\mathbb{C}$. As $p(z)$ is a polynomial, the form of the polynomial $p(z)$ is the same as in (2.3). Thus, $p(z+jd) - p(z) = L(jd)$ for all $j \in \mathbb{N}$.

Therefore, (2.9) yields that

$$\sum_{m=1}^{s} b_m h_1(z+md)e^{p(z+md)+p(z)} + (a_0 - i)e^{2p(z)}$$
$$= a_0 + i + \sum_{m=1}^{s} b_m h_1(z+md)e^{-mL(d)}. \tag{2.10}$$

In view of (2.10) and by similar arguments as in Subcase 1.1, we can conclude that

$$a_0 + i + \sum_{m=1}^{s} b_m h_1(z+md)e^{-mL(d)} \equiv 0. \tag{2.11}$$

Therefore, it follows from (2.10) and (2.11) that

$$a_0 - i + \sum_{m=1}^{s} b_m h_2(z+md)e^{mL(d)} \equiv 0. \tag{2.12}$$

Now, in view of (2.8) and (2.11), we must conclude that $\dfrac{\partial p}{\partial z_1}$ must be constant, say $\alpha \in \mathbb{C}$.

Therefore, in view of (2.3), it follows that

$$\frac{\partial p}{\partial z_1} = \alpha_1 + d_{11}H_1'(s_1) + d_{31}H_3'(s_3) + d_{41}H_4'(s_4) = \alpha. \tag{2.13}$$

In particular, if $d_{11} \neq 0$ then in view of (2.13), we conclude that $H_1(s_1)$ is linear in s_1. Similarly if $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

Therefore, it follows from (2.11) and (2.12) that

$$\begin{cases} a_0 + \sum_{m=1}^{s} b_m \alpha^m e^{mL(d)} = -i \\ a_0 + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(d)} = i. \end{cases}$$

Hence, it follows from (2.1) that

$$f(z) = -i \sinh(L(z) + H_1(s_1) + H_2(s_2) + H_3(s_3) + H_4(s_4) + \xi).$$

Case 2.2. Let $p(z+d)-p(z)$ be non-constant. Then by similar arguments as in Subcase 1.2, we can get a contradiction.

Case 3. Let $L_1^r(f) \not\equiv 0$, $L_2^r(f) \not\equiv 0$ and $c = d \in \mathbb{C}^3$.

In view of (2.1) and (2.7), we obtain that

$$\sum_{j=1}^{k} a_j e^{p(z+jd)+p(z)} - \sum_{j=1}^{k} a_j e^{p(z)-p(z+jd)} + \sum_{m=1}^{s} b_m h_1(z+md) e^{p(z+md)+p(z)}$$
$$- \sum_{m=1}^{s} b_m h_2(z+md) e^{p(z)-p(z+md)} + (a_0 - i) e^{2p(z)} = a_0 + i. \tag{2.14}$$

Now, we consider the following two possible cases.

Subcase 3.1. Let $p(z+d) - p(z) = \eta_2$, where η_2 is a constant in $\mathbb{C}$.

Then, the form of the polynomial $p(z)$ is the same as in (2.3). Then $p(z+\nu d) - p(z) = \nu L(d)$ for all $\nu \in \mathbb{N}$.

Hence, it follows from (2.14) that

$$\sum_{j=1}^{k} a_j e^{p(z+jd)+p(z)} + \sum_{m=1}^{s} b_m h_1(z+md) e^{p(z+md)+p(z)} + (a_0 - i) e^{2p(z)}$$
$$= a_0 + i + \sum_{j=1}^{k} a_j e^{-jL(d)} + \sum_{m=1}^{s} b_m h_2(z+md) e^{-mL(d)}. \tag{2.15}$$

Now, we consider the following two possible subcases.

Subcase 3.1.1. Let

$$a_0 + i + \sum_{j=1}^{k} a_j e^{-jL(d)} + \sum_{m=1}^{s} b_m h_2(z+md) e^{-mL(d)} \equiv 0. \tag{2.16}$$

Then, it follows from (2.16) that

$$\sum_{m=1}^{s} b_m \left[(-1)^m \left(\frac{\partial p}{\partial z_1} \right)^m + H_{2m} \left(\frac{\partial^m p}{\partial z_1^m}, \ldots, \frac{\partial p}{\partial z_1} \right) \right] e^{-mL(d)}$$
$$= -(a_0 + i) - \sum_{j=1}^{k} a_j e^{-jL(d)}. \tag{2.17}$$

In view of (2.15) and (2.16), we obtain that

$$\sum_{j=1}^{k} a_j e^{jL(d)} + \sum_{m=1}^{s} b_m h_1(z+md)e^{mL(d)} = -(a_0 - i). \tag{2.18}$$

Now, in view of (2.17), we can conclude that $\frac{\partial p}{\partial z_1}$ is a constant, say α in $\mathbb{C}$. Then, in view of (2.3), we must get (2.13).

Therefore, from (2.16) and (2.18), we obtain that

$$\begin{cases} a_0 + \sum_{j=1}^{k} a_j e^{-jL(d)} + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(d)} = -i \\ a_0 + \sum_{j=1}^{k} a_j e^{jL(d)} + \sum_{m=1}^{s} b_m \alpha^m e^{mL(d)} = i. \end{cases} \tag{2.19}$$

Hence, it follows from (1.3) that

$$f(z) = -i\sinh(L(z) + H_1(s_1) + H_2(s_2) + H_3(s_3) + H_4(s_4) + \xi),$$

where $L(z)$ satisfies the relation (2.19).

Subcase 3.1.2. Let

$$a_0 + i + \sum_{j=1}^{k} a_j e^{-jL(d)} + \sum_{m=1}^{s} b_m h_2(z+md)e^{-mL(d)} \not\equiv 0.$$

Then, from (2.15), we can easily obtain that

$$\left[a_0 + i + \sum_{j=1}^{k} a_j e^{-jL(d)} + \sum_{m=1}^{s} b_m h_2(z+md)e^{-mL(d)}\right] e^{-2p(z)}$$
$$= a_0 - i + \sum_{j=1}^{k} a_j e^{jL(d)} + \sum_{m=1}^{s} b_m h_1(z+md)e^{mL(d)}.$$

This leads us $p(z)$ is constant, which is a contradiction.

Case 3.2. Let $p(z+d) - p(z)$ be non-constant.

Then, it can be easily deduce that $p(z+jd) - p(z)$ are all non-constants, $j = 1, 2, \ldots, s$.

But, then in view of Lemma 2.3, it follows from (2.15) that

$$a_j = 0, \;\; b_m h_1(z+md) = 0, \;\; b_m h_2(z+md) = 0, \;\; a_0 = \pm i$$

for all $j = 1, 2, \ldots, k$, $m = 1, 2, \ldots, s$. This is a contradiction since atleast one of $a_j \neq 0$. $\square$

Proof of Theorem 1.13. Let f be a transcendental entire solution of (1.8). Then by similar arguments as in Theorem 1.2, we obtain

$$L_3(f)=\frac{e^{p(z)}+e^{-p(z)}}{2},\ \ f(z)=\frac{e^{p(z)}-e^{-p(z)}}{2i}, \tag{2.20}$$

where $p(z)$ is a non-constant polynomial in $\mathbb{C}^n$.

In view of (2.7) and (2.20), we obtain

$$\left[\sum_{l=1}^{t} e_l h_1(z)-i\right] e^{2p(z)}=\sum_{l=1}^{t} e_l h_2(z)+i, \tag{2.21}$$

where $h_1(z)=\left(\frac{\partial p}{\partial z_\lambda}\right)^l+H_{1l}\left(\frac{\partial^l p}{\partial z_\lambda^l},\ldots,\frac{\partial p}{\partial z_\lambda}\right)$ and $h_2(z)=(-1)^l\left(\frac{\partial p}{\partial z_\lambda}\right)^l+H_{2l}\left(\frac{\partial^l p}{\partial z_\lambda^l},\ldots,\frac{\partial p}{\partial z_\lambda}\right)$, H_{1l} and H_{2l} are polynomials of partial derivatives of $p(z)$ of degree less than j.

Since $p(z)$ is a non-constant polynomial, it follows from (2.21) that

$$\sum_{l=1}^{t} e_l h_1(z)=i\ \text{ and }\ \sum_{l=1}^{t} e_l h_2(z)=-i. \tag{2.22}$$

In view of the form of $h_1(z)$ and $h_2(z)$, we conclude from (2.22) that $\dfrac{\partial p}{\partial z_\lambda}$ must be a non-zero constant in $\mathbb{C}$, say α_λ. This implies that

$$p(z)=\alpha_\lambda z_\lambda+\Phi(z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n),$$

$\Phi(z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n)$ is a polynomial in $z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n$

Therefore, from (2.22), we obtain

$$\sum_{l=1}^{t} e_l\alpha_\lambda^l=i\ \text{ and }\ \sum_{l=1}^{t}(-1)^l e_l\alpha_\lambda^l=-i.$$

Hence, it follows from (2.20) that

$$f(z)=-i\sinh(\alpha_\lambda z_\lambda+\Phi(z_1,z_2,\ldots,z_{\lambda-1},z_{\lambda+1},\ldots,z_n)).$$

□

Proof of Theorem 1.16. Let f be a transcendental entire solution of (1.10). Then by similar arguments as in Theorem 1.2, we obtain

$$L_2^r(f)+L_3(f)=\frac{e^{p(z)}+e^{-p(z)}}{2},\ \ f(z+c)=\frac{e^{p(z)}-e^{-p(z)}}{2i}, \tag{2.23}$$

where $p(z)$ is a non-constant polynomial in $\mathbb{C}^3$.

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In view of (2.7) and (2.23), we obtain

$$\sum_{m=1}^{s} b_m h_1(z+mc-c)e^{p(z+mc-c)+p(z)} + \sum_{l=1}^{t} e_l h_1(z-c)e^{p(z-c)+p(z)}$$
$$-\sum_{m=1}^{s} b_m h_2(z+mc-c)e^{-p(z+mc-c)+p(z)} - \sum_{l=1}^{t} e_l h_2(z-c)e^{-p(z-c)+p(z)}$$
$$-ie^{2p(z)} = i. \tag{2.24}$$

Now, we consider two possible cases.

Case 1. Let $p(z-c)-p(z)=\eta$, a constant in $\mathbb{C}$.

Since $p(z)$ is a polynomial, the form of the polynomial $p(z)$ is the same as in (2.3). This implies that $p(z-c)-p(z)=-L(c)$ and $p(z+jc-c)-p(z)=jL(c)-L(c)$ for all $j=1,2,\ldots,s$.

Therefore, (2.24) reduces to

$$\sum_{m=1}^{s} b_m h_1(z+mc-c)e^{p(z+mc-c)+p(z)} + \sum_{l=1}^{t} e_l h_1(z-c)e^{p(z-c)+p(z)} - ie^{2p(z)}$$
$$= \sum_{m=1}^{s} b_m h_2(z+mc-c)e^{-mL(c)+L(c)} + \sum_{l=1}^{t} e_l h_2(z-c)e^{L(c)} + i. \tag{2.25}$$

Now, we discuss the possible two subcases.

Subcase 1.1. Let

$$\sum_{m=1}^{s} b_m h_2(z+mc-c)e^{-mL(c)+L(c)} + \sum_{l=1}^{t} e_l h_2(z-c)e^{L(c)} + i \equiv 0. \tag{2.26}$$

Then, it follows from (2.25) and (2.26) that

$$\sum_{m=1}^{s} b_m h_1(z+mc-c)e^{mL(c)-L(c)} + \sum_{l=1}^{t} e_l h_1(z-c)e^{-L(c)} - i \equiv 0. \tag{2.27}$$

In view of (2.8) and (2.27), we conclude that $\dfrac{\partial p}{\partial z_1}$ is a constant, say α in $\mathbb{C}$. Therefore, in view of (2.3), we obtain (2.13).

Hence, in a similar manner as done in Subcase 2.1 of Theorem 1.2, we can characterize the form of the polynomials $H_1(s_1), H_3(s_3)$ and $H_4(s_4)$.

Thus, we obtain from (2.26) and (2.27) that

$$\begin{cases} \displaystyle\sum_{m=1}^{s}(-1)^m b_m \alpha^m e^{-mL(c)+L(c)} + \sum_{l=1}^{t}(-1)^l e_l \alpha^l e^{L(c)} = -i, \\ \displaystyle\sum_{m=1}^{s} b_m \alpha^m e^{mL(c)-L(c)} + \sum_{l=1}^{t} e_l \alpha^l e^{-L(c)} = i. \end{cases}$$

Hence, it follows from (2.23) that

$$f(z) = -i \sinh\left(L(z) + \sum_{j=1}^{4} H_j(s_j) - L(c) + \xi\right).$$

Subcase 1.2. Let

$$\sum_{m=1}^{s} b_m h_2(z+mc-c)e^{-mL(c)+L(c)} + \sum_{l=1}^{t} e_l h_2(z-c)e^{L(c)} + i \not\equiv 0.$$

Then it follows from (2.25) that

$$\left[\sum_{m=1}^{s} b_m h_2(z+mc-c)e^{-mL(c)+L(c)} + \sum_{l=1}^{t} e_l h_2(z-c)e^{L(c)} + i\right] e^{-2p(z)}$$
$$= \sum_{m=1}^{s} b_m h_1(z+mc-c)e^{mL(c)-L(c)} + \sum_{l=1}^{t} e_l h_1(z-c)e^{-L(c)} - i.$$

This implies that $p(z)$ is constant, which is a contradiction.

Subcase 2. Let $p(z-c)-p(z)$ be non-constant. This implies that $p(z+jc-c)-p(z)$ is also non-constant for all $j \in \mathbb{N}$.

Therefore, by Lemma 2.3, it follows from (2.24) that $i = 0$, which is a contradiction.

This completes the proof of the theorem. □

3. Concluding remark and an open question

Observe that we have been able to exhaustively determine the form of the finite order transcendental entire solutions of equations (1.3) and (1.10) for $L_1^r(f) + L_2^r(f)$ and $L_2^r(f)$, respectively. It is therefore natural to ask the following open question.

Question 3.1. What could be the form of finite order transcendental entire solutions when we replace $L_1^r(f)+L_2^r(f)$ by $L_1(f)+L_2(f)$ in (1.3) and $L_2^r(f)$ by $L_2(f)$ in (1.10)?

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