A note on the Cohen type theorem for uniformly S-Noetherian modules and the Eakin-Nagata type theorem for uniformly S-Noetherian rings

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Abstract

In this note, we give the Cohen type theorem for uniformly S-Noetherian modules and the Eakin-Nagata type theorem for uniformly S-Noetherian rings. We also answer an open question proposed by Kim and Lim [5, Question 4.10].

Key Words: uniformly S-Noetherian ring; uniformly S-Noetherian module; Cohen type theorem; Eakin-Nagata type theorem.
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1. Introduction

Throughout this note, all rings are commutative rings with identity and all modules are unitary. Let R be a ring. We always denote by S a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let M be an R-module. Denote by $\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}$. For a subset U of M, denote by $\langle U \rangle$ the R-submodule of M generated by U.

In the development of Noetherian rings, Cohen theorem and Eakin-Nagata theorem are crucial. In the early 1950s, Cohen [3] showed that a ring R is Noetherian if and only if every prime ideal of R is finitely generated, which is now called Cohen type theorem. Recently, Parkash and Kour [7] provided a Cohen theorem to Noetherian modules: a finitely generated R-module M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}(M) \subseteq \mathfrak{p}$, there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$, where $M(\mathfrak{p}) := \{x \in M \mid sx \in \mathfrak{p}M$ for some $s \in R \setminus \mathfrak{p}\}$. In the late 1960s, Eakin and Nagata independently found that

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if $R \subseteq T$ be an extension of rings with T a finitely generated R-module, then R is a Noetherian ring if and only if so is T (see [4, 6]). This well-known result is now called Eakin-Nagata theorem.

In the past few decades, several generalizations of Noetherian rings (modules) have been extensively studied. In 2002, Anderson and Dumitrescu [1] introduced the notions of S-Noetherian rings and S-Noetherian modules. They also considered the Cohen type theorem and Eakin-Nagata type theorem for S-Noetherian rings [1, Proposition 4, Corollary 7]. Recently, Kim and Lim [5] gave a new proof of the Cohen type theorem for S-Noetherian modules and a generalization of the Eakin-Nagata type theorem for S-Noetherian ring. They also showed that if an R-module M is faithful S-Noetherian with S consisting of non-zero-divisors, then R itself is an S-Noetherian ring, and latter they ask if the regularity of S is essential? (see [5, Proposition 3.7, Question 4.10])

By noticing the elements chosen in S in some concepts of S-versions of classical ones are not "uniform" in general, Zhang [10] recently introduced the notions of uniformly S-torsion modules, uniformly S-exact sequences etc. Utilizing the "uniform" ideas, Qi and Kim etc. [8] introduced the notions of uniformly S-Noetherian rings and uniformly S-Noetherian modules, and then distinguished them from the classical ones. The main motivation of this paper is to investigate some Cohen type theorem and Eakin-Nagata type theorem for uniformly S-Noetherian rings and modules. More precisely, we showed that if S is anti-Archimedean, then an R-module M is u-S-Noetherian if and only if there is an $s \in S$ such that M is s-finite, and for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$, there exists an s-finite submodule $N^{\mathfrak{p}}$ of M satisfying that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$ (see Theorem 2.3); and if $R \subseteq T$ be an extension of rings with T an S-finite R-module, then R is an uniformly S-Noetherian ring if and only if so is T (see Theorem 2.7). Moreover, we obtain that if there exists a faithful R-module M which is also (resp., uniformly) S-Noetherian, then R itself is an (resp., a uniformly) S-Noetherian ring, solving the open problem proposed by [5, Question 4.10] (See Theorem 2.8 and Theorem 2.9).

2. MAIN RESULTS

Let R be a ring. Recall from [1] that an R-module M is *S-finite* if for any submodule N of M, there is an element $s \in S$ and a finitely generated R-module F such that $sN \subseteq F \subseteq N$. In this case, we also say N is s-finite. Moreover, an

R-module M is called an *S-Noetherian module* if every submodule of M is S-finite, and a ring R is called an *S-Noetherian ring* if R itself is an S-Noetherian R-module. Note that the choice of s in these two concepts is decided by the submodules or ideals of the given module or ring.

To fill the gap of "uniformity" in the concept of S-Noetherian rings and S-Noetherian modules, the authors in [8] introduced the notions of uniformly S-Noetherian rings and uniformly S-Noetherian modules, and we restate them as follows.

Definition 2.1. [8, Definition 2.1, Definition 2.6] *Let R be a ring and S a multiplicative subset of R. An R-module M is called a uniformly S-Noetherian R-module* (*with respect to s*) *provided the set of all submodules of M is s-finite for some $s \in S$. A ring R is called a uniformly S-Noetherian ring* (*with respect to s*) *if R itself is a uniformly S-Noetherian R-module* (*with respect to s*).

We obviously have the following implications for both rings and modules:

$$\boxed{\text{Notherian}} \Longrightarrow \boxed{\text{u-S-Notherian}} \Longrightarrow \boxed{\text{S-Notherian}}$$

However, the converses are not correct in general (see [8, Example 2.2, Example 2.5] respectively). Recall that a multiplicative subset S of R is said to be anti-Archimedean if $\bigcap\limits_{n\geq 1} s^nR \bigcap S \neq \emptyset$. The anti-Archimedean condition is very important in some results of S-Noetherian rings, such as Hilbert Theorem for S-Noetherian rings etc. (see [1, Proposition 9, Proposition 10]). It is easy to verify that the multiplicative set given in [8, Example 2.5] is not anti-Archimedean. Now we give an example of S-Noetherian ring which is not uniformly S-Noetherian when S is anti-Archimedean.

Example 2.2. *Let R be a valuation domain whose valuation group is the additive group $G = \mathbb{R}[x]$ of all polynomials with coefficients in the field $\mathbb{R}$ of real numbers, and the order is defined by $f(x) > 0$ if its leading coefficient > 0. Let $S = R \setminus \{0\}$ the set of all nonzero elements of R. Then S is anti-Archimedean, and R is S-Noetherian but not uniformly S-Noetherian.*

Proof. First, we will show S is anti-Archimedean. Denote by v the valuation of $R \setminus \{0\}$ to G. Let s be a nonzero element in R. Let s' be an nonzero element in R such that $\deg(v(s')) > \deg(v(s))$. Then we have $v(s') > nv(s) = v(s^n)$ for any positive integer n. So $s' \in \bigcap\limits_{n\geq 1} s^nR \bigcap S$ for any $s \in S$, that is, S is anti-Archimedean.

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Then, we have that R is S-Noetherian. Indeed, let I be an nonzero ideal of R and $0 \neq s \in I$. Then $sI \subseteq sR \subseteq I$. It follows that R is S-Noetherian.

Finally, we claim that R is not uniformly S-Noetherian. Suppose R is uniformly S-Noetherian with respect to some $s \in S$. Suppose $\deg(v(s)) = n$. Then the R_s-ideal generated by $\{v^{-1}(x^{n+1}), v^{-1}(x^{n+2}), \dots\}$ is not finitely generated, where R_s is the localization of R at $S' = \{1, s, s^2, \dots\}$. So R_s is not Noetherian. Hence R is not uniformly S-Noetherian by [8, Lemma 2.3]. □

Recently, Parkash and Kour [7] provided a Cohen type theorem to Noetherian modules: a finitely generated R-module M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}(M) \subseteq \mathfrak{p}$, there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$, where $M(\mathfrak{p}) := \{x \in M \mid sx \in \mathfrak{p}M \text{ for some } s \in R \setminus \mathfrak{p}\}$. Later, Zhang [11] extended this result to S-Noetherian modules and w-Noetherian modules. In the following, we give the result for uniformly S-Noetherian modules when S is anti-Archimedean.

Theorem 2.3. (***Cohen type theorem for uniformly S-Noetherian modules***) *Let R be a ring and S an anti-Archimedean multiplicative subset of R. Then an R-module M is uniformly S-Noetherian if and only if there exists $s \in S$ such that M is s-finite, and for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$, there exists an s-finite submodule $N^{\mathfrak{p}}$ of M satisfying that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$, where $M(\mathfrak{p}) = \{x \in M \mid sx \in \mathfrak{p}M \text{ for some } s \in R \setminus \mathfrak{p}\}$.*

Proof. Suppose that M is a uniformly S-Noetherian R-module. Then there is $s \in S$ such that the set of all submodules of M is s-finite. Let $\mathfrak{p}$ be a prime ideal with $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$. If we take $N^{\mathfrak{p}} = \mathfrak{p}M$, then $N^{\mathfrak{p}}$ is certainly an s-finite submodule of M satisfying $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.

On the other hand, let $s' \in \bigcap_{n\geq 1} s^n R \bigcap S$. If M is uniformly S-Noetherian with respect to s', then we are done. Otherwise, we will show M is uniformly S-Noetherian with respect to s^n for some positive integer n. On contrary, suppose that M is not uniformly S-Noetherian with respect to s^k for any positive integer k. Let $\mathcal{N}$ be the set of all submodules of M which are not s^k-finite for any positive integer k. We can assume $\mathcal{N}$ is non-empty. Indeed, on contrary assume that for each submodule N of M, there exists a nonnegative integer k_N such that N is s^{k_N}-finite. Since S is anti-Archimedean, then there is an $s' \in \bigcap_{n\geq 1} s^n R \bigcap S$ such that all submodules of

M are s'-finite. Hence M is uniformly S-Noetherian with respect to s', and so the conclusion holds.

Make a partial order on $\mathcal{N}$ by defining $N_1 \leq N_2$ if and only if $N_1 \subseteq N_2$ in $\mathcal{N}$. Let $\{N_i \mid i \in \Lambda\}$ be a chain in $\mathcal{N}$. Set $N := \bigcup\limits_{i\in\Lambda} N_i$. Then N is not s^k-finite for any positive integer k. Indeed, suppose $s^{k_0}N \subseteq \langle x_1, \dots, x_n\rangle \subseteq N$ for some positive integer k_0. Then there exists $i_0 \in \Lambda$ such that $\{x_1, \dots, x_n\} \subseteq N_{i_0}$. Thus $s^{k_0}N_{i_0} \subseteq sN \subseteq \langle x_1, \dots, x_n\rangle \subseteq N_{i_0}$ implying that N_{i_0} is s^{k_0}-finite, which is a contradiction. By Zorn's Lemma $\mathcal{N}$ has a maximal element, which is also denoted by N. Set

$$\mathfrak{p} := (N : M) = \{r \in R \mid rM \subseteq N\}.$$

(1) Claim that $\mathfrak{p}$ is a prime ideal of R. Assume on the contrary that there exist $a, b \in R\setminus\mathfrak{p}$ such that $ab \in \mathfrak{p}$. Since $a, b \in R\setminus\mathfrak{p}$, we have $aM \nsubseteq N$ and $bM \nsubseteq N$. Therefore $N + aM$ is s^{k_0}-finite for some nonnegative integer k_0. Let $\{y_1, \dots, y_m\}$ be a subset of $N + aM$ such that $s^{k_0}(N + aM) \subseteq \langle y_1, \dots, y_m\rangle$. Write $y_i = w_i + az_i$ for some $w_i \in N$ and $z_i \in M$ $(1 \leq i \leq m)$. Set $L := \{x \in M \mid ax \in N\}$. Then $N + bM \subseteq L$, and hence L is also s^{k_1}-finite for some nonnegative integer k_1. Let $\{x_1, \dots, x_k\}$ be a subset of L such that $s^{k_1}L \subseteq \langle x_1, \dots, x_k\rangle$. Let $n \in N$ and write

$$s^{k_0}n = \sum_{i=1}^{m} r_i y_i = \sum_{i=1}^{m} r_i w_i + a\sum_{i=1}^{m} r_i z_i.$$

Then $\sum\limits_{i=1}^{m} r_i z_i \in L$. Thus $s^{k_1}\sum\limits_{i=1}^{m} r_i z_i = \sum\limits_{i=1}^{k} r_i' x_i$ for some $r_i' \in R$ $(i = 1, \dots, k)$. So

$$s^{k_0+k_1}n = \sum_{i=1}^{m} s r_i w_i + \sum_{i=1}^{k} r_i' a x_i.$$

And thus $s^{k_0+k_1}N \subseteq \langle w_1, \dots, w_m, ax_1, \dots, ax_k\rangle \subseteq N$ implying that N is $s^{k_0+k_1}$-finite, which is a contradiction. Hence $\mathfrak{p}$ is a prime ideal of R.

(2) Claim that $M(\mathfrak{p}) \subseteq N$. Suppose on the contrary that there exists $y \in M(\mathfrak{p})$ such that $y \notin N$. Then there exists $t \in R\setminus\mathfrak{p}$ such that $ty \in \mathfrak{p}M = (N : M)M \subseteq N$. As $t \notin \mathfrak{p} = (N : M)$, it follows that $tM \nsubseteq N$. Therefore $N + tM$ is s^{k_2}-finite for some nonnegative integer k_2. Let $\{u_1, \dots, u_m\}$ be a subset of $N + tM$ such that $s^{k_2}(N + tM) \subseteq \langle u_1, \dots, u_m\rangle$ for some $s^{k_2} \in S$. Write $u_i = w_i + tz_i$ $(i = 1, \dots, m)$ with $w_i \in N$ and $z_i \in M$. Set $T := \{x \in M \mid tx \in N\}$. Then $N \subset N + Ry \subseteq T$, and hence T is s^{k_3}-finite for some nonnegative integer k_3. Then there exists a subset

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$\{v_1, \ldots, v_l\}$ of T such that $s^{k_3}T \subseteq \langle v_1, \ldots, v_l\rangle$. Let n be an element in N. Then

$$s^{k_2}n = \sum_{i=1}^{m} r_i u_i = \sum_{i=1}^{m} r_i w_i + t\sum_{i=1}^{m} r_i z_i.$$

Thus $\sum_{i=1}^{m} r_i z_i \in T$. So $s^{k_3}\sum_{i=1}^{m} r_i z_i = \sum_{i=1}^{l} r'_i v_i$ for some $r'_i \in R$ $(i = 1, \ldots, l)$. Hence $s^{k_2+k_3}n = \sum_{i=1}^{m} s_4 r_i w_i + \sum_{i=1}^{l} r'_i t v_i$. Thus $s^{k_2+k_3}N \subseteq \langle w_1, \ldots, w_m, tv_1, \ldots, tv_l\rangle$ implying that N is $s^{k_2+k_3}$-finite, which is a contradiction. Hence $M(\mathfrak{p}) \subseteq N$.

Finally, we will show M is uniformly S-Noetherian. Since M is s-finite, there exists a finitely generated submodule $F = \langle m_1, \ldots, m_k\rangle$ of M such that $sM \subseteq F$. Claim that $\mathfrak{p} \cap S' = \emptyset$ where $S' = \{1, s, s^2, \cdots\}$. Indeed, if $s^{k_4} \in \mathfrak{p}$ for some nonnegative integer k_4, then $s^{k_4}M \subseteq N \subseteq M$. So $s^{1+k_4}N \subseteq s^{1+k_4}M \subseteq s^{k_4}F \subseteq s^{k_4}M \subseteq N$ implies that N is s^{1+k_4}-finite, which is a contradiction. Note that

$$\mathfrak{p} = (N : M) \subseteq (N : F) \subseteq (N : sM) = (\mathfrak{p} : s) = \mathfrak{p}$$

since $\mathfrak{p}$ is a prime ideal of R. So $\mathfrak{p} = (N : F) = (N : \langle m_1, \ldots, m_k\rangle) = \bigcap_{i=1}^{k}(N : Rm_i)$. By [2, Proposition 1.11], $\mathfrak{p} = (N : Rm_j)$ for some $1 \leq j \leq k$. Since $m_j \notin N$, it follows that $N + Rm_j$ is s^{k_5}-finite for some nonnegative integer k_5. Let $\{y_1, \ldots, y_m\}$ be a subset of $N + Rm_j$ such that $s^{k_5}(N + Rm_j) \subseteq \langle y_1, \ldots, y_m\rangle$. Write $y_i = w_i + a_i m_j$ for some $w_i \in N$ and $a_i \in R$ $(i = 1, \ldots, m)$. Let $n \in N$. Then $s^{k_5}n = \sum_{i=1}^{m} r_i(w_i + a_i m_j) = \sum_{i=1}^{m} r_i w_i + (\sum_{i=1}^{m} r_i a_i)m_j$. Thus $(\sum_{i=1}^{m} r_i a_i)m_j \in N$. So $\sum_{i=1}^{m} r_i a_i \in \mathfrak{p}$. Thus $s^{k_5}N \subseteq \langle w_1, \ldots, w_m\rangle + \mathfrak{p}m_j$. As $\mathrm{Ann}_R(M) \subseteq (N : M) = \mathfrak{p}$, there exists an s-finite submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$. Thus

$$\begin{aligned} s^{k_5}N &\subseteq \langle w_1, \ldots, w_m\rangle + \mathfrak{p}m_j \\ &\subseteq \langle w_1, \ldots, w_m\rangle + \mathfrak{p}M \\ &\subseteq \langle w_1, \ldots, w_m\rangle + N^{\mathfrak{p}} \\ &\subseteq \langle w_1, \ldots, w_m\rangle + M(\mathfrak{p}) \\ &\subseteq N \end{aligned}$$

Since $N^{\mathfrak{p}} + \langle w_1, \ldots, w_m\rangle$ is s-finite, it follows that N is s^{1+k_5}-finite, which is a contradiction. Consequently, we have M is uniformly S-Noetherian with respect to $s^{k'}$ for some nonnegative integer k'. □

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Remark 2.4. We do not know whether the condition "S is anti-Archimedean" in Theorem 2.3 can be removed. Note that this condition is mainly used to show the set $\mathcal{N}$ in the proof of Theorem 2.3 can be assumed to be non-empty.

Taking $S = \{1\}$, we can recover Parkash and Kour's result.

Corollary 2.5. [7, Theorem 2.1] *Let R be a ring and M a finitely generated R-module. Then M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$, there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.*

There is a direct corollary of Theorem 2.3.

Corollary 2.6. *Let R be a ring and S an anti-Archimedean multiplicative subset of R. Then an R-module M is uniformly S-Noetherian if and only if there exists $s \in S$ such that M is s-finite and $\mathfrak{p}M$ is s-finite for every prime ideal $\mathfrak{p}$ of R.*

The well-known Eakin-Nagata theorem states that if $R \subseteq T$ is an extension of rings with T a finitely generated R-module, then R is a Noetherian ring if and only if so is T (see [4, 6]). Next, we give the Eakin-Nagata type theorem for uniformly S-Noetherian rings.

Theorem 2.7. (***Eakin-Nagata type theorem for uniformly S-Noetherian rings***) *Let R be a ring, S an anti-Archimedean multiplicative subset of R and T a ring extension of R. If T is S-finite as an R-module. Then the following statements are equivalent.*

(1) *R is a uniformly S-Noetherian ring.*
(2) *T is a uniformly S-Noetherian ring.*
(3) *There is $s \in S$ such that $\mathfrak{p}T$ is an s-finite T-ideal for every prime ideal $\mathfrak{p}$ of R.*
(4) *T is a uniformly S-Noetherian R-module.*

Proof. (1) $\Rightarrow$ (2) Suppose R is a uniformly S-Noetherian ring with respect to some $s_1 \in S$. Let I be an ideal of T. Since $R \subseteq T$, I is an R-submodule of T. Suppose T is s_2-finite as an R-module for some $s_2 \in S$. Then T is the image of a uniformly S-epimorphism $R^n \to T$. One can use the proof of [8, Lemma 2.12] to check R^n is a uniformly S-Noetherian R-module with respect to s_1^n. So T is a uniformly S-Noetherian R-module with respect to $s_1^n s_2$ by [8, Proposition 2.13]. Then there exist

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$a_1, \ldots, a_m \in I$ such that $s_1^n s_2 I \subseteq \langle a_1, \ldots, a_m \rangle R \subseteq I$. Thus $s_1^n s_2 I \subseteq \langle a_1, \ldots, a_m \rangle T \subseteq I$. Consequently, T is a uniformly S-Noetherian ring with respect to $s_1^n s_2$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let $\mathfrak{p}$ be a prime ideal that satisfies $\mathrm{Ann}_R(T) \subseteq \mathfrak{p}$. Then $\mathfrak{p}T$ is s-finite as an T-ideal. So there exist $p_1, \ldots, p_m \in \mathfrak{p}$ such that $s(\mathfrak{p}T) \subseteq \langle p_1, \ldots, p_m \rangle T \subseteq \mathfrak{p}T$. Since T is S-finite, there exist $s' \in S$ and $t_1, \ldots, t_n$ such that $s'T \subseteq \langle t_1, \ldots, t_n \rangle R \subseteq T$. Therefore, we have

$$\begin{aligned} s's(\mathfrak{p}T) &\subseteq s'\langle p_1, \ldots, p_m \rangle T \\ &= s'p_1T + \cdots + s'p_mT \\ &\subseteq p_1(t_1R + \cdots t_nR) + \cdots + p_m(t_1R + \cdots t_nR) \\ &\subseteq \mathfrak{p}T \end{aligned}$$

Hence $\mathfrak{p}T$ is $s's$-finite as an R-module. It follows by Corollary 2.6 that T is a uniformly S-Noetherian R-module.

(4) $\Rightarrow$ (1) Suppose T is a uniformly S-Noetherian R-module. Since R is an R-submodule of T, R is also a uniformly S-Noetherian R-module by [8, Lemma 2.12]. It follows that R is a uniformly S-Noetherian ring. □

Let R be a ring and M an R-module. Recall that M is faithful if $\mathrm{Ann}_R(M) = 0$. We say M is S-faithful if $t\mathrm{Ann}_R(M) = 0$ for some $t \in S$. Hence faithful R-modules are all S-faithful. It is well-known that if a faithful R-module M is Noetherian, then R itself is a Noetherian ring (see [9, Exercise 2.32]).

Theorem 2.8. *Let R be a ring, S a multiplicative subset of R and M an S-faithful R-module. If M is a uniformly S-Noetherian R-module, then R is a uniformly S-Noetherian ring.*

Proof. Suppose M is a uniformly S-Noetherian R-module with respect to some $s \in S$. Then M is s-finite, and so there exist $m_1, \ldots, m_n \in M$ such that $sM \subseteq \langle m_1, \ldots, m_n \rangle \subseteq M$. Consider the R-homomorphism $\phi : R \to M^n$ given by $\phi(r) = (rm_1, \ldots, rm_n)$. We claim that $s\mathrm{Ker}(\phi) = 0$. Indeed, let $r \in \mathrm{Ker}(\phi)$. Then $rm_i = 0$ for each $i = 1, \ldots, n$. Hence $srM \subseteq r\langle m_1, \ldots, m_n \rangle = 0$. And hence $sr \in \mathrm{Ann}_R(M)$. Since M is an S-faithful R-module, we have $tsr = 0$ for some $t \in S$, and so $ts\mathrm{Ker}(\phi) = 0$. Note that the R-module M^n is uniformly S-Noetherian with respect to s^n by the proof of [8, Lemma 2.12]. Hence the R-module $\mathrm{Im}(\phi)$ is also uniformly S-Noetherian with respect to s^n. Considering the exact sequence

$$0 \to \mathrm{Ker}(\phi) \to R \to \mathrm{Im}(\phi) \to 0,$$

we have R is a uniformly S-Noetherian ring with respect to ts^{n+1}. □

Recently, the authors in [5, Proposition 3.7] showed that Theorem 2.8 also holds for S-Noetherian ring (modules) when S consists of non-zero-divisors, and asked if the condition "S consists of non-zero-divisors" is essential (see [5, Question 4.10]). Inspired by the proof of Theorem 2.8, we can show the condition "S consists of non-zero-divisors" in [5, Proposition 3.7] can be removed .

Theorem 2.9. *Let R be a ring, S a multiplicative subset of R and M an S-faithful R-module (for example, M is a faithful R-module). If M is an S-Noetherian R-module, then R is an S-Noetherian ring.*

Proof. Let M be an S-Noetherian faithful R-module. Then M is S-finite, and so there exist $s \in S$ and $m_1, \dots, m_n \in M$ such that $sM \subseteq \langle m_1, \dots, m_n \rangle \subseteq M$. Consider the R-homomorphism $\phi : R \to M^n$ given by $\phi(r) = (rm_1, \dots, rm_n)$. We claim that $s\mathrm{Ker}(\phi) = 0$. Indeed, let $r \in \mathrm{Ker}(\phi)$. Then $rm_i = 0$ for each $i = 1, \dots, n$. Hence $srM \subseteq r\langle m_1, \dots, m_n \rangle = 0$. And hence $sr \in \mathrm{Ann}_R(M)$. Since M is an S-faithful R-module, we have $tsr = 0$ for some $t \in S$, and so $ts\mathrm{Ker}(\phi) = 0$. Note that M^n is also an S-Noetherian R-module, and so is its submodule $\mathrm{Im}(\phi)$. Let I be an ideal of R. Then $\phi(I)$ is a submodule of $\mathrm{Im}(\phi)$, and so is S-finite. Thus there exist $s' \in S$ and $r_1, \cdots r_n \in I$ such that

$$s'\phi(I) \subseteq \phi(r_1R + \cdots + r_nR) \subseteq \phi(I).$$

We claim that $ss'I \subseteq r_1R + \cdots + r_nR$. Indeed, for any $x \in I$, we have $s'\phi(x) = \phi(r_1t_1 + \cdots + r_nt_n)$ for some $t_i \in R$ ($i = 1, \dots, n$). Hence $\phi(r_1t_1 + \cdots + r_nt_n - s'x) = 0$. So $r_1t_1 + \cdots + r_nt_n - s'x \in \mathrm{Ker}(\phi)$, and thus $ts(r_1t_1 + \cdots + r_nt_n) - tss'x = 0$. It follows that $tss'I \subseteq ts(r_1R + \cdots + r_nR) \subseteq r_1R + \cdots + r_nR \subseteq I$. Hence I is S-finite. So R is an S-Noetherian ring. □

References

[1] D. D. Anderson and T. Dumitrescu, S-Noetherian rings, *Comm. Algebra* **30** (2002), 4407-4416.

[2] M. F. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[3] I. S. Cohen, Commutative rings with restricted minimum condition, *Duke Math. J.* **17** (1950), 27-42.

5 Dec 2023 16:46:08 PST
230218-Zhang-2 Version 2 - Submitted to Rocky Mountain J. Math.

[4] P. M. Eakin, Jr., The converse to a well known theorem on Noetherian rings, *Math. Ann.* **177** (1968), 278-282.

[5] D. K. Kim and J. W. Lim, The Cohen type theorem and the Eakin-Nagata type theorem for S-Noetherian rings revisited, *Rocky Mountain J. Math.* **50** (2020), 619-630.

[6] M. Nagata, A type of subrings of a Noetherian ring, *J. Math. Kyoto Univ.* **8** (1968), 465-467.

[7] A. Parkash and S. Kour, On Cohen's theorem for modules, *Indian J. Pure Appl. Math.* **52** (2021), no. 3, 869-871.

[8] W. Qi, H. Kim, F. G. Wang, M. Z. Chen, W. Zhao, Uniformly S-Noetherian rings, to appear in Quaest. Math. https://arxiv.org/abs/2201.07913.

[9] F. G. Wang, H. Kim, *Foundations of Commutative Rings and Their Modules*, Singapore, Springer, 2016.

[10] X. L. Zhang, Characterizing S-flat modules and S-von Neumann regular rings by uniformity, *Bull. Korean Math. Soc.* **59** (2022), no. 3, 643-657.

[11] X. L. Zhang, H. Kim, W. Qi, On two versions of Cohen's theorem for modules, *Kyungpook Math. J.*, **63** (2023), no. 1, 29-36.

5 Dec 2023 16:46:08 PST
230218-Zhang-2 Version 2 - Submitted to Rocky Mountain J. Math.