

SOME PERMANENCE PROPERTIES FOR CROSSED PRODUCTS BY COMPACT GROUP ACTIONS WITH THE TRACIAL ROKHLIN PROPERTY

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ABSTRACT. In this paper, we give some properties of the fixed point algebra and the crossed product of a unital separable simple infinite dimensional C^* -algebra by an action of a second-countable compact group with the tracial Rokhlin property with comparison that could be deduced from the properties of its original algebra: (1) stable rank one; (2) real rank zero; (3) β -comparison; (4) Winter's n -comparison; (5) m -almost divisible; (6) weakly (m,n) -divisible.

1. Introduction

The Rokhlin property for the case of a single automorphism was originally introduced for von Neumann algebras by Connes in [6]. Later, the Rokhlin property for finite group actions on C^* -algebras first appeared in the work of Herman and Jones in [18] and [19]. However, the finite group actions with the Rokhlin property are rare. Phillips, in [27], introduced the tracial Rokhlin property for finite group actions on unital simple C^* -algebras. In [21], Hirshberg and Winter also introduced the Rokhlin property for second-countable compact group actions on unital simple C^* -algebras. Since then, crossed products by compact group actions with the Rokhlin property have been studied by several authors (see [21, 11, 12]). For the non-unital case, Santiago and Gardella studied the Rokhlin property for finite group actions on non-unital simple C^* -algebras in [33] and [16]. Forough and Golestani studied the tracial Rokhlin property for finite group actions on non-unital simple C^* -algebras in [10]. More recently, Mohammadkarimi and Phillips studied the tracial Rokhlin property with comparison for compact group actions and proved that the crossed product of a unital separable simple infinite dimensional C^* -algebra with tracial rank zero by an action of a second-countable compact group with the tracial Rokhlin property with comparison has again tracial rank zero in [25] and some other permanence properties. Moreover, they gave some examples of compact group actions with the tracial Rokhlin property with comparison.

In [9], Fan and Fang proved that for a unital separable simple infinite dimensional C^* -algebra A , a finite group G and an action $\alpha: G \rightarrow \text{Aut}(A)$ with the tracial Rokhlin property, if A has stable rank one then the crossed product $A \rtimes_{\alpha} G$ has stable rank one, and if A has real rank zero then the crossed product $A \rtimes_{\alpha} G$ has real rank zero. In [11], Gardella proved that for a σ -unital C^* -algebra A , a second-countable compact group G and an action $\alpha: G \rightarrow \text{Aut}(A)$ with the Rokhlin property, if A has stable rank one then the fixed point algebra A^{α} and the crossed product $A \rtimes_{\alpha} G$ have stable rank one, and if A has real rank zero then the fixed point algebra A^{α} and the crossed product $A \rtimes_{\alpha} G$ have real rank zero. In this paper, we prove that for a unital separable simple infinite dimensional C^* -algebra A ,

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1 a second-countable compact group G and an action $\alpha: G \rightarrow \text{Aut}(A)$ with the tracial tracial Rokhlin
 2 property with comparison, if A has stable rank one then the fixed point algebra A^α and the crossed
 3 product $A \rtimes_\alpha G$ have stable rank one, and if A has real rank zero then the fixed point algebra A^α and
 4 the crossed product $A \rtimes_\alpha G$ have real rank zero.

5 Besides, comparison is an important property of C^* -algebra. Toms and Winter conjecture that strict
 6 comparison of positive elements, finite nuclear dimension and \mathcal{L} -stability are equivalent for unital
 7 separable nuclear infinite dimensional C^* -algebras in [35]. Kirchberg and Rørdam introduced the
 8 property of β -comparison in [23]. The property of n -comparison was introduced by Winter in [36].
 9 Gardella, Hirshberg, Santiago and Vaccaro explored connections between comparison and the tracial
 10 Rokhlin property in [13] and [14]. In [25], Mohammadkarimi and Phillips proved that the radius of
 11 comparison of the fixed point algebra of an unital separable simple infinite dimensional C^* -algebra
 12 by an action of a second-countable compact group with the tracial Rokhlin property with comparison
 13 is no larger than the radius of comparison of the original algebra. In this paper, we prove that for
 14 a unital separable simple infinite dimensional C^* -algebra A , a second-countable compact group G
 15 and an action $\alpha: G \rightarrow \text{Aut}(A)$ with the tracial tracial Rokhlin property with comparison, if A has
 16 β -comparison then the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ have β -comparison, and
 17 if A has Winter's n -comparison then the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ have
 18 Winter's n -comparison.

19 Divisibility is also important for C^* -algebra. The property of m -almost divisibility was introduced
 20 by Robert and Tikuisis in [30]. The property of weakly (m,n) -divisibility was introduced by Robert
 21 and Rørdam in [31]. In this paper, we prove that for a unital separable simple infinite dimensional
 22 C^* -algebra A , a second-countable compact group G and an action $\alpha: G \rightarrow \text{Aut}(A)$ with the tracial
 23 tracial Rokhlin property with comparison, if A is m -almost divisible then the fixed point algebra A^α
 24 and the crossed product $A \rtimes_\alpha G$ are m -almost divisible, and if A is weakly (m,n) -divisible then the
 25 fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ are weakly (m,n) -divisible.

26 The regular properties considered in this paper are not at all independent. In [1, Theorem 8.12] it is
 27 shown that a separable C^* -algebra with stable rank one and no nonzero elementary ideal-quotients
 28 (for example, a simple, separable, stable rank one C^* -algebra) has strict comparison whenever it
 29 has β -comparison for some β , or Winter's n -comparison for some n . The simple case was proved
 30 earlier in [34]. Following terminology of Winter [35], one says that a C^* -algebra is (n,m) -pure if its
 31 Cuntz semigroup has Winter's n -comparison and is m -almost divisible. By [2, Theorem D], a simple
 32 C^* -algebra is pure (that is, $(0,0)$ -pure) whenever it is (n,m) -pure for some n, m .

33 To be precise, we get the following results.

34 **Theorem 1.1.** *The following properties of unital separable simple infinite dimensional C^* -algebras by*
 35 *actions of second-countable compact groups with the tracial Rokhlin property with comparison pass*
 36 *from the original algebras to the fixed point algebras and the crossed products:*
 37

- 38 (1) *stable rank one;*
- 39 (2) *real rank zero;*
- 40 (3) *β -comparison;*
- 41 (4) *Winter's n -comparison;*
- 42 (5) *m -almost divisible;*

1 (6) weakly (m,n) -divisible.

2 The paper is organized as follows. Section 2 contains some preliminaries about central sequence
3 algebras, Cuntz subequivalence and actions of second-countable compact groups with the tracial
4 Rokhlin property with comparison. Section 3 shows the permanence of stable rank one and real rank
5 zero. Section 4 shows the permanence of β -comparison and Winter's n -comparison. Section 5 shows
6 the permanence of m -almost divisibility and weakly (m,n) -divisibility.
7

8 2. Preliminaries and Definitions

9
10 In this section, we recall some definitions and known facts about central sequence algebras, Cuntz
11 subequivalence and the tracial Rokhlin property with comparison for second-countable compact group
12 actions.

13 **Definition 2.1.** Let A be a unital C^* -algebra. We use $l^\infty(\mathbb{N}, A)$ to denote the set of all bounded
14 sequences in A with the supremum norm which is a unital C^* -algebra with the unit as the constant
15 sequence 1. Let

$$16 c_0(\mathbb{N}, A) = \{(a_n)_{n \in \mathbb{N}} \in l^\infty(\mathbb{N}, A) : \lim_{n \rightarrow \infty} \|a_n\| = 0\}.$$

17
18 It is obvious that $c_0(\mathbb{N}, A)$ is a closed two-side ideal in $l^\infty(\mathbb{N}, A)$, and we use the notation A_∞ to
19 denote the quotient $l^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$. Denoted by $\kappa_A: l^\infty(\mathbb{N}, A) \rightarrow A_\infty$ the quotient map. Define
20 $\iota: A \rightarrow l^\infty(\mathbb{N}, A)$ by $\iota(a) = (a, a, a, \dots)$, the constant sequence, for all $a \in A$. Identify A with $\kappa_A \circ \iota(A)$.
21 Denoted by $A_\infty \cap A'$ the relative commutant of A inside of A_∞ .

22 Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A , then it induced actions of G on $l^\infty(\mathbb{N}, A)$ and on A_∞ ,
23 denoted by α^∞ and α_∞ . Since for any $g \in G$,

$$24 (\alpha_\infty)_g(A_\infty \cap A') \subseteq A_\infty \cap A',$$

25
26 so we also use α_∞ to denote the restricted action on $A_\infty \cap A'$. These actions are not necessarily
27 continuous when G is not discrete. Therefore, we set

$$28 l_\alpha^\infty(\mathbb{N}, A) = \{a \in l^\infty(\mathbb{N}, A) : g \mapsto \alpha_g^\infty(a) \text{ is continuous}\},$$

29
30 and $A_{\infty, \alpha} = \kappa_A(l_\alpha^\infty(\mathbb{N}, A))$. Then, $A_{\infty, \alpha}$ is invariant under α_∞ and the action α_∞ is continuous.

31
32 **Definition 2.2.** Let G be a locally compact group, we denote the action induced by left translation of
33 G on itself by $\text{Lt}: G \rightarrow \text{Aut}(C_0(G))$.

34 **Definition 2.3.** Let A be a C^* -algebra, $a \in A_+$ and $\varepsilon > 0$. Then we denote $f(a)$ by $(a - \varepsilon)_+$, where
35 $f(t) = \max\{0, t - \varepsilon\}$ is continuous from $[0, \infty)$ to $[0, \infty)$.
36

37 The following definitions related to Cuntz comparison are from [7], for more information, you can
38 refer to [15] and [3].

39 **Definition 2.4.** Let A be a C^* -algebra. Let $a, b \in (A \otimes K)_+$.

40 (1) We say that a is Cuntz subequivalent to b (written $a \precsim_A b$), if there is a sequence $(r_n)_{n=1}^\infty$ in
41 $A \otimes K$ such that $\lim_{n \rightarrow \infty} \|r_n^* b r_n - a\| = 0$.
42

1 (2) We say that a is Cuntz equivalent to b (written $a \sim_A b$), if $a \preceq_A b$ and $b \preceq_A a$. This is an
 2 equivalence relation, we use $\langle a \rangle_A$ to denote the equivalence class of a . With the addition
 3 operation $\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$ and the order operation $\langle a \rangle_A \leq \langle b \rangle_A$ if $a \preceq_A b$, $\text{Cu}(A) = (A \otimes$
 4 $K)_+ / \sim_A$ is an ordered semigroup which we called Cuntz semigroup. $W(A) = M_\infty(A)_+ / \sim_A$
 5 is also an ordered semigroup with the same operation and order as above.

6 If B is a hereditary C^* -subalgebra of A , and $a, b \in B_+$, then it is easy to check that $a \preceq_A b \iff a \preceq_B b$.

7
 8 We give the following known facts about Cuntz subequivalence. Part (1) is contained in [22,
 9 Proposition 2.6] (and in a slightly different form in the earlier [32, Proposition 2.4]). Part (2) is
 10 contained in [32, Proposition 2.4]. Part (3) is immediate (and is [22, Lemma 2.5(i)]). Part (4) is [28,
 11 Corollary 1.6]. Part (5) is [22, Lemma 2.8(ii)]. Part (6) is [22, Lemma 2.8(iii)], and Part (7) is [22,
 12 Lemma 2.9].

13 **Lemma 2.5.** *Let A be a C^* -algebra.*

- 14 (1) *Let $a, b \in A_+$. Then the following are equivalent:*
 15 (a) $a \preceq_A b$;
 16 (b) $(a - \varepsilon)_+ \preceq_A b$ for all $\varepsilon > 0$;
 17 (c) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $(a - \varepsilon)_+ \preceq_A (b - \delta)_+$.
 18 (2) *Let $a, b \in A_+$. If $a \leq b$, then we have $a \preceq_A b$.*
 19 (3) *Let $a \in A_+$ and $\varepsilon_1, \varepsilon_2 > 0$. Then we have $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.*
 20 (4) *Let $\varepsilon > 0$ and $\lambda \geq 0$, $a, b \in A_+$. If $\|a - b\| < \varepsilon$, then we have $(a - \varepsilon - \lambda)_+ \preceq_A (b - \lambda)_+$.*
 21 (5) *Let $a, b \in A_+$, then we have $a + b \preceq_A a \oplus b$.*
 22 (6) *Let $a, b \in A_+$ with $ab = 0$, then we have $a + b \sim_A a \oplus b$.*
 23 (7) *Let $a_1, a_2, b_1, b_2 \in A_+$. If $a_1 \preceq_A a_2$ and $b_1 \preceq_A b_2$, then we have $a_1 \oplus a_2 \preceq_A b_1 \oplus b_2$.*

25 **Definition 2.6.** [20, Definition 1.3] Let G be a compact group, and let A be a C^* -algebra, B be a
 26 C^* -algebra. Let $\alpha: G \rightarrow \text{Aut}(A)$ and $\gamma: G \rightarrow \text{Aut}(B)$ be actions of G on A and B . Let $F \subseteq A$ and
 27 $S \subseteq B$ be subsets, and let $\varepsilon > 0$. A completely positive contractive map $\varphi: A \rightarrow B$ is said to be an
 28 (F, S, ε) -approximately central equivariant multiplicative map if:

- 29 (1) $\|\varphi(xy) - \varphi(x)\varphi(y)\| < \varepsilon$ for all $x, y \in F$.
 30 (2) $\|\varphi(x)a - a\varphi(x)\| < \varepsilon$ for all $x \in F$ and all $a \in S$.
 31 (3) $\sup_{g \in G} \|\varphi(\alpha_g(x)) - \gamma_g(\varphi(x))\| < \varepsilon$ for all $x \in F$.
 32

33 **Definition 2.7.** [25, Definition 1.4] Let A and B be C^* -algebras, and let $F \subseteq A$. A completely positive
 34 contractive map $\varphi: A \rightarrow B$ is said to be an (n, F, ε) -approximately multiplicative map if whenever
 35 $m \in \{1, 2, \dots, n\}$ and $x_1, x_2, \dots, x_m \in F$, we have

$$36 \|\varphi(x_1 x_2 \cdots x_m) - \varphi(x_1)\varphi(x_2)\cdots\varphi(x_m)\| < \varepsilon.$$

37
 38 If $S \subseteq B$ is also given, then φ is said to be an (n, F, S, ε) -approximately central multiplicative map if, in
 39 addition, $\|\varphi(x)a - a\varphi(x)\| < \varepsilon$ for all $x \in F$ and all $a \in S$.
 40

41 Now, let us recall the notion of the tracial Rokhlin property with comparison for second-countable
 42 compact group actions defined by Mohammadkarimi and Phillips in [25].

Definition 2.8. [25, Definition 2.4] Let G be a second-countable compact group, let A be a unital simple infinite dimensional C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. We say that the action α has the tracial Rokhlin property with comparison if for any $\varepsilon > 0$, any finite set $F \subseteq A$, any finite set $S \subseteq C(G)$, any $x \in A_+$ with $\|x\| = 1$, and any $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely positive map $\psi: C(G) \rightarrow pAp$ such that

- (1) ψ is an (F, S, ε) -approximately central equivariant multiplicative map.
- (2) $1 - p \underset{A}{\sim} x$.
- (3) $1 - p \underset{A^\alpha}{\sim} y$.
- (4) $1 - p \underset{A^\alpha}{\sim} p$.
- (5) $\|pxp\| > 1 - \varepsilon$.

The next theorem is the key tool for transferring properties from the original algebra to the fixed point algebra.

Theorem 2.9. [25, Theorem 2.17] Let G be a second-countable compact group, let A be a unital separable simple infinite dimensional C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the tracial Rokhlin property with comparison. Then for any $\varepsilon > 0$, any $n \in \mathbb{N}$, any compact subset $F_1 \subseteq A$, any compact subset $F_2 \subseteq A^\alpha$, any $x \in A_+$ with $\|x\| = 1$, and any $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely positive map $\varphi: A \rightarrow pA^\alpha p$ such that

- (1) φ is an $(n, F_1 \cup F_2, \varepsilon)$ -approximately multiplicative map.
- (2) $\|pa - ap\| < \varepsilon$ for all $a \in F_1 \cup F_2$.
- (3) $\|\varphi(a) - pap\| < \varepsilon$ for all $a \in F_2$.
- (4) $\|\varphi(a)\| \geq \|a\| - \varepsilon$ for all $a \in F_1 \cup F_2$.
- (5) $1 - p \underset{A}{\sim} x$.
- (6) $1 - p \underset{A^\alpha}{\sim} y$.
- (7) $1 - p \underset{A^\alpha}{\sim} p$.
- (8) $\|pxp\| > 1 - \varepsilon$.

The Condition $1 - p \underset{A^\alpha}{\sim} p$ is just used to prove that the algebras $A \rtimes_\alpha G$ and A^α are Morita equivalent (see [25, Proposition 3.7]), without it, we can not transfer properties from the original algebra to the crossed product. However, the other conclusions still hold if we omit this condition. In other words, without $1 - p \underset{A^\alpha}{\sim} p$, we can still transfer properties from the original algebra to the fixed point algebra. So we consider the following version of tracial Rokhlin property for compact group actions.

Definition 2.10. Let G be a second-countable compact group, let A be a unital simple infinite dimensional C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. We say that the action α has the tracial Rokhlin property if for any $\varepsilon > 0$, any finite set $F \subseteq A$, any finite set $S \subseteq C(G)$, any $x \in A_+$ with $\|x\| = 1$, and any $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely positive map $\psi: C(G) \rightarrow pAp$ such that

- (1) ψ is an (F, S, ε) -approximately central equivariant multiplicative map.
- (2) $1 - p \underset{A}{\sim} x$.
- (3) $1 - p \underset{A^\alpha}{\sim} y$.
- (4) $\|pxp\| > 1 - \varepsilon$.

Lemma 2.11. *Let G be a second-countable compact group, let A be a unital separable simple infinite dimensional C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A with the tracial Rokhlin property with comparison. Let q be an α -invariant projection in A . Set $B = qAq$, and denote by $\beta: G \rightarrow \text{Aut}(B)$ the compressed action of G . Then β has the tracial Rokhlin property.*

Proof. Let $\varepsilon > 0$, let $F \subseteq B$, and let $S \subseteq C(G)$ be finite subsets, let $x \in B_+$ with $\|x\| = 1$ and let $y \in (B^\beta)_+ \setminus \{0\}$. Without loss of generality, we assume that F and S contain only contractions. We will show that there exist a projection $p \in B^\beta$ and a unital completely positive map $\psi: C(G) \rightarrow pBp$ such that

- (1) ψ is an (F, S, ε) -approximately central equivariant multiplicative map.
- (2) $1 - p \underset{B}{\prec} x$.
- (3) $1 - p \underset{B^\beta}{\prec} y$.
- (4) $\|pxp\| > 1 - \varepsilon$.

Since α has the tracial Rokhlin property, for $F_1 = F \cup \{q\} \subseteq A$, for $S \subseteq C(G)$, for $x \in B_+ \subseteq A_+$ and for $y \in (B^\beta)_+ \setminus \{0\} \subseteq (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p_1 \in A^\alpha$ and a unital completely positive map $\psi_1: C(G) \rightarrow p_1 A p_1$ such that

- (5) ψ_1 is an $(F_1, S, \frac{\varepsilon}{3})$ -approximately central equivariant multiplicative map.
- (6) $1 - p_1 \underset{A}{\prec} x$.
- (7) $1 - p_1 \underset{A^\alpha}{\prec} y$.
- (8) $\|p_1 x p_1\| > 1 - \frac{\varepsilon}{3}$.

Set $p = qp_1q \in B^\beta$ and define a unital completely positive map $\psi: C(G) \rightarrow pBp$ by

$$\psi(f) = q\psi_1(f)q \quad \text{for all } f \in C(G).$$

For condition (1), for $f \in S$ and $a \in F$, we have

$$\begin{aligned} \|\psi(f)a - a\psi(f)\| &= \|q\psi_1(f)qa - aq\psi_1(f)q\| \\ &\leq \|q\psi_1(f)qa - \psi_1(f)qa\| + \|\psi_1(f)qa - aq\psi_1(f)\| + \|aq\psi_1(f) - aq\psi_1(f)q\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

For $f \in S$ and $g \in G$, we have

$$\begin{aligned} \|\psi(\text{Lt}_g(f)) - \beta_g(\psi(f))\| &= \|q\psi_1(\text{Lt}_g(f))q - \beta_g(q\psi_1(f)q)\| \\ &= \|q(\psi_1(\text{Lt}_g(f)) - \alpha_g(\psi_1(f)))q\| \\ &\leq \|\psi_1(\text{Lt}_g(f)) - \alpha_g(\psi_1(f))\| < \frac{\varepsilon}{3}. \end{aligned}$$

Thus for $f \in S$, we have

$$\sup_{g \in G} \|\psi(\text{Lt}_g(f)) - \beta_g(\psi(f))\| \leq \frac{\varepsilon}{3} < \varepsilon.$$

1 For $f_1, f_2 \in S$, we have

$$\begin{aligned} 2 \quad & \|\psi(f_1 f_2) - \psi(f_1)\psi(f_2)\| = \|q\psi_1(f_1 f_2)q - q\psi_1(f_1)q\psi_1(f_2)q\| \\ 3 \quad & \leq \|q\psi_1(f_1 f_2)q - q\psi_1(f_1)\psi_1(f_2)q\| + \|q\psi_1(f_1)\psi_1(f_2)q - q\psi_1(f_1)q\psi_1(f_2)q\| \\ 4 \quad & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon. \end{aligned}$$

5 Therefore, condition (1) holds.

6 For condition (2), note that $1_B - p = q(1 - p_1)q \lesssim_A 1 - p_1 \lesssim_A x$ and B is a hereditary C^* -algebra of A , it follows that $1_B - p \lesssim_B x$.

7 For condition (3), note that $1_B - p = q(1 - p_1)q \lesssim_{A^\alpha} 1 - p_1 \lesssim_{A^\alpha} y$ and B^β is a hereditary C^* -algebra of A^α , it follows that $1_B - p \lesssim_{B^\beta} y$.

8 For condition (4), we have

$$9 \quad \|pxp - x\| = \|qp_1qxqp_1q - x\| = \|qp_1xp_1q - qxq\| \leq \|p_1xp_1 - x\| < \frac{\varepsilon}{3}.$$

10 Thus, $\|pxp\| > 1 - \frac{\varepsilon}{3} > 1 - \varepsilon$. □

11 *Remark 2.12.* In general, the tracial Rokhlin property with comparison usually can not pass to corners since we can not always get $q(1 - p)q \lesssim_{B^\beta} qpq$ from $1 - p \lesssim_{A^\alpha} p$ for any α -invariant projection q in A and $B = qAq$.

12 **Theorem 2.13.** (cf. [25, Theorem 2.17]) *Let G be a second-countable compact group, let A be a unital separable simple infinite dimensional C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the tracial Rokhlin property. Then for any $\varepsilon > 0$, any $n \in \mathbb{N}$, any compact subset $F_1 \subseteq A$, any compact subset $F_2 \subseteq A^\alpha$, any $x \in A_+$ with $\|x\| = 1$, and any $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely positive map $\varphi: A \rightarrow pA^\alpha p$ such that*

- 13 (1) φ is an $(n, F_1 \cup F_2, \varepsilon)$ -approximately multiplicative map.
- 14 (2) $\|pa - ap\| < \varepsilon$ for all $a \in F_1 \cup F_2$.
- 15 (3) $\|\varphi(a) - pap\| < \varepsilon$ for all $a \in F_2$.
- 16 (4) $\|\varphi(a)\| \geq \|a\| - \varepsilon$ for all $a \in F_1 \cup F_2$.
- 17 (5) $1 - p \lesssim_A x$.
- 18 (6) $1 - p \lesssim_{A^\alpha} y$.
- 19 (7) $\|pxp\| > 1 - \varepsilon$.

20 *Proof.* The proof is essentially the same as that of [25, Theorem 2.17], we only omit the condition $1 - p \lesssim_{A^\alpha} p$. □

21 **Theorem 2.14.** [25, Theorem 3.9, Corollary 3.10] *Let G be a second-countable compact group, let A be a unital separable simple infinite dimensional C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the tracial Rokhlin property with comparison. Then the crossed product $A \rtimes_\alpha G$ is simple. Moreover, the algebras $A \rtimes_\alpha G$ and A^α are Morita equivalent and stably isomorphic.*

39 3. Stable rank one and Real rank zero

40 In this section, we prove that the fixed point algebra and the crossed product of a unital separable simple infinite dimensional C^* -algebra with stable rank one (respectively, real rank zero) by an action

1 of a second-countable compact group with the tracial Rokhlin property with comparison have again
 2 stable rank one (respectively, real rank zero). First, we introduce the definitions of stable rank one and
 3 real rank zero.

4 **Definition 3.1.** Let A be a C^* -algebra.

- 5 (1) If A is unital, it is said to have stable rank one if the set of invertible elements is dense in A ,
 6 written $\text{tsr}(A) = 1$. If A is not unital, it is said to have stable rank one if for its unitization \tilde{A} ,
 7 we have $\text{tsr}(\tilde{A}) = 1$.
 8
 9 (2) If A is unital, it is said to have real rank zero if the set of invertible self-adjoint elements is
 10 dense in A_{sa} , written $\text{RR}(A) = 0$. If A is not unital, it is said to have real rank zero if for its
 11 unitization \tilde{A} , we have $\text{RR}(\tilde{A}) = 0$.

12 **Definition 3.2.** A C^* -algebra A is said to have the Property (SP), if every nonzero hereditary C^* -
 13 subalgebra of A contains a nonzero projection.

14 **Definition 3.3.** Let $\varepsilon > 0$. Define

$$f_\varepsilon(t) = \begin{cases} 0, & t \in [0, \varepsilon], \\ \text{linear}, & t \in [\varepsilon, 2\varepsilon]. \\ 1, & t \in [2\varepsilon, +\infty). \end{cases}$$

15
 16
 17
 18
 19
 20 Now, we will show the permanence of stable rank one. Given an element $x \in A^\alpha$, we will find an
 21 invertible element in A^α to approximate it. First, we find a zero-divisor $c \in A^\alpha$ to approximate x . Next,
 22 we show that there exists a projection $p_1 \in A^\alpha$ and find an invertible element $b \in A_1 = (1 - p_1)A(1 - p_1)$
 23 to approximate c . Then, we can use Theorem 2.13 to get a projection $p \in (A_1)^\alpha$ and an approximately
 24 multiplicative map $\varphi: A_1 \rightarrow p(A_1)^\alpha p$ and the sum of two elements to approximate c . One of them is
 25 $\varphi(b)$ which is invertible in $p(A_1)^\alpha p$, the other is Cuntz subequivalent to p_1 . Last, we find an invertible
 26 element in A^α to approximate c by matrix decomposition and some perturbations.

27
 28 **Theorem 3.4.** Let A be a unital separable simple infinite dimensional C^* -algebra with stable rank one.
 29 Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin
 30 property with comparison. Then the fixed point algebra A^α has stable rank one.

31 *Proof.* If A does not have the Property (SP), by [25, Lemma 2.8], α has the Rokhlin property. It
 32 follows that A^α has stable rank one (see [11, Proposition 4.13(2)]).

33 Thus we may assume that A has the Property (SP). Let $a \in A^\alpha$ and let $\varepsilon \in (0, 1)$. Without loss of
 34 generality, we may assume that a is not invertible and $\|a\| = 1$. A is stably finite since it has stable
 35 rank one (see [29, Lemma 3.5]). Thus A^α is stably finite. Therefore a is not one-sided invertible. By
 36 [24, Lemma 3.6.9], we may assume that there exists a zero-divisor $c \in A^\alpha$ such that
 37

$$\|a - c\| < \frac{\varepsilon}{4}.$$

38
 39
 40 Since A has the Property (SP), by [25, Corollary 4.4], so does A^α . Then, there exists a nonzero
 41 projection $e \in A^\alpha$ such that $ce = ec = 0$. Since A^α is not of type I, simple and unital (see [25, Theorem
 42 3.2] and [25, Proposition 3.3]), by [27, Lemma 1.10], there are nonzero projections $p_1, p_2 \in \text{Her}(e)$

1 such that $p_1 p_2 = 0$ and $p_1 \sim_{A^\alpha} p_2$. Set $A_1 = (1 - p_1)A(1 - p_1)$. Since A has stable rank one, it follows
 2 that A_1 has stable rank one. Note that $c \in A_1$, there exists an invertible element $b \in A_1$ such that

$$3 \quad \|c - b\| < \frac{\varepsilon}{8}.$$

4
 5
 6 Then, by Theorem 2.13, for $F_1 = \{b\}$, $F_2 = \{c\}$, there exist a projection $p \in (A_1)^\alpha$ and a unital
 7 completely positive map $\varphi: A_1 \rightarrow p(A_1)^\alpha p$ such that

- 8 (1) φ is an $(2, F_1 \cup F_2, \frac{\varepsilon}{8})$ -approximately multiplicative map.
 9 (2) $\|px - xp\| < \frac{\varepsilon}{8}$ for all $x \in F_1 \cup F_2$.
 10 (3) $\|\varphi(x) - pxp\| < \frac{\varepsilon}{8}$ for all $x \in F_2$.
 11 (4) $1_{A_1} - p \precsim_{(A_1)^\alpha} p_2$.

12
 13 Set $c_1 = pc p$ and $c_2 = (1_{A_1} - p)c(1_{A_1} - p)$. Then

$$14 \quad \|c - (c_1 + c_2)\| < \frac{\varepsilon}{4}.$$

15
 16
 17 Note that

$$18 \quad \|\varphi(b)\varphi(b^{-1}) - p\| = \|\varphi(b^{-1})\varphi(b) - p\| < \frac{\varepsilon}{8} < 1,$$

$$19 \quad \|\varphi(c) - pc p\| < \frac{\varepsilon}{8}.$$

20
 21
 22
 23 Then $\varphi(b)\varphi(b^{-1})$ and $\varphi(b^{-1})\varphi(b)$ are invertible in $p(A_1)^\alpha p = p(1 - p_1)A^\alpha(1 - p_1)p$. It follows that
 24 $\varphi(b)$ is invertible. Thus we have

$$25 \quad \|c_1 - \varphi(b)\| \leq \|c_1 - \varphi(c)\| + \|\varphi(c) - \varphi(b)\|$$

$$26 \quad < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

27
 28
 29
 30 By (4), there exists $u \in A^\alpha$ such that

$$31 \quad u^* u = 1_{A_1} - p = (1 - p)(1 - p_1) \text{ and } uu^* \leq p_1.$$

32
 33 Set $d = c_2 + \frac{\varepsilon}{16}u + \frac{\varepsilon}{16}u^* + \frac{\varepsilon}{4}(p_1 - uu^*)$. Therefore d , in a matrix form respect to $(1 - p)(1 - p_1)$,
 34 $p(1 - p_1)$, p_1 ;

$$35 \quad \begin{pmatrix} c_2 & \frac{\varepsilon}{16} & 0 \\ \frac{\varepsilon}{16} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon}{4} \end{pmatrix},$$

36
 37
 38
 39 is an invertible element in $(1 - p)A^\alpha(1 - p) = (1 - p)(1 - p_1)A^\alpha(1 - p_1)(1 - p) + p_1A^\alpha p_1$. Moreover,

$$40 \quad \|d - c_2\| < \frac{\varepsilon}{4}.$$

1 Hence, $\varphi(b) + d$ is invertible in A^α . Finally,

$$\begin{aligned}
 2 \quad & \|a - (\varphi(b) + d)\| \leq \|a - c\| + \|c - (\varphi(b) + d)\| \\
 3 \quad & < \frac{\varepsilon}{4} + \|c - (c_1 + c_2)\| + \|(c_1 + c_2) - (\varphi(b) + d)\| \\
 4 \quad & < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \|c_1 - \varphi(b)\| + \|c_2 - d\| \\
 5 \quad & < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
 6 \quad & = \varepsilon. \\
 7 \quad & \\
 8 \quad & \\
 9 \quad & \\
 10 \quad & \square
 \end{aligned}$$

11 **Corollary 3.5.** *Let A be a unital separable simple infinite dimensional C^* -algebra with stable rank*
 12 *one. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial*
 13 *Rokhlin property with comparison. Then the crossed product $A \rtimes_\alpha G$ has stable rank one.*

14 *Proof.* The algebra A^α has stable rank one by Theorem 3.4. Theorem 2.14 implies that $A \rtimes_\alpha G$ is
 15 Morita equivalent to A^α , so [29, Theorem 3.6] implies that $A \rtimes_\alpha G$ has stable rank one. \square

16 Now, we will show the permanence of real rank zero. The idea of the proof is essentially the same
 17 as that of Theorem 3.4, the only difference is replacing invertible elements with invertible self-adjoint
 18 elements.

19 **Theorem 3.6.** *Let A be a unital separable simple infinite dimensional C^* -algebra with real rank zero.*
 20 *Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin*
 21 *property with comparison. Then the fixed point algebra A^α has real rank zero.*

22 *Proof.* Since A has real rank zero, we know that A has the Property (SP). Let $a \in (A^\alpha)_{sa}$ and let
 23 $\varepsilon \in (0, 1)$. Without loss of generality, we may assume that a is not invertible and $\|a\| = 1$. Define a
 24 continuous function $f \in C([-1, 1])$ such that $0 \leq f \leq 1$ and $f(t) = 0$ if $|t| \geq \frac{\varepsilon}{4}$ and $f(t) = 1$ if $|t| < \frac{\varepsilon}{8}$.
 25 Since $0 \in \text{sp}(a)$, $f(a) \neq 0$. By [25, Corollary 4.4], there exists a nonzero projection $e \in \text{Her}(f(a)) \subseteq A^\alpha$.
 26 Since A^α is not of type I, simple and unital (see [25, Theorem 3.2] and [25, Proposition 3.3]), by [27,
 27 Lemma 1.10], there are nonzero projections $p_1, p_2 \in \text{Her}(e)$ such that $p_1 p_2 = 0$ and $p_1 \sim_{A^\alpha} p_2$. Let
 28 $c = f_{\frac{\varepsilon}{4}}(a)a$. Then $p_1 c = c p_1 = 0$ and

$$29 \quad \|a - c\| < \frac{\varepsilon}{4}.$$

30 Set $A_1 = (1 - p_1)A(1 - p_1)$. Since A has real rank zero, it follows that A_1 has real rank zero. Note that
 31 $c \in A_1$, there exists an invertible self-adjoint element $b \in A_1$ such that

$$32 \quad \|c - b\| < \frac{\varepsilon}{8}.$$

33 As in the same step in the argument in Theorem 3.4, we get a projection $p \in (A_1)^\alpha$ and a unital
 34 completely positive map $\varphi: A_1 \rightarrow p(A_1)^\alpha p$ such that $\varphi(b) \in (p(A_1)^\alpha p)_{sa} = (p(1 - p_1)A^\alpha(1 - p_1)p)_{sa}$
 35 is invertible and $d \in ((1 - p)(A^\alpha)(1 - p))_{sa} = ((1 - p)(1 - p_1)A^\alpha(1 - p_1)(1 - p) + p_1 A^\alpha p_1)_{sa}$ which
 36 is invertible such that

$$37 \quad \|c - (\varphi(b) + d)\| < \frac{3\varepsilon}{4}.$$

1 Moreover, we have $\varphi(b) + d$ is invertible in $(A^\alpha)_{sa}$ and

$$2 \quad \|a - \varphi(b) + d\| < \varepsilon.$$

3
4 □

5 **Corollary 3.7.** *Let A be a unital separable simple infinite dimensional C^* -algebra with real rank zero.*
6 *Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin*
7 *property with comparison. Then the crossed product $A \rtimes_\alpha G$ has real rank zero.*

8 *Proof.* The algebra A^α has real rank zero by Theorem 3.6. Theorem 2.14 implies that $A \rtimes_\alpha G$ is Morita
9 equivalent to A^α , so [5, Theorem 3.8] implies that $A \rtimes_\alpha G$ has real rank zero. □

11 4. β -comparison and Winter's n -comparison

12
13 In this section, we prove that the fixed point algebra and the crossed product of a unital separable simple
14 infinite dimensional stably finite C^* -algebra with β -comparison (respectively, Winter's n -comparison)
15 by an action of a second-countable compact group with the tracial Rokhlin property with comparison
16 have again β -comparison (respectively, Winter's n -comparison). First, we introduce the definitions of
17 β -comparison and Winter's n -comparison.

18 **Definition 4.1.** [23, Definition 2.2] Let A be a C^* -algebra and let $1 \leq \beta < \infty$. We say that A has
19 β -comparison if for all $\langle x \rangle, \langle y \rangle \in \text{Cu}(A)$ and all integers $k, l \geq 1$ with $k > \beta l$, the inequality $k\langle x \rangle \leq l\langle y \rangle$
20 implies $\langle x \rangle \leq \langle y \rangle$.

21
22 **Definition 4.2.** [36, Definition 3.1] Let A be a C^* -algebra and write $x <_s y$ to mean $(k+1)\langle x \rangle \leq k\langle y \rangle$
23 for some $k \in \mathbb{N}$. We say that A has Winter's n -comparison, if, whenever $\langle x \rangle, \langle y_0 \rangle, \langle y_1 \rangle, \dots, \langle y_n \rangle \in \text{Cu}(A)$
24 such that $\langle x \rangle <_s \langle y_j \rangle$ for all $j = 0, 1, \dots, n$, then $\langle x \rangle \leq \langle y_0 \rangle + \langle y_1 \rangle + \dots + \langle y_n \rangle$.

25 The proof of the following lemma is contained in the [25, Proposition 4.22] which is the same as
26 that of [4, Lemma 3.10].

27
28 **Lemma 4.3.** *Let G be a second-countable compact group, let A be a unital separable simple infinite*
29 *dimensional C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the tracial Rokhlin property*
30 *with comparison. Then, for every $x \in (A^\alpha)_+ \setminus \{0\}$, there exists $c \in (A^\alpha)_+$ such that $c \preceq_{A^\alpha} x$ and*
31 *$\text{sp}(c) = [0, 1]$.*

32 Now, we will show the permanence of β -comparison. The key of the proof is using [25, Proposition
33 4.21] to transfer the Cuntz subequivalence from the original algebra to the fixed point algebra. We use
34 [28, Lemma 2.7] to work with details.

35
36 **Theorem 4.4.** *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra with*
37 *β -comparison. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the*
38 *tracial Rokhlin property with comparison. Then the fixed point algebra A^α has β -comparison*

39 *Proof.* Let $a, b \in (A^\alpha \otimes K)_+$ and let $k, l \geq 1$ be integers with $k > \beta l$ such that

$$40 \quad k\langle a \rangle_{A^\alpha} \leq l\langle b \rangle_{A^\alpha}.$$

41
42 Since $\langle a \rangle_{A^\alpha} = \sup_{\varepsilon > 0} \langle (a - \varepsilon)_+ \rangle_{A^\alpha}$, we need to show that $\langle (a - \varepsilon)_+ \rangle_{A^\alpha} \preceq_{A^\alpha} b$ for any $\varepsilon > 0$.

1 Fix $0 < \varepsilon < \frac{1}{2}$. Choose $m \in \mathbb{N}$ such that

$$2 \frac{m}{m+1} \frac{k}{l} > \beta.$$

4 Then we have

$$5 mk \langle a \rangle_{A^\alpha} \leq ml \langle b \rangle_{A^\alpha}.$$

6 Therefore, by Lemma 2.5(1), there is $\delta > 0$ such that

$$8 mk \langle (a - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq ml \langle (b - \delta)_+ \rangle_{A^\alpha}.$$

10 Choose $n \in \mathbb{N}$ and $a_0, b_0 \in M_n(A^\alpha)_+$ such that

$$11 \|a_0 - (a - \frac{\varepsilon}{3})_+\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|b_0 - (b - \frac{\delta}{3})_+\| < \frac{\delta}{6}.$$

13 Then we have

$$14 (a - \varepsilon)_+ \preceq_{A^\alpha} (a_0 - \frac{\varepsilon}{3})_+ \preceq_{A^\alpha} (a - \frac{\varepsilon}{3})_+$$

15 and

$$17 (b - \delta)_+ \preceq_{A^\alpha} (b_0 - \frac{\delta}{3})_+ \preceq_{A^\alpha} (b_0 - \frac{\delta}{6})_+ \preceq_{A^\alpha} (b - \frac{\delta}{3})_+.$$

18 Thus we have

$$20 mk \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq ml \langle (b_0 - \frac{\delta}{3})_+ \rangle_{A^\alpha}.$$

21 By [25, Proposition 4.20], the action $\alpha \otimes \text{id}_{M_n} : G \rightarrow \text{Aut}(A \otimes M_n)$ also has the tracial Rokhlin
 22 property with comparison. Thus $M_n(A^\alpha)$ is not of type I, simple and infinite dimensional, by [28,
 23 Lemma 2.7], there are positive elements $c \in M_n(A^\alpha)_+$ and $d \in M_n(A^\alpha)_+ \setminus \{0\}$ such that

$$25 m \langle (b_0 - \frac{\delta}{3})_+ \rangle_{A^\alpha} \leq (m+1) \langle c \rangle_{A^\alpha} \quad \text{and} \quad \langle c \rangle_{A^\alpha} + \langle d \rangle_{A^\alpha} \leq \langle (b_0 - \frac{\delta}{6})_+ \rangle_{A^\alpha}.$$

26 By Lemma 4.3, there is $d_0 \in M_n(A^\alpha)_+$ such that $d_0 \preceq_{A^\alpha} d$ and $\text{sp}(d_0) = [0, 1]$. Thus we have

$$28 \langle c \rangle_{A^\alpha} + \langle d_0 \rangle_{A^\alpha} \leq \langle (b_0 - \frac{\delta}{6})_+ \rangle_{A^\alpha},$$

30 and

$$31 mk \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq (m+1)l \langle c \rangle_{A^\alpha}.$$

32 Since $A^\alpha \subset A$ and A has β -comparison and

$$34 mk > \beta(m+1)l,$$

35 we have $(a_0 - \frac{\varepsilon}{3})_+ \preceq_A c$. Therefore,

$$37 (a - \varepsilon)_+ \preceq_{A^\alpha} (a_0 - \frac{\varepsilon}{3})_+ \preceq_A c \oplus d_0.$$

38 Note that 0 is a limit point of $\text{sp}(c + d_0)$, by [25, Proposition 4.21],

$$40 (a - \varepsilon)_+ \preceq_{A^\alpha} (a_0 - \frac{\varepsilon}{3})_+ \preceq_{A^\alpha} c \oplus d_0 \preceq_{A^\alpha} (b_0 - \frac{\delta}{6})_+ \preceq_{A^\alpha} (b - \frac{\delta}{3})_+ \preceq_{A^\alpha} b.$$

42 □

Corollary 4.5. Let A be a unital separable simple infinite dimensional stably finite C^* -algebra with β -comparison. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin property with comparison. Then the crossed product $A \rtimes_{\alpha} G$ has β -comparison

Proof. The algebra A^{α} has β -comparison by Theorem 4.4. Theorem 2.14 implies that $A \rtimes_{\alpha} G$ is stably isomorphic to A^{α} . Since two stably isomorphic C^* -algebras have canonically isomorphic Cuntz semigroups, we know that $A \rtimes_{\alpha} G$ has β -comparison. \square

Now, we will show the permanence of Winter's n -comparison. The idea of the proof is essentially the same as that of Theorem 4.4.

Theorem 4.6. Let A be a unital separable simple infinite dimensional stably finite C^* -algebra with Winter's n -comparison. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin property with comparison. Then fixed point algebra A^{α} has Winter's n -comparison.

Proof. Let $a, b_0, b_1, \dots, b_n \in (A^{\alpha} \otimes K)_{+}$ and let $k_i \in \mathbb{N}$ such that

$$(k_i + 1)\langle a \rangle_{A^{\alpha}} \leq k_i \langle b_i \rangle_{A^{\alpha}}$$

in $\text{Cu}(A)$ for all $0 \leq i \leq n$. Since $\langle a \rangle_{A^{\alpha}} = \sup_{\varepsilon > 0} \langle (a - \varepsilon)_{+} \rangle_{A^{\alpha}}$, we need to show that $(a - \varepsilon)_{+} \precsim_{A^{\alpha}} b_0 \oplus b_1 \oplus \dots \oplus b_n$ for any $\varepsilon > 0$.

Fix $0 < \varepsilon < \frac{1}{2}$. Note that k_i can be chosen to be the same for all b_i , as, with $k = (k_0 + 1)(k_1 + 1) \dots (k_n + 1) - 1$, one has

$$(k + 1)\langle a \rangle_{A^{\alpha}} \leq k \langle b_i \rangle_{A^{\alpha}}$$

for all $0 \leq i \leq n$. Choose $m \in \mathbb{N}$ such that

$$m - k \geq 1.$$

Then we have

$$m(k + 1)\langle a \rangle_{A^{\alpha}} \leq mk \langle b_i \rangle_{A^{\alpha}}$$

for all $0 \leq i \leq n$. Therefore, by Lemma 2.5(1), there is $\delta > 0$ such that

$$m(k + 1)\langle (a - \frac{\varepsilon}{3})_{+} \rangle_{A^{\alpha}} \leq mk \langle (b_i - \delta)_{+} \rangle_{A^{\alpha}}$$

for all $0 \leq i \leq n$.

Choose $l \in \mathbb{N}$ and $a', b'_i \in M_l(A^{\alpha})_{+}$ for $0 \leq i \leq n$ such that

$$\|a' - (a - \frac{\varepsilon}{3})_{+}\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|b'_i - (b_i - \frac{\delta}{3})_{+}\| < \frac{\delta}{6}.$$

Then we have

$$(a - \varepsilon)_{+} \precsim_{A^{\alpha}} (a' - \frac{\varepsilon}{3})_{+} \precsim_{A^{\alpha}} (a - \frac{\varepsilon}{3})_{+}$$

and

$$(b_i - \delta)_{+} \precsim_{A^{\alpha}} (b'_i - \frac{\delta}{3})_{+} \precsim_{A^{\alpha}} (b'_i - \frac{\delta}{6})_{+} \precsim_{A^{\alpha}} (b_i - \frac{\delta}{3})_{+}.$$

for all $0 \leq i \leq n$. Thus we have

$$m(k + 1)\langle (a' - \frac{\varepsilon}{3})_{+} \rangle_{A^{\alpha}} \leq mk \langle (b'_i - \frac{\delta}{3})_{+} \rangle_{A^{\alpha}}$$

1 for all $0 \leq i \leq n$.

2 By [25, Proposition 4.20], the action $\alpha \otimes \text{id}_{M_l}: G \rightarrow \text{Aut}(A \otimes M_l)$ also has the tracial Rokhlin
3 property with comparison. Thus $M_l(A^\alpha)$ is not of type I, simple and infinite dimensional, by [28,
4 Lemma 2.7], there are positive elements $c_i \in M_l(A^\alpha)_+$ and $d_i \in M_l(A^\alpha)_+ \setminus \{0\}$ such that

$$5 \quad m \langle (b'_i - \frac{\delta}{3})_+ \rangle_{A^\alpha} \leq (m+1) \langle c_i \rangle_{A^\alpha} \quad \text{and} \quad \langle c_i \rangle_{A^\alpha} + \langle d_i \rangle_{A^\alpha} \leq \langle (b'_i - \frac{\delta}{6})_+ \rangle_{A^\alpha}$$

7 for all $0 \leq i \leq n$. By Lemma 4.3, there are $d'_i \in M_l(A^\alpha)_+$ such that $d'_i \lesssim_{A^\alpha} d_i$ and $\text{sp}(d'_i) = [0, 1]$ for all
8 $0 \leq i \leq n$. Thus we have

$$10 \quad \langle c_i \rangle_{A^\alpha} + \langle d'_i \rangle_{A^\alpha} \leq \langle (b'_i - \frac{\delta}{6})_+ \rangle_{A^\alpha},$$

11 and

$$13 \quad m(k+1) \langle (a' - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq (m+1)k \langle c_i \rangle_{A^\alpha}$$

14 for all $0 \leq i \leq n$.

15 Since $A^\alpha \subset A$ and A has Winter's n -comparison and

$$17 \quad m - k \geq 1,$$

18 we have

$$20 \quad ((m+1)k+1) \langle (a' - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq m(k+1) \langle (a' - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq (m+1)k \langle c_i \rangle_{A^\alpha}.$$

21 Thus,

$$22 \quad (a' - \frac{\varepsilon}{3})_+ \lesssim_A c_0 \oplus c_1 \oplus \cdots \oplus c_n.$$

24 Therefore,

$$25 \quad (a - \varepsilon)_+ \lesssim_{A^\alpha} (a' - \frac{\varepsilon}{3})_+ \lesssim_A c_0 \oplus d'_0 \oplus c_1 \oplus d'_1 \oplus \cdots \oplus c_n \oplus d'_n.$$

27 Note that 0 is a limit point of $\text{sp}(c_0 \oplus d'_0 \oplus c_1 \oplus d'_1 \oplus \cdots \oplus c_n \oplus d'_n)$, by [25, Proposition 4.21],

$$\begin{aligned} 29 \quad (a - \varepsilon)_+ &\lesssim_{A^\alpha} (a' - \frac{\varepsilon}{3})_+ \\ 30 \quad &\lesssim_{A^\alpha} c_0 \oplus d'_0 \oplus c_1 \oplus d'_1 \oplus \cdots \oplus c_n \oplus d'_n \\ 31 \quad &\lesssim_{A^\alpha} (b'_0 - \frac{\delta}{6})_+ \oplus \cdots \oplus (b'_n - \frac{\delta}{6})_+ \\ 32 \quad &\lesssim_{A^\alpha} (b_0 - \frac{\delta}{3})_+ \oplus \cdots \oplus (b_n - \frac{\delta}{3})_+ \\ 33 \quad &\lesssim_{A^\alpha} b_0 \oplus \cdots \oplus b_n. \end{aligned}$$

37 \square

39 **Corollary 4.7.** *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra with
40 Winter's n -comparison. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which
41 has the tracial Rokhlin property with comparison. Then the crossed product $A \rtimes_\alpha G$ has Winter's
42 n -comparison*

1 *Proof.* The algebra A^α has Winter's n -comparison by Theorem 4.6. Theorem 2.14 implies that $A \rtimes_\alpha G$
 2 is stably isomorphic to A^α . Since two stably isomorphic C^* -algebras have canonically isomorphic
 3 Cuntz semigroups, we know that $A \rtimes_\alpha G$ has Winter's n -comparison. \square

4 **Corollary 4.8.** *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra with
 5 strict comparison. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which
 6 has the tracial Rokhlin property with comparison. Then the fixed point algebra A^α and the crossed
 7 product $A \rtimes_\alpha G$ have strict comparison.*

8
 9 *Proof.* Apply Theorem 4.4 when $\beta = 1$ or Theorem 4.6 when $n = 0$. \square

10 5. m -almost divisibility and weakly (m,n) -divisibility

11
 12 In this section, we prove that the fixed point algebra and the crossed product of a unital separable
 13 simple infinite dimensional stably finite C^* -algebra which is m -almost divisible (respectively, weakly
 14 (m,n) -divisible) by an action of a second-countable compact group with the tracial Rokhlin property
 15 with comparison which are again m -almost divisible (respectively, weakly (m,n) -divisible). First, we
 16 introduce the definitions of m -almost divisibility and weakly (m,n) -divisibility.
 17

18 **Definition 5.1.** [30, 2.3] Let $m \in \mathbb{N}$. We say that the C^* -algebra A is m -almost divisible if for
 19 each $\langle a \rangle \in \text{Cu}(A)$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\langle b \rangle \in \text{Cu}(A)$ such that $k\langle b \rangle \leq \langle a \rangle$ and $\langle (a - \varepsilon)_+ \rangle \leq$
 20 $(k + 1)(m + 1)\langle b \rangle$.

21 **Definition 5.2.** [31, Definition 3.1] Let A be a C^* -algebra. Let $m, n \geq 1$ be integers. A is said to be
 22 weakly (m,n) -divisible, if for every $\langle u \rangle \in \text{Cu}(A)$ and any $\varepsilon > 0$, there exist elements $\langle x_1 \rangle, \langle x_2 \rangle, \dots, \langle x_n \rangle \in$
 23 $\text{Cu}(A)$ such that $m\langle x_j \rangle \leq \langle u \rangle$ for all $j = 1, 2, \dots, n$ and $\langle (u - \varepsilon)_+ \rangle \leq \langle x_1 \rangle + \dots + \langle x_n \rangle$.
 24

25 Now, we will show the permanence of m -almost divisibility. Given an element $a \in A^\alpha$. First, we
 26 divide the proof into two cases and in each case we find a bit of extra room in the subequivalence coming
 27 from comparison. Next, we use Theorem 2.9 and Theorem 2.13 to get approximately multiplicative
 28 maps φ and φ' and the sum of three elements to approximate a . Then, we can use φ and φ' to transfer
 29 two of them from A to A^α . The extra room is used to contral the third one. Last, we can transfer
 30 m -almost divisibility from A to A^α .
 31

32 **Theorem 5.3.** *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra which is
 33 m -almost divisible. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has
 34 the tracial Rokhlin property with comparison. Then the fixed point algebra A^α is m -almost divisible.*

35 *Proof.* We need to show that for each $a \in (A^\alpha \otimes K)_+$, any $\varepsilon > 0$, there exists $b \in (A^\alpha \otimes K)_+$ such that
 36 $k\langle b \rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha}$ and $\langle (a - \varepsilon)_+ \rangle_{A^\alpha} \leq (k + 1)(m + 1)\langle b \rangle_{A^\alpha}$.

37 Choose $n \in \mathbb{N}$ and $a_0 \in M_n(A^\alpha)_+$ such that

$$38 \quad \left\| a_0 - \left(a - \frac{\varepsilon}{3} \right)_+ \right\| < \frac{\varepsilon}{6}.$$

39
 40 Then we have

$$41 \quad \left(a - \varepsilon \right)_+ \preceq_{A^\alpha} \left(a_0 - \frac{\varepsilon}{3} \right)_+ \preceq_{A^\alpha} \left(a_0 - \frac{\varepsilon}{6} \right)_+ \preceq_{A^\alpha} \left(a - \frac{\varepsilon}{3} \right)_+ \preceq_{A^\alpha} a.$$

1 By [25, Proposition 4.20], the action $\alpha \otimes \text{id}_{M_n} : G \rightarrow \text{Aut}(A \otimes M_n)$ also has the tracial Rokhlin property
 2 with comparison. We may assume that $n = 1$.

3 Set $a_1 = (a_0 - \frac{\varepsilon}{6})_+$. Since A is m -almost divisible, and $a_1 \in A^\alpha \subset A$, with any $\varepsilon' > 0$, there exists
 4 $b_1 \in A$ such that $k\langle b_1 \rangle_A \leq \langle a_1 \rangle_A$ and $\langle (a_1 - \varepsilon')_+ \rangle_A \leq (k+1)(m+1)\langle b_1 \rangle_A$.

5 We divide the proof into two cases.

6 Case (1), we assume that $(a_1 - \varepsilon')_+$ is Cuntz equivalent to a projection. Without loss of generality,
 7 we may assume that $(a_1 - \varepsilon')_+$ is a projection.

8 By [26, Proposition 2.2], we may assume that there exists non-zero $c \in A_+$ such that $c(a_1 - \varepsilon')_+ = 0$
 9 and $\langle (a_1 - \varepsilon')_+ \rangle_A + \langle c \rangle_A \leq (k+1)(m+1)\langle b_1 \rangle_A$.

10 Since $k\langle b_1 \rangle_A \leq \langle a_1 \rangle_A$, there exists $v = (v_{i,j}) \in M_k(A)_+$ such that

$$11 \quad \quad \quad 12 \quad \quad \quad 13 \quad \quad \quad \left\| v^* \text{diag}(a_1, \underbrace{0, \dots, 0}_{k-1}) v - b_1 \otimes 1_k \right\| < \frac{\varepsilon'}{2}.$$

14 Since $\langle (a_1 - \varepsilon')_+ \rangle_A + \langle c \rangle_A \leq (k+1)(m+1)\langle b_1 \rangle_A$, there exists $w = (w_{s,t}) \in M_{(k+1)(m+1)}(A)_+$ such
 15 that

$$16 \quad \quad \quad 17 \quad \quad \quad 18 \quad \quad \quad \left\| w^* (b_1 \otimes 1_{(k+1)(m+1)}) w - \text{diag}((a_1 - \varepsilon')_+ + c, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1}) \right\| < \frac{\varepsilon'}{2}.$$

19 By Theorem 2.9, with $F_1 = \{a_1, b_1, (a_1 - \varepsilon')_+, c, v_{i,j}, v_{i,j}^*, w_{s,t}, w_{s,t}^* : i, j = 1, 2, \dots, k \text{ and } s, t = 1, 2, \dots, (k+1)(m+1)\}$, $F_2 = \{a_1\}$, $\frac{\varepsilon'}{2((k+1)(m+1))^2} > 0$ and $n = 3$, there exist a projection $p \in A^\alpha$ and a unital com-
 20 pletely positive map $\varphi : A \rightarrow pA^\alpha p$ such that

21 (1) φ is an $(3, F_1 \cup F_2, \frac{\varepsilon'}{2((k+1)(m+1))^2})$ -approximately multiplicative map.

22 (2) $\|px - xp\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_1 \cup F_2$.

23 (3) $\|\varphi(x) - p\varphi(x)p\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_2$.

24 Thus we have

$$25 \quad \quad \quad 26 \quad \quad \quad 27 \quad \quad \quad \left\| \varphi \otimes \text{id}_{M_k}(v^*) \text{diag}(\varphi(a_1), \underbrace{0, \dots, 0}_{k-1}) \varphi \otimes \text{id}_{M_k}(v) - \varphi(b_1) \otimes 1_k \right\| < \varepsilon',$$

28 and

$$29 \quad \quad \quad 30 \quad \quad \quad 31 \quad \quad \quad \left\| \varphi \otimes \text{id}_{(k+1)(m+1)}(w^*) (\varphi(b_1) \otimes 1_{(k+1)(m+1)}) \varphi \otimes \text{id}_{(k+1)(m+1)}(w) \right. \\ 32 \quad \quad \quad 33 \quad \quad \quad 34 \quad \quad \quad \left. - \text{diag}(\varphi((a_1 - \varepsilon')_+ + \varphi(c)), \underbrace{0, \dots, 0}_{(k+1)(m+1)-1}) \right\| < \varepsilon'.$$

35 Therefore we have

$$36 \quad \quad \quad 37 \quad \quad \quad 38 \quad \quad \quad k\langle (\varphi(b_1) - 2\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi(a_1) - \varepsilon')_+ \rangle_{A^\alpha},$$

39 and

$$40 \quad \quad \quad \langle (\varphi(a_1) - 4\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi(c) - 3\varepsilon')_+ \rangle_{A^\alpha} \leq (k+1)(m+1)\langle (\varphi(b_1) - 2\varepsilon')_+ \rangle_{A^\alpha}.$$

41 Let $a_2 = (1-p)a_1(1-p)$, for $a_2 \in A^\alpha \subset A$, since A is m -almost divisible, there exists $b_2 \in A$ such
 42 that $k\langle b_2 \rangle_A \leq \langle a_2 \rangle_A$ and $\langle (a_2 - \varepsilon')_+ \rangle_A \leq (k+1)(m+1)\langle b_2 \rangle_A$.

Since $k\langle b_2 \rangle_A \leq \langle a_2 \rangle_A$, there exist $v' = (v'_{i,j}) \in M_k(A)_+$ such that

$$\|v'^* \text{diag}(a_2, \underbrace{0, \dots, 0}_{k-1})v' - b_2 \otimes 1_k\| < \varepsilon'.$$

Since $\langle (a_2 - \varepsilon')_+ \rangle_A \leq (k+1)(m+1)\langle b_2 \rangle_A$, there exists $w' = (w'_{s,t}) \in M_{(k+1)(m+1)}(A)_+$ such that

$$\|w'^*(b_2 \otimes 1_{(k+1)(m+1)})w' - \text{diag}((a_2 - \varepsilon')_+, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1})\| < \varepsilon'.$$

For $A_1 = (1-p)A(1-p)$, by Theorem 2.13, with $F_1 = \{b_2, a_2, (a_2 - \varepsilon')_+, v'_{i,j}, v'^*_{i,j}, w'_{s,t}, w'^*_{s,t} : i, j = 1, 2, \dots, k \text{ and } s, t = 1, 2, \dots, (k+1)(m+1)\}$, $F_2 = \{a_2\}$, $\frac{\varepsilon'}{2((k+1)(m+1))^2} > 0$ and $n = 3$, there exist a projection $q \in (A_1)^\alpha$ and a unital completely positive map $\varphi' : A_1 \rightarrow q(A_1)^\alpha q$ such that

- (4) φ' is an $(3, F_1 \cup F_2, \frac{\varepsilon'}{2((k+1)(m+1))^2})$ -approximately multiplicative map.
- (5) $\|qx - xq\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_1 \cup F_2$.
- (6) $\|\varphi'(x) - qxq\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_2$.
- (7) $1_{A_1} - q \preceq_{(A_1)^\alpha} (\varphi(c) - 3\varepsilon')_+$.

Thus we have

$$\|\varphi' \otimes \text{id}_{M_k}(v'^*) \text{diag}(\varphi'(a_2), \underbrace{0, \dots, 0}_{k-1})\varphi' \otimes \text{id}_{M_k}(v') - \varphi'(b_2) \otimes 1_k\| < \varepsilon',$$

and

$$\|\varphi' \otimes \text{id}_{(k+1)(m+1)}(w'^*) (\varphi'(b_2) \otimes 1_{(k+1)(m+1)})\varphi' \otimes \text{id}_{(k+1)(m+1)}(w') - \text{diag}(\varphi'((a_2 - \varepsilon')_+, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1})\| < \varepsilon'.$$

Therefore we have

$$k\langle (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi'(a_2) - 2\varepsilon')_+ \rangle_{A^\alpha},$$

and

$$\langle (\varphi'(a_2) - 6\varepsilon')_+ \rangle_{A^\alpha} \leq (k+1)(m+1)\langle (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha}.$$

Therefore, with ε' sufficiently small, we have

$$\begin{aligned} & k\langle (\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha} \\ &= k\langle (\varphi(b_1) - 2\varepsilon')_+ \rangle_{A^\alpha} + k\langle (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha} \\ &\leq \langle (\varphi(a_1) - \varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi'(a_2) - 2\varepsilon')_+ \rangle_{A^\alpha} \\ &\leq \langle (\varphi(a_1) - \varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi'(a_2) - 2\varepsilon')_+ \rangle_{A^\alpha} + \langle ((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+ \rangle_{A^\alpha} \\ &= \langle (\varphi(a_1) - \varepsilon')_+ \oplus (\varphi'(a_2) - 2\varepsilon')_+ \oplus ((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+ \rangle_{A^\alpha} \\ &= \langle (\varphi(a_1) - \varepsilon')_+ + (\varphi'(a_2) - 2\varepsilon')_+ + ((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+ \rangle_{A^\alpha} \\ &\leq \langle (\varphi(a_1) - \varepsilon')_+ + ((1-p)a_1(1-p) - \varepsilon')_+ \rangle_{A^\alpha} \leq \langle a_1 \rangle_{A^\alpha} = \langle (a_0 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha}, \end{aligned}$$

1 and we also have

$$\begin{aligned}
 & \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} = \langle (a_1 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha} \\
 & \leq \langle (\varphi(a_1) - 4\varepsilon')_+ + (\varphi'(a_2) - 6\varepsilon')_+ + ((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+ \rangle_{A^\alpha} \\
 & = \langle (\varphi(a_1) - 4\varepsilon')_+ \oplus (\varphi'(a_2) - 6\varepsilon')_+ \oplus ((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+ \rangle_{A^\alpha} \\
 & = \langle (\varphi(a_1) - 4\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi'(a_2) - 6\varepsilon')_+ \rangle_{A^\alpha} + \langle ((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+ \rangle_{A^\alpha} \\
 & \leq \langle (\varphi(a_1) - 4\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi'(a_2) - 6\varepsilon')_+ \rangle_{A^\alpha} + \langle 1_{A_1} - q \rangle_{A^\alpha} \\
 & \leq \langle (\varphi(a_1) - 4\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi'(a_2) - 6\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi(c) - 3\varepsilon')_+ \rangle_{A^\alpha} \\
 & \leq (k+1)(m+1)\langle (\varphi(b_1) - 2\varepsilon')_+ \rangle_{A^\alpha} + (k+1)(m+1)\langle (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha} \\
 & = (k+1)(m+1)\langle (\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+ \rangle_{A^\alpha}.
 \end{aligned}$$

13 Set $b = (\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+$, we have

$$15 \quad k\langle b \rangle_{A^\alpha} \leq \langle (a_0 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha} \text{ and } \langle (a - \varepsilon)_+ \rangle_{A^\alpha} \leq \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq (k+1)(m+1)\langle b \rangle_{A^\alpha}.$$

17 Case (2), we assume that $(a_1 - \varepsilon')_+$ is not Cuntz equivalent to a projection.

18 By [8, Theorem 2.1(4)], there is a non-zero positive element $d \in A_+$ such that $d(a_1 - 2\varepsilon')_+ = 0$ and
 19 $\langle (a_1 - 2\varepsilon')_+ \rangle_A + \langle d \rangle_A \leq \langle (a_1 - \varepsilon')_+ \rangle_A$.

20 Since $k\langle b_1 \rangle_A \leq \langle a_1 \rangle_A$, there exists $v = (v_{i,j}) \in M_k(A)_+$ such that

$$22 \quad \|v^* \text{diag}(a, \underbrace{0, \dots, 0}_{k-1})v - b_1 \otimes 1_k\| < \frac{\varepsilon'}{2}.$$

25 Since $\langle (a_1 - 2\varepsilon')_+ \rangle_A + \langle d \rangle_A \leq \langle (a_1 - \varepsilon')_+ \rangle_A \leq (k+1)(m+1)\langle b_1 \rangle_A$, there exists $w = (w_{s,t}) \in$
 26 $M_{(k+1)(m+1)}(A)_+$ such that

$$28 \quad \|w^*(b_1 \otimes 1_{(k+1)(m+1)})w - \text{diag}((a_1 - 2\varepsilon')_+ + d, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1})\| < \frac{\varepsilon'}{2}.$$

31 By Theorem 2.9, with $F_1 = \{a_1, b_1, (a_1 - 2\varepsilon')_+, d, v_{i,j}, v_{i,j}^*, w_{s,t}, w_{s,t}^* : i, j = 1, 2, \dots, k \text{ and } s, t =$
 32 $1, 2, \dots, (k+1)(m+1)\}$, $F_2 = \{a\}$, $\frac{\varepsilon'}{2((k+1)(m+1))^2} > 0$ and $n = 3$, there exist a projection $p \in A^\alpha$ and
 34 a unital completely positive map $\varphi : A \rightarrow pA^\alpha p$ such that

35 (8) φ is an $(3, F_1 \cup F_2, \frac{\varepsilon'}{2((k+1)(m+1))^2})$ -approximately multiplicative map.

36 (9) $\|px - xp\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_1 \cup F_2$.

37 (10) $\|\varphi(x) - pxp\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_2$.

39 Thus we have

$$41 \quad \|\varphi \otimes \text{id}_{M_k}(v^*) \text{diag}(\varphi(a_1), \underbrace{0, \dots, 0}_{k-1}) \varphi \otimes \text{id}_{M_k}(v) - \varphi(b_1) \otimes 1_k\| < \varepsilon',$$

1 and

$$\begin{aligned} & \|\varphi \otimes \text{id}_{(k+1)(m+1)}(w^*)(\varphi(b_1) \otimes 1_{(k+1)(m+1)})\varphi \otimes \text{id}_{(k+1)(m+1)}(w) \\ & \quad - \text{diag}(\varphi((a_1 - 2\varepsilon')_+ + d, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1}))\| < \varepsilon'. \end{aligned}$$

2 Therefore we have

$$k\langle(\varphi(b_1) - 2\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi(a_1) - \varepsilon')_+\rangle_{A^\alpha},$$

3 and

$$\langle(\varphi(a_1) - 5\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi(d) - 3\varepsilon)_+\rangle_{A^\alpha} \leq (k+1)(m+1)\langle(\varphi(b_1) - 2\varepsilon')_+\rangle_{A^\alpha}.$$

4 Let $a_2 = (1-p)a_1(1-p)$, for $a_2 \in A^\alpha \subset A$, since A is m -almost divisible, there exists $b_2 \in A$ such that $k\langle b_2 \rangle_A \leq a_2$ and $\langle(a_2 - \varepsilon')_+\rangle_A \leq (k+1)(m+1)b_2$.

5 Since $k\langle b_2 \rangle_A \leq \langle a_2 \rangle_A$, there exist $v' = (v'_{i,j}) \in M_k(A)_+$ such that

$$\|v'^* \text{diag}(a_2, \underbrace{0, \dots, 0}_{k-1})v' - b_2 \otimes 1_k\| < \frac{\varepsilon'}{2},$$

6 Since $\langle(a_2 - \varepsilon')_+\rangle_A \leq (k+1)(m+1)\langle b_2 \rangle_A$, there exists $w' = (w'_{s,t}) \in M_{(k+1)(m+1)}(A)_+$ such that

$$\|w'^*(b_2 \otimes 1_{(k+1)(m+1)})w' - \text{diag}((a_2 - \varepsilon')_+, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1})\| < \frac{\varepsilon'}{2}.$$

7 For $A_1 = (1-p)A(1-p)$, by Theorem 2.13, with $F_1 = \{b_2, a_2, (a_1 - \varepsilon')_+, v'_{i,j}, v'^*_{i,j}, w'_{s,t}, w'^*_{s,t} : i, j = 1, 2, \dots, k \text{ and } s, t = 1, 2, \dots, (k+1)(m+1)\}$, $F_2 = \{a_2\}$, $\frac{\varepsilon'}{2((k+1)(m+1))^2} > 0$ and $n = 3$, there exist a projection $q \in (A_1)^\alpha$ and a unital completely positive map $\varphi' : A_1 \rightarrow q(A_1)^\alpha q$ such that

8 (11) φ' is an $(3, F_1 \cup F_2, \frac{\varepsilon'}{2((k+1)(m+1))^2})$ -approximately multiplicative map.

9 (12) $\|qx - xq\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_1 \cup F_2$.

10 (13) $\|\varphi'(x) - qxq\| < \frac{\varepsilon'}{2((k+1)(m+1))^2}$ for all $x \in F_2$.

11 (14) $1_{A_1} - q \preceq_{(A_1)^\alpha} (\varphi(d) - 3\varepsilon')_+$.

12 Thus we have

$$\|\varphi' \otimes \text{id}_{M_k}(v'^*) \text{diag}(\varphi'(a_2), \underbrace{0, \dots, 0}_{k-1})\varphi' \otimes \text{id}_{M_k}(v') - \varphi'(b_2) \otimes 1_k\| < \varepsilon',$$

13 and

$$\begin{aligned} & \|\varphi' \otimes \text{id}_{(k+1)(m+1)}(w'^*)(\varphi'(b_2) \otimes 1_{(k+1)(m+1)})\varphi' \otimes \text{id}_{(k+1)(m+1)}(w') \\ & \quad - \text{diag}(\varphi'((a_2 - \varepsilon')_+, \underbrace{0, \dots, 0}_{(k+1)(m+1)-1}))\| < \varepsilon'. \end{aligned}$$

14 Therefore we have

$$k\langle(\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi'(a_2) - 2\varepsilon')_+\rangle_{A^\alpha},$$

15 and

$$\langle(\varphi'(a_2) - 6\varepsilon')_+\rangle_{A^\alpha} \leq (k+1)(m+1)\langle(\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha}.$$

Therefore, with ε' sufficiently small, we have

$$\begin{aligned}
 & k\langle(\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha} \\
 &= k\langle(\varphi(b_1) - 2\varepsilon')_+\rangle_{A^\alpha} + k\langle(\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - \varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi'(a_2) - 2\varepsilon')_+\rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - \varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi'(a_2) - 2\varepsilon')_+\rangle_{A^\alpha} + \langle((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+\rangle_{A^\alpha} \\
 &= \langle(\varphi(a_1) - \varepsilon')_+ \oplus (\varphi'(a_2) - 2\varepsilon')_+ \oplus ((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+\rangle_{A^\alpha} \\
 &= \langle(\varphi(a_1) - \varepsilon')_+ + (\varphi'(a_2) - 2\varepsilon')_+ + ((1_{A_1} - q)a_2(1_{A_1} - q) - 2\varepsilon')_+\rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - \varepsilon')_+ + ((1 - p)a_1(1 - p) - \varepsilon')_+\rangle_{A^\alpha} \leq \langle a_1 \rangle_{A^\alpha} = \langle(a_0 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha},
 \end{aligned}$$

and we also have

$$\begin{aligned}
 & \langle(a_0 - \frac{\varepsilon}{3})_+\rangle_{A^\alpha} = \langle(a_1 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - 5\varepsilon')_+ + (\varphi'(a_2) - 6\varepsilon')_+ + ((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+\rangle_{A^\alpha} \\
 &= \langle(\varphi(a_1) - 5\varepsilon')_+ \oplus (\varphi'(a_2) - 6\varepsilon')_+ \oplus ((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+\rangle_{A^\alpha} \\
 &= \langle(\varphi(a_1) - 5\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi'(a_2) - 6\varepsilon')_+\rangle_{A^\alpha} + \langle((1_{A_1} - q)a_2(1_{A_1} - q) - 6\varepsilon')_+\rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - 5\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi'(a_2) - 6\varepsilon')_+\rangle_{A^\alpha} + \langle 1_{A_1} - q \rangle_{A^\alpha} \\
 &\leq \langle(\varphi(a_1) - 5\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi'(a_2) - 6\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi(d) - 3\varepsilon')_+\rangle_{A^\alpha} \\
 &\leq (k+1)(m+1)\langle(\varphi(b_1) - 2\varepsilon')_+\rangle_{A^\alpha} + (k+1)(m+1)\langle(\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha} \\
 &= (k+1)(m+1)\langle(\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+\rangle_{A^\alpha}.
 \end{aligned}$$

Set $b = (\varphi(b_1) - 2\varepsilon')_+ \oplus (\varphi'(b_2) - 4\varepsilon')_+$, we have

$$k\langle b \rangle_{A^\alpha} \leq \langle(a_0 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha} \text{ and } \langle(a - \varepsilon)_+\rangle_{A^\alpha} \leq \langle(a_0 - \frac{\varepsilon}{3})_+\rangle_{A^\alpha} \leq (k+1)(m+1)\langle b \rangle_{A^\alpha}.$$

□

Corollary 5.4. *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra which is m -almost divisible. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin property with comparison. Then the crossed product $A \rtimes_\alpha G$ is m -almost divisible.*

Proof. The algebra A^α is m -almost divisible by Theorem 5.3. Theorem 2.14 implies that $A \rtimes_\alpha G$ is stably isomorphic to A^α . Since two stably isomorphic C^* -algebras have canonically isomorphic Cuntz semigroups, we know that $A \rtimes_\alpha G$ is m -almost divisible. □

Now, we will show the permanence of weakly (m, n) -divisibility. The idea of the proof is essentially the same as that of Theorem 5.3.

Theorem 5.5. *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra which is weakly (m, n) -divisible. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group which has the tracial Rokhlin property with comparison. Then the fixed point algebra A^α is weakly (m, n) -divisible.*

1 *Proof.* We need to show that for each $a \in (A^\alpha \otimes K)_+$, any $\varepsilon > 0$, there exists $x_1, x_2, \dots, x_n \in (A^\alpha \otimes K)_+$
 2 such that $m\langle x_j \rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha}$ for all $1 \leq j \leq n$, and $\langle (a - \varepsilon)_+ \rangle_{A^\alpha} \leq \langle \bigoplus_{i=1}^n x_i \rangle_{A^\alpha}$.

3 Choose $n \in \mathbb{N}$ and $a_0 \in M_n(A^\alpha)_+$ such that

$$4 \quad \|a_0 - (a - \frac{\varepsilon}{3})_+\| < \frac{\varepsilon}{6}.$$

6 Then we have

$$7 \quad (a - \varepsilon)_+ \precsim_{A^\alpha} (a_0 - \frac{\varepsilon}{3})_+ \precsim_{A^\alpha} (a_0 - \frac{\varepsilon}{6})_+ \precsim_{A^\alpha} (a - \frac{\varepsilon}{3})_+ \precsim_{A^\alpha} a.$$

8 By [25, Proposition 4.20], the action $\alpha \otimes \text{id}_{M_n}: G \rightarrow \text{Aut}(A \otimes M_n)$ also has the tracial Rokhlin property
 9 with comparison. We may assume that $n = 1$.

10 Set $a_1 = (a_0 - \frac{\varepsilon}{6})_+$. Since A is weakly (m, n) -divisible, and $a_1 \in A^\alpha \subset A$, with any $\varepsilon' > 0$, there
 11 exists $y_1, y_2, \dots, y_n \in A$ such that $m\langle y_j \rangle_A \leq \langle a_1 \rangle_A$ for all $1 \leq j \leq n$, and $\langle (a_1 - \varepsilon')_+ \rangle_A \leq \langle \bigoplus_{i=1}^n y_i \rangle_A$.

12 We divide the proof into two case.

13 Case (1), we assume that $(a_1 - \varepsilon')_+$ is Cuntz equivalent to a projection. Without loss of generality,
 14 we may assume that $(a_1 - \varepsilon')_+$ is a projection.

15 As in the same step in the argument in Theorem 5.3, we get a projection $p \in A^\alpha$ and a unital
 16 completely positive map $\varphi: A \rightarrow pA^\alpha p$ such that

$$17 \quad m\langle (\varphi(y_j) - 2\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi(a_1) - \varepsilon')_+ \rangle_{A^\alpha},$$

19 for all $1 \leq j \leq n$, and

$$20 \quad \langle (\varphi(a_1) - 4\varepsilon')_+ \rangle_{A^\alpha} + \langle (\varphi(c) - 3\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi(\bigoplus_{i=1}^n y_i) - 2\varepsilon')_+ \rangle_{A^\alpha} = \langle \bigoplus_{i=1}^n (\varphi(y_i) - 2\varepsilon')_+ \rangle_{A^\alpha}.$$

22 Let $a_2 = (1 - p)a_1(1 - p)$, for $a_2 \in A^\alpha \subset A$, since A is weakly (m, n) -divisible, there exists
 23 $z_1, z_2, \dots, z_n \in A$ such that $m\langle z_j \rangle_A \leq \langle a_2 \rangle_A$ for all $1 \leq j \leq n$ and $\langle (a_2 - \varepsilon')_+ \rangle_A \leq \langle \bigoplus_{i=1}^n z_i \rangle_A$.

24 As in the same step in the argument in Theorem 5.3, we get a projection $p \in A^\alpha$ and a unital
 25 completely positive map $\varphi': A_1 \rightarrow q(A_1)^\alpha q$ such that $1_{A_1} - q \precsim_{(A_1)^\alpha} (\varphi(c) - 3\varepsilon')_+$ and

$$26 \quad m\langle (\varphi'(z_j) - 4\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi'(a_2) - 2\varepsilon')_+ \rangle_{A^\alpha}.$$

28 for all $1 \leq j \leq n$, and

$$29 \quad \langle (\varphi'(a_2) - 6\varepsilon')_+ \rangle_{A^\alpha} \leq \langle (\varphi'(\bigoplus_{i=1}^n z_i) - 4\varepsilon')_+ \rangle_{A^\alpha} = \langle \bigoplus_{i=1}^n (\varphi'(z_i) - 4\varepsilon')_+ \rangle_{A^\alpha}.$$

30 Therefore, with ε' sufficiently small, as in the same step in the argument in Theorem 5.3, we have

$$32 \quad m\langle (\varphi(y_j) - 2\varepsilon')_+ \oplus (\varphi'(z_j) - 4\varepsilon')_+ \rangle_{A^\alpha} \leq \langle a_1 \rangle_{A^\alpha} = \langle (a_0 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha},$$

34 for all $1 \leq j \leq n$, and

$$35 \quad \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} = \langle (a_1 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha} \leq \langle \bigoplus_{i=1}^n ((\varphi(y_i) - 2\varepsilon')_+ \oplus (\varphi'(z_i) - 4\varepsilon')_+) \rangle_{A^\alpha}.$$

37 Set $x_j = (\varphi(y_j) - 2\varepsilon')_+ \oplus (\varphi'(z_j) - 4\varepsilon')_+$ for all $1 \leq j \leq n$, we have

$$38 \quad m\langle x_j \rangle_{A^\alpha} \leq \langle (a_0 - \frac{\varepsilon}{6})_+ \rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha} \text{ and } \langle (a - \varepsilon)_+ \rangle_{A^\alpha} \leq \langle (a_0 - \frac{\varepsilon}{3})_+ \rangle_{A^\alpha} \leq \langle \bigoplus_{i=1}^n x_i \rangle_{A^\alpha}.$$

40 Case (2), we assume that $(a - \varepsilon')_+$ is not Cuntz equivalent to a projection.

41 By [8, Theorem 2.1(4)], there is a non-zero positive element $d \in A_+$ such that $d(a - 2\varepsilon')_+ = 0$ and
 42 $\langle (a - 2\varepsilon')_+ \rangle_A + \langle d \rangle_A \leq \langle (a - \varepsilon')_+ \rangle_A$.

1 As in the same step in the argument in Theorem 5.3, we get a projection $p \in A^\alpha$ and a unital
2 completely positive map $\varphi: A \rightarrow pA^\alpha p$ such that

$$3 \quad m\langle(\varphi(y_j) - 2\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi(a) - \varepsilon')_+\rangle_{A^\alpha},$$

4 for all $1 \leq j \leq n$, and

$$5 \quad \langle(\varphi(a) - 5\varepsilon')_+\rangle_{A^\alpha} + \langle(\varphi(d) - 3\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi(\bigoplus_{i=1}^n y_i) - 2\varepsilon')_+\rangle_{A^\alpha} = \langle\bigoplus_{i=1}^n (\varphi(y_i) - 2\varepsilon')_+\rangle_{A^\alpha}.$$

6 As in the same step in the argument in Theorem 5.3, we get a projection $p \in A^\alpha$ and a unital
7 completely positive map $\varphi': A_1 \rightarrow q(A_1)^\alpha q$ such that $1_{A_1} - q \preceq_{(A_1)^\alpha} (\varphi(d) - 3\varepsilon')_+$ and

$$8 \quad m\langle(\varphi'(z_j) - 4\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi'(a_1) - 2\varepsilon')_+\rangle_{A^\alpha}.$$

9 for all $1 \leq j \leq n$, and

$$10 \quad \langle(\varphi'(a_1) - 6\varepsilon')_+\rangle_{A^\alpha} \leq \langle(\varphi'(\bigoplus_{i=1}^n z_i) - 4\varepsilon')_+\rangle_{A^\alpha} = \langle\bigoplus_{i=1}^n (\varphi'(z_i) - 4\varepsilon')_+\rangle_{A^\alpha}.$$

11 Therefore, with ε' sufficiently small, as in the same step in the argument in Theorem 5.3, we have

$$12 \quad m\langle(\varphi(y_j) - 2\varepsilon')_+ \oplus (\varphi'(z_j) - 4\varepsilon')_+\rangle_{A^\alpha} \leq \langle a_1 \rangle_{A^\alpha} = \langle(a_0 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha},$$

13 for all $1 \leq j \leq n$, and

$$14 \quad \langle(a_0 - \frac{\varepsilon}{3})_+\rangle_{A^\alpha} = \langle(a_1 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha} \leq \langle\bigoplus_{i=1}^n ((\varphi(y_i) - 2\varepsilon')_+ \oplus (\varphi'(z_i) - 4\varepsilon')_+)\rangle_{A^\alpha}.$$

15 Set $x_j = (\varphi(y_j) - 2\varepsilon')_+ \oplus (\varphi'(z_j) - 4\varepsilon')_+$ for all $1 \leq j \leq n$, we have

$$16 \quad m\langle x_j \rangle_{A^\alpha} \leq \langle(a_0 - \frac{\varepsilon}{6})_+\rangle_{A^\alpha} \leq \langle a \rangle_{A^\alpha} \text{ and } \langle(a - \varepsilon)_+\rangle_{A^\alpha} \leq \langle(a_0 - \frac{\varepsilon}{3})_+\rangle_{A^\alpha} \leq \langle\bigoplus_{i=1}^n x_i \rangle_{A^\alpha}.$$

17 \square

18 **Corollary 5.6.** *Let A be a unital separable simple infinite dimensional stably finite C^* -algebra which
19 is weakly (m,n) -divisible. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a second-countable compact group
20 which has the tracial Rokhlin property with comparison. Then the crossed product $A \rtimes_\alpha G$ is weakly
21 (m,n) -divisible.*

22 *Proof.* The algebra A^α is weakly (m,n) -divisible by Theorem 5.5. Theorem 2.14 implies that $A \rtimes_\alpha G$ is
23 stably isomorphic to A^α . Since two stably isomorphic C^* -algebras have canonically isomorphic Cuntz
24 semigroups, we know that $A \rtimes_\alpha G$ is weakly (m,n) -divisible. \square

32 6. Futher work

33 Gardella and Santiago introduced the equivariant Cuntz semigroups for compact group actions on
34 C^* -algebras in [17]. They proved that the equivariant Cuntz semigroup, as a functor, is continuous and
35 stable. One of their main results is an analog of Julg's theorem as following.

36 **Theorem 6.1.** [17, Theorem 5.14] *Let G be a compact group, let A be a C^* -algebra, and let $\alpha: G \rightarrow$
37 $\text{Aut}(A)$ be a continuous action. Then there exists a natural Cu^G -isomorphism*

$$38 \quad \text{Cu}^G(A, \alpha) \cong \text{Cu}(A \rtimes_\alpha G),$$

39 where the $\text{Cu}(G)$ -semimodule structure on $\text{Cu}(A \rtimes_\alpha G)$ is given by [17, Definition 5.10].

1 Moreover, they computed the equivariant Cuntz semigroups of a number of dynamical systems,
2 particularly, the Rokhlin property for an action of a finite group and got the following result.

3 **Proposition 6.2.** [17, Proposition 6.2] *Let G be a finite group, let A be a C^* -algebra, and let $\alpha: G \rightarrow$
4 $\text{Aut}(A)$ be an action with the Rokhlin property. Then there exists a natural Cu^G -isomorphism*

$$5 \text{Cu}^G(A, \alpha) \cong \text{Cu}(A)_{\mathbb{N}}^{\text{Cu}(\alpha)},$$

6 where the induced $\text{Cu}(G)$ -semimodule structure on $\text{Cu}(A)_{\mathbb{N}}^{\text{Cu}(\alpha)}$ is trivial.
7

8 For the compact group action with the Rokhlin property, this is relatively easy, and we imagine that
9 the outcome for the tracial Rokhlin property (with comparison) will be much more interesting since the
10 $\text{Cu}(G)$ -semimodule structure will not be trivial in general. We will try to compute it in the future work.

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