Invariant Subspaces for Operators Commuting with a Bounded Operator having a Compact Hermitian Component

Michael Gil'
Department of Mathematics
Ben Gurion University of the Negev
P.0. Box 653, Beer-Sheva 84105, Israel
E-mail: gilmi@bezeqint.net

Abstract

Let A be a bounded linear operator in a Hilbert space whose Hermitian component $A_I = (A - A^*)/2i$ (A^* is the adjoint operator), belongs to the Macaev ideal, i.e. the eigenvalues $\lambda_k(A_I)$ $(k = 1, 2, ...)$ of A_I taken with the multiplicities satisfy the condition $\sum_{k=1}^{\infty}(2k-1)^{-1}|\lambda_k(A_I)| < \infty$. Under certain restrictions it is proved that each operator commuting with A has a closed nontrivial invariant subspace. An illustrative example is presented.

Key words: Hilbert space, linear operators, invariant subspaces

AMS (MOS) subject classification: 47A15, 47A46

1 Introduction and statement of the main result

Let $\mathcal{H}$ be a Hilbert space with a scalar product $(.,.)_{\mathcal{H}}$ and the unit operator I, and $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators in $\mathcal{H}$. For an $A \in \mathcal{L}(\mathcal{H})$, A^* is the adjoint operator, $\sigma(A)$ is the spectrum, and $A_I = (A - A^*)/2$.

A subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is an invariant one for A, if for any $x \in \mathcal{H}_1$, we have $Ax \in \mathcal{H}_1$. It is said to be nontrivial if $\mathcal{H}_1 \neq 0$ and $\mathcal{H}_1 \neq \mathcal{H}$.

The problem of the existence of nontrivial closed invariant subspaces for linear operators is one of the most important problems of the theory of linear operators. In spite of its long history that problem continues to attract the attention of many scientists because of the absence of its complete solution. The classical results connected with the problem of the existence of invariant subspaces are presented in the well-known book [2]; the recent results can be found in the survey [18] and papers [4, 6, 14, 15, 16, 23, 24], and the references, which are given therein.

One of the celebrated results here belongs to V. Lomonosov [17] who has proved, in particular, that each bounded linear operator in a Banach space, commuting with a compact one has

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.

nontrivial closed invariant subspaces. That result has caused a great interest of mathematicians [1, 7, 8, 9, 21, 22, 25], etc. Below we prove the existence of nontrivial invariant subspaces for a class of operators in $\mathcal{H}$ commuting with a non-compact linear operator having a compact Hermitian component. That class contains various integro-differential operators.

Introduce the notations. For two orthogonal projections P_1, P_2 in $\mathcal{H}$ we write $P_1 < P_2$ if $P_1\mathcal{H} \subset P_2\mathcal{H}$. A set $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$ containing at least two orthogonal projections is called *a chain*, if from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it follows that either $P_1 < P_2$ or $P_1 > P_2$. For two chains $\mathcal{P}_1, \mathcal{P}_2$ we write $\mathcal{P}_1 < \mathcal{P}_2$ if from $P \in \mathcal{P}_1$ it follows that $P \in \mathcal{P}_2$. In this case we say that $\mathcal{P}_1$ precedes $\mathcal{P}_2$. The chain that precedes only itself is called *a maximal chain.*

For a chain $\mathcal{P}$, let $P^-, P^+ \in \mathcal{P}$, and $P^- < P^+$. If for every $P \in \mathcal{P}$ we have either $P < P^-$ or $P > P^+$, then the pair (P^+, P^-) is called a gap of $\mathcal{P}$. Besides, dim $(P_+\mathcal{H}) \ominus (P_-\mathcal{H})$ is the dimension of the gap.

An orthogonal projection P in $\mathcal{H}$ is called a limit projection of a chain $\mathcal{P}$ if there is a sequence $P_k \in \mathcal{P}$ $(k = 1, 2, ...)$ which strongly converges to P. A chain is said to be closed if it contains all its limit projections.

As it is shown in [11, Proposition XX.4.1, p. 478], [3, Theorem II.14.1], a chain is maximal if and only if it is closed, contains 0 and I, and all its gaps (if they exist) are one dimensional. A maximal chain is said to be *continuous*, if it does not have gaps.

We will say that *a maximal chain $\mathcal{P}$ is proper of $A \in \mathcal{L}(\mathcal{H})$, or A has a maximal proper chain $\mathcal{P}$*, if $PAP = AP$ for any $P \in \mathcal{P}$. Let $\sigma(A)$ be real. Then a proper chain $\mathcal{P}$ of A is said to be *separating* $\sigma(A)$, if for any $t \in \sigma(A)$ exists $P_t \in \mathcal{P}$, such that

$$\sigma(AP_t|P_t\mathcal{H}) \subset (-\infty, t] \text{ and } \sigma((I - P_t)A|(I - P_t)\mathcal{H}) \subset [t, \infty). \tag{1.1}$$

Here and below $C|\mathcal{H}_1$ $(C \in \mathcal{L}(\mathcal{H}))$ means the restriction of C onto a subspace $\mathcal{H}_1$.

The aim of this paper is to prove the following theorem.

Theorem 1.1 *Let $A \in \mathcal{L}(\mathcal{H})$ with a real spectrum have a continuous maximal proper chain $\mathcal{P}$ separating its spectrum. Then $\mathcal{P}$ is also a proper chain of any $X \in \mathcal{L}(\mathcal{H})$ commuting with A.*

The proof of this theorem is presented in the next section.

Recall that the Macaev ideal $\mathcal{S}_\omega$ is defined as the set of compact operators K satisfying the condition

$$\sum_{k=1}^{\infty} \frac{s_k(K)}{2k-1} < \infty,$$

where $s_k(K)$ are the eigenvalues of $(K^*K)^{1/2}$ enumerated in the non-increasing order with the multiplicities taken into account, cf. [13, 20].

V. Macaev and Ju. Ljubich [19, 20] have shown that $A \in \mathcal{L}(\mathcal{H})$ with a real spectrum has a maximal chain separating $\sigma(A)$, provided $A_I \in \mathcal{S}_\omega$. Now Theorem 1.1 implies.

Corollary 1.2 *Let the maximal proper chain $\mathcal{P}$ of $A \in \mathcal{L}(\mathcal{H})$ with a real spectrum and $A_I \in \mathcal{S}_\omega$ be continuous. Then $\mathcal{P}$ is also a proper chain of any $X \in \mathcal{L}(\mathcal{H})$ commuting with A.*

This corollary can be considered as a particular generalization of the Lomonosov theorem for operators in a Hilbert space.

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.

2 Proofs of Theorems 1.1

1. Following the definitions from [13, Sec. V.1], we will say that an operator-valued function $P(t)$ defined on a bounded closed set Ω of real numbers is a standard projection function if

a) all values $P(t)$ $(t \in \Omega)$ are orthogonal projections in $\mathcal{H}$,

b) for all $t_1 < t_2$ $(t_1, t_2 \in \Omega)$ $P(t_1) < P(t_2)$,

c) $P(t)$ is strongly continuous on Ω, i.e. for any $f \in \mathcal{H}$, $P(t)f$ is continuous in the norm on Ω.

Let $\mathcal{P}$ be some chain and $P(t)$ some standard projection function. We will say that that $P(t)$ is obtained by the parameterization of $\mathcal{P}$ on Ω, if $\mathcal{P}$ coincides with the set of all values of $P(t)$. Naturally, $P(t)$ is said to be invariant for A if $P(t)AP(t) = AP(t)$ for all $t \in \Omega$.

A standard projection function is called continuous if the set of its values is a continuous chain. Recall that under consideration it is assumed that $\sigma(A)$ is real and $\mathcal{P}$ is continuous. Put $a = \inf \sigma(A)$ and $b = \sup \sigma(A)$. Since $\mathcal{P}$ is continuous, due to [13, Theorem V.1.1], $\mathcal{P}$ admits a parameterization $P(t)$ on $[a, b]$, such that $(P(t)f, f)_{\mathcal{H}}$ is absolutely continuous for any $f \in \mathcal{H}$.

If $P(t)$ is a parameterization of the maximal chain $\mathcal{P}$ separating $\sigma(A)$, we will say that $P(t)$ separates $\sigma(A)$. In this case, according to (1.1),

$$\sigma(AP(t)|P(t)\mathcal{H}) = [a, t] \text{ and } \sigma((I - P(t))A|(I - P(t))\mathcal{H}) = [t, b] \ \ (t \in [a, b]). \tag{2.1}$$

So

$$\sigma(AP(t)|P(t)\mathcal{H}) \cap \sigma((I - P(s))A|(I - P(s)) = \emptyset \ \ (a \le t < s \le b). \tag{2.2}$$

2. Let H_1, H_2 be Hilbert spaces, $C_k \in \mathcal{L}(H_k)$ $(k = 1, 2)$, and M be a bounded linear operator acting from H_1 into H_2. Assume that $\sigma(C_1) \cap \sigma(C_2) = \emptyset$. Then due to Theorem I.3.1 [5] (see also equation (3.10) from [5, Section I.3]), the operator equation

$$YC_1 - C_2Y = M$$

has a solution Y, which is a a bounded linear operator acting from H_1 into H_2 and representable as

$$Y = -\frac{1}{4\pi^2} \int_{L_2} \int_{L_1} \frac{1}{\lambda - \mu} R_\mu(C_2) M R_\lambda(C_1) d\lambda \, d\mu, \tag{2.3}$$

where L_k is a Jordan contour surrounding $\sigma(C_k)$ $(k = 1, 2)$.

If, in particular, $YC_2 = C_1Y$, then $M = 0$ and from (3.3) it follows that $Y = 0$.

To prove Theorem 1.1 we need the following lemma.

Lemma 2.1 *Let $A \in \mathcal{L}(\mathcal{H})$ with a real $\sigma(A)$ have a continuous invariant standard projection function $P(t)$ $(a \le t \le b)$. Let there be a $Z \in \mathcal{L}(\mathcal{H})$, such that $P(t)$ is invariant also for the operator $T = ZA - AZ$. Then $P(t)$ is invariant for Z.*

Proof: Since $P(t)$ is invariant for T, we have

$$(I - P(s))TP(s) = 0 \ \ (a \le s \le b),$$

and therefore,

$$(I - P(s))TP(s)P(t) = (I - P(s))TP(t) = 0 \ \ (a \le t < s \le b). \tag{2.4}$$

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.

For fixed $s, t \in (a, b)$ with $s > t$, put $P = P(t)$ and $Q = I - P(s)$. Then $QP = 0$. Since $P(s)$ is invariant for A, we can write $QAQ = QA$. Due to (2.2)

$$\sigma(AP|P\mathcal{H}) \cap \sigma(QA|Q\mathcal{H}) = \emptyset.$$

According to (2.4) $QTP = 0$, i.e.

$$Q(AZ - ZA)P = QAQZP - QZPAP = 0.$$

Making use of (2.3) with $M = 0$, $Y = QZP$, $C_2 = QA$ and $C_1 = AP$, we obtain $QXP = 0$, or

$$(I - P(s))ZP(t) = 0 \ \ (b > s > t \geq a).$$

Letting $s \to t$, and taking into account that $P(t)$ is continuous, we get $(I - P(t))ZP(t) = 0$, or $P(t)ZP(t) = ZP(t)$ for any $t \in (a, b)$. This proves the lemma. $\square$

Proof of Theorem 1.1: Take $Z = X$. Then $T = AX - XA = 0$. So $P(t)TP(t) = TP(t) = 0$. Now the required result follows from Lemma 2.1. $\square$

3 Example

Let $L^2(0, 1)$ be the Hilbert space of complex functions f defined on $[0, 1]$ with the traditional scalar product and norm $\|f\| = (\int_0^1 |f(s)|^2 ds)^{1/2}$. Consider the operator C defined by

$$(Cf)(x) = a(x)f(x) + \int_x^1 K(x, y)f(y)dy \ \ (x \in [0, 1];\ f \in L^2(0, 1)), \tag{3.1}$$

where $a(.)$ is a real continuous nondecreasing function defined on $[0, 1]$ and the kernel $K(x, y) : \{0 \leq x \leq y \leq 1\} \to \mathbb{C}$ satisfies the condition

$$\int_0^1 \int_x^1 |K(x, y)|^2 dy\, dx < \infty. \tag{3.2}$$

So the Volterra operator

$$V : f \to \int_x^1 K(x, y)f(y)dy \ \ (x \in [0, 1]; f \in L^2(0, 1))$$

is a Hilbert-Schmidt one. Note that condition (3.1) is imposed for the simplicity. Similarly one can consider the Schatten-von Neumann Volterra operators.

Introduce the projection function $\hat{P}(s)$ by

$$(\hat{P}(s)f)(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq s, \\ 0 & \text{if } s < x \leq 1 \end{cases} \tag{3.3}$$

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.

$(f \in L^2(0,1),\ s \in (0,1))$, $\hat{P}(0) = 0$, $\hat{P}(1) = I$. We have

$$(C\hat{P}(s)f)(x) = a(x)f(x) + \int_x^1 K(x,y)f(y)dy \text{ for } x \leq s, \text{ and } (A\hat{P}(s)f)(x) = 0 \text{ for } x > s.$$

Therefore, $C\hat{P}(s) = \hat{P}(s)C\hat{P}(s)$ $(0 \leq s \leq 1)$, i.e. the values of $\hat{P}(s)$ form the maximal continuous chain of C.

Due to Corollary 8.2 from [10], $\sigma(C) = \sigma(D)$, where D is defined by

$$(Df)(x) = a(x)f(x) \ \ (x \in [0,1], f \in L^2(0,1)),$$

and thus

$$\sigma(C) = \{z \in \mathbb{R} : z = a(x), 0 \leq x \leq 1\} = [a,b]$$

with $a = \inf_{a \leq x \leq b} a(x), b = \sup_{a \leq x \leq b} a(x)$. According to (3.2), $C - C^* = V - V^*$ is a Hilbert-Schmidt operator. Now Corollary 1.2 implies.

Corollary 3.1 *Let C be defined by (3.1) and condition (3.2) hold. Then for any $X \in \mathcal{L}(L^2(0,1))$ commuting with C the projection function $\hat{P}(s)$ defined by (3.3) is invariant.*

References

[1] Arias de Reyna, J.; Diestel, J.; Lomonosov, V. and Rodriguez-Piazza, L. Some observations about the space of weakly continuous functions from a compact space into a Banach space. *Quaestiones Math. 15* (1992), no. 4, 415–425.

[2] Beauzamy, B., *Introduction to Operator Theory and Invariant Subspaces.* North-Holland Mathematical Library, 42. North-Holland Publishing Co., Amsterdam, 1988.

[3] Brodskii, M.S. *Triangular and Jordan Representations of Linear Operators*, Transl. Math. Mongr., Vol. 32, Amer. Math. Soc., Providence, R. I., 1971.

[4] Chalendar, I. and Partington, J. R., Invariant subspaces for products of Bishop operators. *Acta Sci. Math. (Szeged) 74* (2008), no. 3-4, 719-727.

[5] Daleckii, Yu. L. and Krein, M.G. *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R. I., 1974.

[6] Gallardo-Gutiérrez, E. A. and Gorkin, P., Minimal invariant subspaces for composition operators. *J. Math. Pures Appl. (9) 95* (2011), no. 3, 245-259.

[7] Gamal', M.F. On existence of shift-type invariant subspaces for polynomially bounded operators. *J. Operator Theory, 84* (2020), no. 1, 3-34.

[8] Gil', M.I. Perturbations of invariant subspaces of operators with Hilbert - Schmidt Hermitian components *Arch. Math. 105* (2015), 447-452

[9] Gil', M.I. Perturbations of invariant subspaces of compact operators. *Acta Sci. Math. (Szeged) 82* (2016), no. 1-2, 271-279.

[10] Gil, M.I. *Operator Functions and Operator Equations.* World Scientific, New Jersey, 2018.

[11] Gohberg, I.C., Goldberg. S. and Kaashoek M.A. *Classes of Linear Operators, Vol. 2*, Birkhäuser Verlag, Basel, 1993.

[12] Gohberg, I. C. and Krein, M. G. *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, Vol. 18, Amer. Math. Soc., R.I. 1969.

[13] Gohberg, I.C. and Krein, M.G. *Theory and Applications of Volterra Operators in a Hilbert Space*. Translated from the Russian. In: Translations of Mathematical Monographs, vol. 24. Amer. Math. Soc., Providence, R. I. 1970.

[14] Gurdal, M. Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener, algebra, *Expo. Math. 27* (2009) 153-160

[15] Kim, Jaewoong, On invariant subspaces of operators in the class θ . *J. Math. Anal. Appl. 396* (2012), no. 2, 562-568.

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.

[16] Liu, Mingxue, Common invariant subspaces for finitely quasinilpotent collections of positive operators on a Banach space with a Schauder basis. *Rocky Mountain J. Math. 37* (2007), no. 4, 1187-1193.

[17] Lomonosov, V.I. Invariant subspaces of the family of operators which commute with a completely continuous operator, *Funkctional Anal. and Prilozen., 7(3)* (1973), 55-56.

[18] Lomonosov, V.I. and Shul'man, V. S. Halmos problems and related results in the theory of invariant subspaces. (Russian) *Uspekhi Mat. Nauk 73* (2018), no. 1(439), 35-98; translation in *Russian Math. Surveys 73* (2018), no. 1, 31-90.

[19] Lyubich, Yu. I. and Macaev, V.I. On operators with separable spectrum, *Mathem. Sbornik*, **56 (98)**, 4, (1962). 433-468 (Russian). English translation: *Amer. Math. Soc. Transl* (Series 2), **47** (1965), 89–129.

[20] Macaev, V.I. A class of completely continuous operators, *Dokl. Akad. Nauk SSSR* **139 (2)**, (1961) 548-551 (Russian); English translation: *Soviet Math. Dokl.* **1**, (1961), 972–975.

[21] Maji, A. and Sankar, T. R. Doubly commuting mixed invariant subspaces in the polydisc. *Bull. Sci. Math. 172* (2021), Paper No. 103051, 22 pp.

[22] Maltese, G., The role of convexity in existence theorems for invariant and hyperinvariant subspaces in Hilbert space, *Rend. Circ. Mat. Palermo (2), 49* (2000), no. 2, 381-390.

[23] Partington, J. and Smith, R. C., L_1-factorizations and invariant subspaces for weighted composition operators. *Arch. Math. (Basel) 87* (2006), no. 6, 564-571.

[24] Ringel, C. M. and Schmidmeier, M., Invariant subspaces of nilpotent linear operators. I. *J. Reine Angew. Math. 614* (2008), 1-52.

[25] Sasvári, Z. New proof of Naĭmark's theorem on the existence of nonpositive invariant subspaces for commuting families of unitary operators in Pontryagin spaces. *Monatsh. Math. 109* (1990), no. 2, 153-156.

16 Feb 2023 01:43:10 PST
230216-Gil Version 1 - Submitted to Rocky Mountain J. Math.