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ROTHE TIME-DISCRETIZATION METHOD FOR A FRACTIONAL CONTACT PROBLEM WITH COULOMB'S FRICTION IN ELECTRO-VISCOELASTICITY

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ABSTRACT. In this paper, we study a quasistatic contact problem involving fractional calculus, friction, and an electro-viscoelastic body supported by a conductive foundation. The contact is characterized by Signorini's conditions, and friction is governed by Coulomb's law. The weak formulation takes the form of a system that couples a fractional variational inequality for displacement with an elliptic variational equality for the electric potential. We establish the existence of a weak solution, relying on techniques such as monotone operators, Abstract Volterra theory, Caputo derivatives, the Rothe method, and the Banach fixed-point theorem in our proofs.

1. Introduction

The field of fractional calculus has emerged as a new area in various branches of applied mathematics, serving as a modeling tool for numerous physical phenomena across mechanics, technology, electronics, energy, and more (see, for example, [3, 7, 9, 14]). The quasistatic frictionless contact problem for a viscoelastic body with the fractional Kelvin-Voigt law was addressed in [5].The authors in [17, 19] addressed a class of elliptic and parabolic differential hemivariational inequalities involving the time fractional order integral operator, applied in the context of contact problems.

References [6, 15, 16] describe fractional mathematical models for materials exhibiting viscoelastic behavior, and more recent developments are discussed in [12, 13]. Bouallala et al. [1] addressed a thermo-viscoelastic fractional contact problem with normal compliance and Coulomb's friction. They demonstrated the existence of a weak solution using monotone operators, the Galerkin method, abstract Caputo derivatives, and fixed point theorems. More recently, the authors of [20] have initiated the study of a new frictionless dynamic contact problem model for a viscoelastic body with normal compliance, taking into account the Kelvin-Voigt constitutive law with a time fractional component.

In this paper, we study a new mathematical model that describes the time fractional process of contact between an electro-viscoelastic body and an electrically conductive support. The constitutive relation is modeled with the following fractional Kelvin-Voigt law.

$$\sigma(t) = \mathscr{F}\varepsilon\left({}_0^C D_t^{\alpha} u(t)\right) + \mathscr{B}\varepsilon\left(u(t)\right) - \mathscr{E}^* E(\varphi(t)) \quad \text{in} \quad \Omega\times(0,T), \tag{1}$$

$$D(t) = \mathscr{E}\varepsilon(u(t)) + \beta E(\varphi(t)) \quad \text{in} \quad \Omega\times(0,T), \tag{2}$$

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where $\alpha \in (0,1]$ and $t \in [0,T]$.
The case where $\alpha = 1$ has been studied in various contexts, incorporating different contact and friction laws; however, the specific conditions used in this study have not been explored. References related to this framework can be found in [21, 22, 23, 24].

The motivation behind choosing this model is to generalize the behavior laws of rheological models used in linear viscoelasticity, which consist of springs and dampers. This also allows describing viscoelastic behavior across a wide range of frequencies with few parameters. This combination with piezoelectricity has several applications in physics, engineering, and applied sciences, such as biological models aimed at capturing long-term memory properties observed in biological systems, as well as mechanical modeling of rubbers and elastomers. This is particularly relevant for materials capable of memorizing past deformations and exhibiting viscoelastic behavior.
Moreover, we assume that the contact is described with the Signorini's condition and the Coulomb's friction including. We derive the variational formulation of this problem, and we prove the existence of a weak solution.
The major difficulties in this work lie mainly in the estimations for the convergence of the Rothe method, as well as in the mechanical-electrical coupling.

The paper is structured as follows: In section 2, we present the model for the equilibrium process of the electro-viscoelastic body in frictional contact with a conductive foundation involving time fractional. In section 3, we outline the assumptions regarding the data, develop a variational formulation of the friction contact problem, and present our existence results. In Section 4, we establish the weak solvability of this hemivariational inequality by applying abstract results on monotone operators, the Rothe method, and the Banach fixed-point theorem. Finally, in the Appendix, we provide some necessary definitions and results that are useful in proving the main result.

2. The Fractional Contact Problem

We begin by defining the mechanical setting of the contact problem. We assume the presence of an electro-viscoelastic body occupying a domain Ω in $\mathbb{R}^d$, where $d = 1,2,3$. The boundary $\partial\Omega$ is Lipschitz continuous and can be divided into three disjoint, open, and measurable parts: Γ_D, Γ_N, and Γ_C. Additionally, we partition $\partial\Omega \backslash \overline{\Gamma}_C$ into two open parts, Γ_a and Γ_b, with the conditions that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$. The fractional problem will be discussed within the finite time interval $[0,T]$, where $T > 0$.
The body is assumed to be clamped in $\Gamma_D \times (0,T)$ and subjected to a volume force f_1 and a volume electric charge density q_1 in $\Omega \times (0,T)$. Additionally, it is subject to mechanical and electrical constraints on the boundary. Furthermore, we assume that a density of traction forces, denoted as f_2, acts on the boundary segment $\Gamma_N \times (0,T)$. We also impose the condition that the electrical potential is zero on Γ_a, and a surface electrical charge density of q_2 is prescribed on $\Gamma_b \times (0,T)$.
In the reference configuration, the body may come into contact with an electrically conductive foundation over Γ_C. We assume that the foundation's potential is held constant at φ_F. This contact is characterized by friction, and there may be electrical charges present on the contact surface. The

normalized gap denoted by g exists between Γ_C and the rigid foundation.

We denote the unit outward normal vector as $\nu = (\nu_i)$, and we will use the standard notation for the tangential components of the displacement field vector u and the stress tensor σ

$$u_\nu = u \cdot \nu, \;\; u_\tau = u - u_\nu \cdot \nu, \;\; \text{and} \;\; \sigma_\nu = \sigma\nu \cdot \nu, \;\; \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \tag{3}$$

We introduce the notation $\mathbb{S}^d$ to represent the linear space of second-order symmetric tensors on $\mathbb{R}^d$. The symbols "$\cdot$" and $\|\cdot\|$ represent the inner products and Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. In other words, for all $u, v \in \mathbb{R}^d$ and for all $\sigma, \tau \in \mathbb{S}^d$

$$\begin{aligned} &u \cdot v = u_i v_i, \;\; \|v\|_{\mathbb{R}^d} = (v,v)^{\frac{1}{2}}, \;\; \text{for all } u = (u_i),\; v = (v_i) \in \mathbb{R}^d, \\ &\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \;\; \|\tau\|_{\mathbb{S}^d} = (\tau,\tau)^{\frac{1}{2}}, \;\; \text{for all } \sigma = (\sigma_{ij}),\; \tau = (\tau_{ij}) \in \mathbb{S}^d. \end{aligned} \tag{4}$$

We introduce the following notation $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ the displacement field, $\sigma = (\sigma_{ij}) : \Omega \times [0,T] \longrightarrow \mathbb{S}^d$ the stress tensor, $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}$ is the electric potential and $D = (D_i) : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ the electric displacement. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ where $u_{ij} = \partial u_i / \partial x_j$. Here and below "$Div$" and "$div$" denote the divergence operators for tensor and vector valued functions, respectively, i.e., $Div\ \sigma = (\sigma_{ij,j})$ and $div\ \xi = (\xi_{j,j})$.

The classical formulation of the fractional contact problem is as follows:
Problem (P): Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ and an electric potential $\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}$ for all $\alpha \in]0,1]$ such that

$$\sigma(t) = \mathscr{F}\varepsilon\left({}_0^C D_t^\alpha u(t)\right) + \mathscr{B}\varepsilon(u(t)) - \mathscr{E}^* E(\varphi(t)) \quad \text{in } \Omega \times (0,T), \tag{5}$$

$$D(t) = \mathscr{E}\varepsilon(u(t)) + \beta E(\varphi(t)) \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$Div\ \sigma(t) + f_1(t) = 0 \quad \text{in } \Omega \times (0,T), \tag{7}$$

$$div\ D(t) = q_1(t) \quad \text{in } \Omega \times (0,T), \tag{8}$$

$$u = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{9}$$

$$\sigma(t) \cdot \nu = f_2(t) \quad \text{on } \Gamma_N \times (0,T), \tag{10}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{11}$$

$$D(t) \cdot \nu = q_2(t) \quad \text{on } \Gamma_b \times (0,T), \tag{12}$$

$$u(0,x) = u_0 \quad \text{in } \Omega, \tag{13}$$

$$\sigma_\nu(u(t)) \le 0, \;\; u_\nu(t) \le g, \;\; \sigma_\nu(u(t))(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_C \times (0,T), \tag{14}$$

$$\left.\begin{aligned} &\|\sigma_\tau(t)\| \le \nu_f \|\sigma_\nu(t)\|, \\ &\|\sigma_\tau(t)\| < \nu_f \|\sigma_\nu(t)\| \Longrightarrow u_\tau(t) = 0, \\ &\|\sigma_\tau(t)\| = \nu_f \|\sigma_\nu(t)\| \Longrightarrow \exists \lambda \neq 0 \text{ such that } \sigma_\tau(t) = -\lambda\, u_\tau(t) \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{15}$$

$$D(t) \cdot \nu = \psi(u_\nu(t) - g)\, \phi_L(\varphi(t) - \varphi_F) \quad \text{on } \Gamma_C \times (0,T), \tag{16}$$

In the equations (5)-(6), we represent the fractional Kelvin-Voigt electro-viscoelastic constitutive law of the Caputo type, as described in [18]. Here, $\mathscr{B} = (b_{ijkl})$, $\beta = (\beta_{ij})$, $\mathscr{F} = (f_{ijkl})$, and $\mathscr{E} = (e_{ijk})$ denote the elastic tensor, electric permittivity tensor, viscosity tensor, and third-order piezoelectric tensor, respectively. The transpose of $\mathscr{E}$ is denoted as $\mathscr{E}^*$ and is defined as follows:

$$\mathscr{E}^* = \left(e^*_{ijk}\right), \text{ where } e^*_{ijk} = e_{kij}, \text{ and } \mathscr{E}\sigma\upsilon = \sigma\mathscr{E}^*\upsilon, \text{ for all } \sigma \in \mathbb{S}^d,\ \upsilon \in \mathbb{R}^d. \tag{17}$$

Equations of stress equilibrium are given in (7)-(8), while (9)-(12) describe the mechanical and electrical boundary conditions. Furthermore, the initial condition is described in equation (13). The unilateral boundary condition, (14), represents Signorini's law, and (15) represents Coulomb's friction law, where ν_f is the friction coefficient. Finally, equation (16) is the regularized electrical contact on Γ_C such that

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad -L \le s \le L, \\ L & \text{if} \quad s > L, \end{cases} \quad \psi(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ k_e \delta r & \text{if} \quad 0 \le r \le \frac{1}{\delta}, \\ k_e & \text{if} \quad r > \frac{1}{\delta}. \end{cases} \tag{18}$$

Here, L is a large positive constant, $\delta > 0$ is a small parameter, and $k_e \ge 0$ represents the electrical conductivity coefficient as discussed in [8].

3. Weak Formulation and Existence Result

To formulate the weak formulation, we will make use of function spaces.

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathscr{H} = \{\sigma \in \mathbb{S}^d,\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$\mathscr{W} = \{D = (D)_i \in H_1,\ D_i \in L^2(\Omega),\ div(D) \in L^2(\Omega)\}.$$

These are real Hilbert spaces with inner products.

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$
$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx,\ (D,E)_{\mathscr{W}} = (D,E)_H + (div\, D,\ div\, E)_{L^2(\Omega)},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{W}}$, respectively.
Based on the boundary conditions (9) and (11), we introduce closed subspaces of H_1.

$$V = \{v \in H_1,\ v = 0 \text{ on } \Gamma_D\} \text{ and } W = \left\{\xi \in H^1(\Omega),\ \xi = 0 \text{ on } \Gamma_a\right\},$$

endowed with the inner product given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \|v\|_V = (v,v)_V^{\frac{1}{2}},$$
$$(\varphi,\xi)_W = (\nabla\varphi, \nabla\xi)_H, \quad \|\xi\|_W = (\xi,\xi)_W^{\frac{1}{2}}.$$

Define V_{ad} as the set of admissible displacements given by.

$$V_{ad} = \{v \in V,\ v_\nu \le g \text{ on } \Gamma_C\}.$$

Since V is dense in H, we identify H with its dual space H^*, and we write $V \subset H \equiv H^* \subset V^*$, where V is a reflexive and separable Banach space, H is a separable Hilbert space, and the embedding $V \to H$ is dense and continuous. The compact embedding operator between V and H is denoted by ι.

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The dual space to V is V^*, and the dual mapping $\imath^* : H \to V^*$ of $\imath$ is also linear, continuous, and compact.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma : H \to H_\Gamma$ be the trace map. For every element $v \in H$, we also use the notation v to denote the trace γv of v on Γ.
Let H_Γ^* be the dual of H_Γ and $\langle \cdot, \cdot \rangle$ denote the duality pairing between H_Γ^* and H_Γ. For every $\sigma \in \mathscr{H}$, $\sigma\nu$ can be defined as the element in H_Γ^* satisfying

$$\langle \sigma\nu, \gamma v\rangle = (\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\, \sigma, v)_H, \; \forall v \in H_1. \tag{19}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu \cdot v da, \; \forall v \in H_1, \tag{20}$$

where da is the surface measure element.
Combining (19) through (20), we have the following Green's formula in elasticity

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\, \sigma, v)_H = \int_\Gamma \sigma\nu . v da. \tag{21}$$

When $D \in \mathscr{W}$ is a sufficiently regular function, we obtain the following Green's formula

$$(D, \nabla\xi)_{L^2(\Omega)} + (div\, D, \xi)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu \xi \, da, \; \forall \xi \in H^1(\Omega). \tag{22}$$

Recall that since $meas(\Gamma_D) > 0$, Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq c_K \|v\|_{H_1}, \text{ for all } v \in V, \tag{23}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_D.
According to the Sobolev trace theorem, there exist constants c_d and c_e that depend solely on Ω, Γ_D, Γ_C, and Γ_a, such that

$$\|v\|_{L^2(\Gamma_C)^d} \leq c_d \|v\|_V, \text{ and } \|\xi\|_{L^2(\Gamma_C)} \leq c_e \|\xi\|_W, \tag{24}$$

for all $v \in V$ and $\xi \in W$.
Moreover, due to the positive measure of Γ_a, the Friedrichs-Poincar inequality is applicable.

$$\|\nabla\xi\|_{\mathscr{W}} \geq c_F \|\xi\|_W, \text{ for all } \xi \in W. \tag{25}$$

Next, let $(X, \|.\|_X)$ be a real Banach space. For $1 \leq p \leq +\infty$, $k = 1, 2, \ldots$ we use the usual notations for the spaces $L^p(0,T;X)$, $C(0,T;X)$ and $W^{k,p}(0,T;X)$ the space of all measurable functions on $[0,T]$ with values in X, endowed with the canonical inner product.
In the study of **Problem (P)**, we define the following bilinear forms:

$$\begin{aligned}
&a : V \times V \to \mathbb{R}, a(u,v) := (\mathscr{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \\
&b : V \times V \to \mathbb{R}, b(u,v) := (\mathscr{B}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \\
&c : W \times W \to \mathbb{R}, c(\varphi, \xi) := (\beta\nabla\varphi, \nabla\xi)_H, \\
&e : V \times W \to \mathbb{R}, e(v, \xi) := (\mathscr{E}\varepsilon(u), \nabla\xi)_H = (\mathscr{E}^*\nabla\xi, \varepsilon(v))_V.
\end{aligned} \tag{26}$$

Using Riesz's representation theorem, we define the elements $f(t) \in V$ and $q(t) \in W$ for all $(v,\xi) \in V \times W$ such that

$$(f(t),v)_{V^*\times V} = \int_\Omega f_1(t)\cdot v\,dx + \int_\Gamma f_2(t)\cdot v\,da, \tag{27}$$

$$(q(t),\xi)_W = \int_\Omega q_1(t)\cdot \xi\,dx + \int_{\Gamma_b} q_2(t)\cdot \xi\,da. \tag{28}$$

Also, we define the mappings $j: V \to \mathbb{R}^+$ and $\ell: V\times W\times W \to \mathbb{R}$ by

$$j(v) := \int_{\Gamma_C} \nu_f \sigma_\nu(\|v_\tau\|)\,da, \tag{29}$$

$$\ell(u(t),\varphi(t),\xi) := \int_{\Gamma_C} \psi(u_\nu(t)-g)\phi_L(\varphi(t)-\varphi_F)\xi\,da. \tag{30}$$

Next, let's introduce the following assumptions:

(H1) i) The forms a, b, and c are bilinear and satisfy the following symmetry property:

$$f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega),\ b_{ijkl} = b_{jikl} = b_{lkij} \in L^\infty(\Omega) \text{ and } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega).$$

ii) The forms a and c satisfy the property of ellipticity:

$$a(v,v) \geq m_a\|v\|_V^2 \text{ and } c(\xi,\xi) \geq m_c\|\xi\|_W^2,$$

where $m_a, m_c > 0$.

(H2) The forms a, b, c, and e satisfy the usual boundedness property:

$$|a(u,v)| \leq M_a\|u\|_V\|v\|_V,\quad |b(u,v)| \leq M_b\|u\|_V\|v\|_V,$$
$$|c(\varphi,\xi)| \leq M_c\|\varphi\|_W\|\xi\|_W,\quad |e(v,\xi)| \leq M_e\|v\|_V\|\xi\|_W,$$

where $M_a, M_b, M_c, M_e > 0$.

(H3) The forces, traction, volume, and surface free charge densities satisfy the following inclusions:

$$f_1 \in C\left(0,T;L^2(\Omega)^d\right),\quad f_2 \in C\left(0,T;L^2(\Gamma_N)^d\right),$$
$$q_1 \in L^2\left(0,T;L^2(\Omega)\right),\quad q_2 \in L^2\left(0,T;L^2(\Gamma_b)\right).$$

(H4) The coefficient of friction, the gap function, the initial condition, and the potential satisfy the following conditions:

$$\nu_f \geq 0,\ \nu_f \in L^\infty(\Gamma_C),\ \ g \geq 0,\ g \in L^\infty(\Gamma_C),\ \ u_0 \in V_{ad},\ \ \varphi_F \in L^2\left(0,T;L^2(\Gamma_C)\right).$$

(H5) The surface electrical conductivity $\psi: \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies

i) there exists $L_\psi > 0$ such that $|\psi(x,u_1)-\psi(x,u_2)| \leq L_\psi|u_1-u_2|$, for all $u_1,u_2 \in \mathbb{R}$, and a.e. $x \in \Gamma_C$;

ii) there exists $M_\psi > 0$ such that $|\psi(x,u)| < M_\psi$ for all $u \in \mathbb{R}$ and $x \in \Gamma_C$;

iii) $x \mapsto \psi(x,u)$ is measurable on Γ_C for all $x \in \mathbb{R}$;

iv) $x \mapsto \psi(x,u) = 0$, for all $u \leq 0$.

(H6) For all $v \in V$, and a.e. $t \in (0,T)$

i) j is locally Lipschitz on Γ_C;

ii) there exists $c_j > 0$ such that

$$\|\partial j(v)\|_{V^*} \leq c_j(1+\|v\|_V).$$

By utilizing the above assumptions, notations, and equations (19)(28), along with the equation $E(\varphi) = -\nabla\varphi$, we can derive the following variational formulation for **Problem (P)** in terms of the displacement field and electric potential:

Problem (PV): Find a displacement field $u : \Omega\times[0,T] \to \mathbb{R}^d$, and an electric potential $\varphi : \Omega\times[0,T] \to \mathbb{R}$ a.e. $t \in]0,T[$ for all $v \in V$, $\xi \in W$ and $\alpha \in]0,1]$ such that

$$(31)\quad \begin{aligned} &a\left({}_0^C D_t^\alpha u(t), v-u(t)\right) + b\left(u(t), v-u(t)\right) + e\left(v-u(t), \varphi(t)\right) \\ &+ j(v) - j(u(t)) \geq (f(t), v-u(t))_{V^*\times V}, \end{aligned}$$

$$(32)\quad c\left(\varphi(t), \xi\right) - e\left(u(t), \xi\right) + \ell\left(u(t), \varphi(t), \xi\right) = (q(t), \xi)_W,$$

$$(33)\quad u(0) = u_0.$$

Finally, we have the following existence result

Theorem 3.1. *Assuming hypotheses (H1)-(H6), then **Problem (PV)** has at least one solution that satisfies the following regularity*

$$(34)\quad u \in W^{1,2}(0,T;V) \text{ and } \varphi \in L^2(0,T;W).$$

4. Proof of main result

The proof of Theorem 3.1 will be carried out in several steps and is based on results for time fractional hemivariational inequalities, variational equalities, the Rothe method, and the Banach fixed point.

In the first step, let $\beta \in L^2(0,T;V^*)$, and we will consider the following problem.

Problem (PV1): Find a displacement field $u : \Omega\times[0,T] \to \mathbb{R}^d$ a.e. $t \in]0,T[$ for all $v \in V$ and $\alpha \in]0,1]$ such that

$$(35)\quad \begin{aligned} &a\left({}_0^C D_t^\alpha u_\beta(t), v-u_\beta(t)\right) + b\left(u_\beta(t), v-u_\beta(t)\right) + \left(\beta(t), v-u_\beta(t)\right)_{V^*\times V} \\ &+ j(v) - j\left(u_\beta(t)\right) \geq \left(f(t), v-u_\beta(t)\right)_{V^*\times V}. \end{aligned}$$

Using Riesz's representation theorem, we define the following operator.

$$(36)\quad \left(f_\beta(t), v-u_\beta(t)\right)_{V^*\times V} = \left(f(t), v-u_\beta(t)\right)_{V^*\times V} - \left(\beta(t), v-u_\beta(t)\right)_{V^*\times V}.$$

Problem (PV1) can be reformulated as follows:

Find $w \in V$ for a.e., $t \in]0,T[$ such that

$$(37)\quad a(w(t)) + b\left(u_0 + {}_0I_t^\alpha u_\beta(t)\right) + \partial j\left(u_0 + {}_0I_t^\alpha u_\beta(t)\right) \ni f(t),$$

where $w(t) = {}_0^C D_t^\alpha u_\beta(t)$ and $u_\beta(t) = u_0 + {}_0I_t^\alpha w(t)$.

Let $N \in \mathbb{N}$ be fixed, and $\tau = \Delta t = \dfrac{T}{N}$. We can consider the following approximation of the fractional integral operator ${}_0I^{\alpha}_{t_n}w(t)$ by:

$$ {}_0\tilde{I}^{\alpha}_{t_n}w(t) = \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{\alpha-1}w(t_i)ds \\ = \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n}w(t_i)\left[(n-i+1)^{\alpha}-(n-i)^{\alpha}\right], \tag{38} $$

for a sufficiently small time step τ and $t_k = k\tau$.
We also define the functional f^k_τ as follows for $k = 1,\ldots,N$

$$ f^k_\tau = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(s)ds. \tag{39} $$

Now, by applying the Rothe method to (37), we obtain the following fractional Rothe problem:
Problem (FR): Find $\left\{w^k_\tau\right\}_{k=1}^{N} \subset V$ such that

$$ a\left(w^k_\tau\right) + b\left(u^k_{\beta_\tau}\right) + \partial j\left(u^k_{\beta_\tau}\right) \ni f^k_\tau, \tag{40} $$

where $u^k_{\beta_\tau}$ is defined by

$$ u^k_{\beta_\tau} = u_0 + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{k} w^i_\tau\left[(k-i+1)^{\alpha}-(k-i)^{\alpha}\right], \text{ for } k = 1,\cdots,N. \tag{41} $$

Then, we have the following Lemma.

Lemma 4.1. *There exists $\tau_0 > 0$ such that if $\tau \in (0,\tau_0)$, then **Problem (FR)** has at least one solution.*

Proof. We suppose that $\left\{w^k_\tau\right\}_{k=0}^{n-1}$ is given, and we will find $w^n_\tau \in V$ that satisfies (40) and (41).
For $w \in V$, we consider the multivalued operators $R : V \to V^*$ and $P : V \to V^*$ are given by

$$ R(w) = P(w) + \partial j(w), \tag{42} $$

where

$$ P(w) = a(w) + b\left(u_0 + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1} w^i_\tau\left[(n-i+1)^{\alpha}-(n-i)^{\alpha}\right] + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}w\right). \tag{43} $$

Now, we will prove that R is a surjective operator.
First, we will establish that the operator R is coercive. Let $c_\tau > 0$ be the constant defined by

$$ c_\tau = \|u_0\| + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}\left\|w^i_\tau\right\|\left[(n-i+1)^{\alpha}-(n-i)^{\alpha}\right]. \tag{44} $$

From the hypothesis (H6) and (44), we obtain

$$\begin{aligned}
&\left|\partial j\left(u_0+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1} w_\tau^i\left[(n-i+1)^\alpha-(n-i)^\alpha\right]+\frac{\tau^\alpha}{\Gamma(\alpha+1)}w\right)\right|_{V^*}\\
&\le c_\tau\left(1+\|u_0\|+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}\|w_\tau^i\|\left[(n-i+1)^\alpha-(n-i)^\alpha\right]+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w\|_V\right)\\
&\le c_\tau\left(1+c_\tau+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w\|_V\right).
\end{aligned} \tag{45}$$

Due to (H1) and (44), we have

$$\begin{aligned}
R(w,w)&\ge m_a\|w\|_V^2-M_b\left(c_\tau+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w\|_V\right)\|w\|_V\\
&\quad -c_j\left(1+c_\tau+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w\|_V\right)\\
R(w,w)&\ge\left(m_a-\frac{\tau^\alpha(M_b+c_j)}{\Gamma(\alpha+1)}\right)\|w\|_V^2-\left(M_bc_\tau+\frac{c_j\tau^\alpha}{\Gamma(\alpha+1)}\right)\|w\|_V-c_j(1+c_\tau).
\end{aligned} \tag{46}$$

We choose $\tau_0=\left(\frac{m_a\Gamma(\alpha+1)}{\tau^\alpha(M_b+c_j)}\right)^{1/\alpha}$ to demonstrate that R is a coercive operator. Next, we apply the assumptions regarding a and b to deduce

$$P(w,w)\ge\left(m_a-\frac{M_b\tau^\alpha}{\Gamma(\alpha+1)}\right)\|w\|_V^2. \tag{47}$$

Then, P is a bounded, continuous, and coercive operator. Hence, P is pseudomonotone.
Now, we will prove that the multivalued operator $G:V\to V^*$ is given by

$$G(w)=\partial j\left(u_0+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}w_\tau^i\left[(n-i+1)^\alpha-(n-i)^\alpha\right]+\frac{\tau^\alpha}{\Gamma(\alpha+1)}w\right), \tag{48}$$

for $w\in V$, is pseudomonotone.
It follows from the properties of j and the reflexivity of V that $G(w)$ is nonempty, convex, and weakly compact for all $w\in V$.
Also, by (H6), G is bounded. Let $\{w_m\}\subset V$ be such that $w_m\to w$ weakly in V, as $m\to\infty$ and

$$\theta_m\in\partial j\left(u_0+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}w_\tau^i\left[(n-i+1)^\alpha-(n-i)^\alpha\right]+\frac{\tau^\alpha}{\Gamma(\alpha+1)}w_m\right). \tag{49}$$

Since the operator ∂j is bounded, the sequence $\{\theta_m\}$ is bounded in V^*. Therefore, by passing to a subsequence, if necessary, we have $\theta_m\to\theta$ weakly in V^* as $m\to\infty$. Since the graph of the multivalued mapping

$$V\ni w\mapsto\partial j\left(u_0+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}w_\tau^i\left[(n-i+1)^\alpha-(n-i)^\alpha\right]+\frac{\tau^\alpha}{\Gamma(\alpha+1)}w\right), \tag{50}$$

is closed with respect to $V \times V^*$ topology (see [11, Proposition 3.23(v)]), we deduce that

$$\theta \in \partial j \left(u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n-1} w_\tau^i \left[(n-i+1)^\alpha - (n-i)^\alpha \right] + \frac{\tau^\alpha}{\Gamma(\alpha+1)} w \right). \tag{51}$$

Furthermore, it is clear that $\theta \in G(w)$ and we have

$$\langle \theta_m, w_m \rangle \to \langle \theta, w \rangle_{V^*,V}, \text{ as } m \to \infty. \tag{52}$$

Using Proposition 4.1, we deduce that the operator G is pseudomonotone. Hence, the operator R is pseudomonotone. Therefore, **Problem (FR)** has at least one solution. □

Now, we will present estimates for the sequence of solutions of the fractional Rothe problem (40).

Lemma 4.2. *Under assumptions (H1)-(H6) and (44), there exists $\tau_0 > 0$ and $C > 0$ independent of τ, such that $0 < \tau < \tau_0$, solutions of **Problem (FR)** satisfy*

$$\max_k \left\| w_\tau^k \right\| \le C, \tag{53}$$

$$\max_k \left\| u_{\beta_\tau}^k \right\| \le C, \tag{54}$$

$$\max_k \left\| \theta_\tau^k \right\| \le C, \tag{55}$$

for $k = 1, \ldots, N$ and $\theta_\tau^k \in \partial j \left(w_\tau^k \right)$ and

$$a\left(w_\tau^k \right) + b\left(w_\tau^k \right) + \theta_\tau^k = f_\tau^k. \tag{56}$$

Proof. For all $1 \le n \le N$, multiply (56) by w_τ^n. We obtain

$$a\left(w_\tau^n, w_\tau^n \right) + b\left(u_{\beta_\tau}^n, w_\tau^n \right) + \langle \theta_\tau^n, w_\tau^n \rangle_{V^* \times V} = \langle f_\tau^n, w_\tau^n \rangle. \tag{57}$$

From (41) and hypotheses (H1), (H2), and (H6), we have

$$\begin{aligned}
\langle f_\tau^n, w_\tau^n\rangle &\geq m_a \|w_\tau^n\|_V^2 - M_b \left\|u_{\beta_\tau}^n\right\|_V \|w_\tau^n\|_V - c_j\left(1+\left\|u_{\beta_\tau}^n\right\|_V\right)\|w_\tau^n\|_V \\
&\geq m_a \|w_\tau^n\|_V - M_b\left(\|u_0\|_V + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n \|w_\tau^i\|_V [(n-i+1)^\alpha-(n-i)^\alpha]\right)\|w_\tau^n\|_V \\
&-\left(1+\|u_0\|_V + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n \|w_\tau^i\|_V [(n-i+1)^\alpha-(n-i)^\alpha]\right)\|w_\tau^n\|_V \\
&\geq m_a \|w_\tau^n\|_V^2 - \frac{M_b\tau^\alpha}{\Gamma(\alpha+1)}\|w_\tau^n\|_V^2 - M_b\|u_0\|_V\|w_\tau^n\|_V \\
&- c_j\|w_\tau^n\|_V - c_j\|u_0\|_V\|w_\tau^n\|_V - \frac{c_j\tau^\alpha}{\Gamma(\alpha+1)}\|w_\tau^n\|_V^2 \\
&- \frac{M_b\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha]\|w_\tau^i\|_V\|w_\tau^n\|_V \\
&- \frac{c_j\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha]\|w_\tau^i\|_V\|w_\tau^n\|_V .
\end{aligned} \tag{58}$$

Therefore, based on the previous analysis, we deduce that

$$\begin{aligned}
&\|f_\tau^n\|_V + \frac{\tau^\alpha(M_b+c_j)}{\Gamma(\alpha+1)}\|w_\tau^i\|_V\sum_{i=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha] \\
&+c_j+(c_j+M_b)\|u_0\|_V \geq \left(m_a - \frac{\tau^\alpha(M_b+c_j)}{\Gamma(\alpha+1)}\right)\|w_\tau^n\|_V .
\end{aligned} \tag{59}$$

We choose $\tau_0 = \left(\frac{m_a\Gamma(\alpha+1)}{2(M_b+c_j)}\right)^{1/\alpha}$. We deduce that $m_a - \frac{\tau^\alpha(M_b+c_j)}{\Gamma(\alpha+1)} \geq \frac{m_a}{2}$ for all $0<\tau<\tau_0$. Thus,

$$\begin{aligned}
&2\frac{\|f_\tau^n\|_V}{m_a} + 2\frac{c_j+c_j\|u_0\|+M_b\|u_0\|}{m_a} \\
&+2\frac{\tau^\alpha(M_b+c_j)}{m_a\Gamma(\alpha+1)}\sum_{i=1}^{n-1}\|w_\tau^i\|_V[(n-i+1)^\alpha-(n-i)^\alpha] \geq \|w_\tau^n\|_V .
\end{aligned} \tag{60}$$

Using hypothesis (H3), for all $\tau>0$ and $n\in\mathbb{N}$, there exists a constant $c_f>0$ such that $\|f_\tau^n\|_V \leq c_f$. Naming

$$c_0 = \frac{2}{m_a}\left(c_f + M_b\|u_0\| + c_j\|u_0\| + c_j\right). \tag{61}$$

Applying the generalized discrete Gronwall's inequality of Lemma 4.6, we can see that

$$\begin{aligned} \|f_\tau^n\|_V &\le c_0 \exp\left(\frac{2\tau^\alpha (M_b+c_j)}{m_a \Gamma(\alpha+1)} \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha]\right) \\ &= c_0 \exp\left(\frac{2t_n^\alpha (M_b+c_j)}{m_a \Gamma(\alpha+1)}\right) \le C, \end{aligned} \tag{62}$$

so, the estimate (53) is true.
Furthermore, considering (62) and (41), we obtain the following estimate

$$\begin{aligned} \left\|u_{\beta_\tau}^n\right\|_V &= \left\| u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} w_\tau^i [(n-i+1)^\alpha - (n-i)^\alpha] \right\|_V \\ &\le \|u_0\|_V + \sum_{i=1}^{n} \left(t_{n-i+1}^\alpha - t_{n-i}^\alpha\right) \\ &\le \|u_0\|_V + \frac{CT^\alpha}{\Gamma(\alpha+1)} \le C. \end{aligned} \tag{63}$$

Finally, by (H6), we obtain the following estimate for θ_τ^n

$$\|\theta_\tau^n\|_{V^*} \le c_j(1+\|u_\tau^n\|_V) \le c_j(1+C). \tag{64}$$

Thus, Lemma 4.2 is proven □

The solvability of **Problem (PV1)** follows from the following result

Theorem 4.3. *For all $v \in V$ and a.e., $t \in (0,T)$, **Problem (PV1)** has at least one solution $u \in V$.*

Proof. Let $\{\tau_n\}$ be a sequence such that $\tau_n \to 0$, as $n \to \infty$.
Based on the estimate (53)-(55) the sequence $\{\overline{w}_\tau\}$, $\{\overline{u}_{\beta_\tau}\}$ and $\{\overline{\theta}_\tau\}$ interpolate to $\{w_\tau\}$, $\{u_{\beta_\tau}\}$ and $\{\theta_\tau\}$ respectively are bounded for $k = 1, \ldots, N$.
Therefore, there exist $w \in V$, $u \in V$ and $\theta \in V^*$ such that

$$\overline{w}_\tau \to w \text{ weakly in } V, \text{ as } \tau \to 0, \tag{65}$$

$$\overline{u}_{\beta_\tau} \to u \text{ weakly in } V, \text{ as } \tau \to 0, \tag{66}$$

$$\overline{\theta}_\tau \to \theta \text{ weakly in } V^*, \text{ as } \tau \to 0. \tag{67}$$

Using [10, Lemma 4(a)], we obtain that

$${}_0I_t^\alpha \overline{w}_\tau \to {}_0I_t^\alpha w \text{ weakly in } V, \text{ as } \tau \to 0. \tag{68}$$

By using (53), and for all $t \in (0,T)$, it follows that

$$\begin{aligned}
\left\|\overline{u}_{\beta_\tau}(t) - u_0 - {}_0I_t^{\alpha}\overline{w}_\tau(t)\right\| &= \Bigg\| \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} w_\tau^i \left[(n-i+1)^\alpha - (n-i)^\alpha\right] \\
&- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\overline{w}_\tau(s)ds \Bigg\| \\
&\le \frac{C}{\Gamma(\alpha)} \int_t^{t_n} (t_n-s)^{\alpha-1} ds + \int_0^t \left|(t-s)^{\alpha-1} - (t_n-s)^{\alpha-1}\right| ds \\
&\le \frac{C}{\Gamma(\alpha)} \left[(t_n-t)^\alpha + t^\alpha + (t_n-t)^\alpha - t_n^\alpha\right],
\end{aligned} \tag{69}$$

for $t \in [t_{n-1}, t_n]$. Then

$$\overline{u}_{\beta_\tau}(t) - u_0 - {}_0I_t^{\alpha}\overline{w}_\tau(t) \to 0 \text{ strongly in } V, \text{ as } \tau \to 0. \tag{70}$$

This, combined with (68), leads to

$$\overline{u}_{\beta_\tau}(t) \to u_0 + {}_0I_t^{\alpha} w(t) \text{ weakly in } V, \text{ as } \tau \to 0. \tag{71}$$

Since the mapping $v \mapsto \partial j(v)$ is upper semi-continuous from V to V^*, and based on (67) and [11, Theorem 3.13], we have

$$\theta(t) \in \partial j\left(u_0 + {}_0I_t^{\alpha} w_\tau\right), \text{ for a.e., } t \in]0,T[. \tag{72}$$

Now, we define the Nemytskii operators $\overline{a}$ and $\overline{b}$ corresponding to a and b as follows:

$$(\overline{a}w)(t) = aw(t), \text{ and } (\overline{b}w)(t) = b\left(u_0 + {}_0I_t^{\alpha} w(t)\right), \tag{73}$$

for all $w \in V$ and a.e., $t \in]0,T[$.

Given the assumption (H2), as well as (65) and (66), we have for $t \in (0,T)$ that

$$\overline{a}\,\overline{w}_\tau \to \overline{a}w \text{ weakly in } V, \text{ as } \tau \to 0, \tag{74}$$

and

$$b\left(u_0 + {}_0I_t^{\alpha}\overline{w}_\tau(t)\right) \to b\left(u_0 + {}_0I_t^{\alpha} w(t)\right) \text{ weakly in } V, \text{ as } \tau \to 0. \tag{75}$$

From (H2) and (53), we have

$$\begin{aligned}
\int_0^T \left\|b\left(u_0 + {}_0I_t^{\alpha}\overline{w}_\tau(t)\right)\right\|_V dt &\le \frac{M_b C}{\Gamma(\alpha+1)} \int_0^T t^\alpha dt + TM_b\|u_0\| \\
&= \frac{M_b C T^{\alpha+1}}{\Gamma(\alpha+2)} + TM_b\|u_0\|.
\end{aligned} \tag{76}$$

Applying the Lebesgue dominated convergence theorem, we can write

$$\begin{aligned}
\lim_{\tau\to 0} b(\overline{w}_\tau, v) &= \lim_{\tau\to 0} \int_0^T b\left(u_0 + {}_0I_t^{\alpha}\overline{w}_\tau(t), v(t)\right) dt \\
&= \int_0^T \lim_{\tau\to 0} b\left(u_0 + {}_0I_t^{\alpha}\overline{w}_\tau(t), v(t)\right) dt \\
&= \int_0^T b\left(u_0 + {}_0I_t^{\alpha} w(t), v(t)\right) dt = b(w,v).
\end{aligned} \tag{77}$$

On the other hand, from [2, Lemma 3.3], we know that

$$f_\tau \to f \text{ strongly in } V, \text{ as } \tau \to 0. \tag{78}$$

Finally, by utilizing (72), (74), and (76), we can pass to the limit in equation (40), which implies that $w \in L^2(0,T;V)$ is a solution to the problem (37).
Hence, we infer that $u_\beta \in W^{1,2}(0,T;V)$ given by $u_\beta(t) = u_0 + {}_0I_t^\alpha w(t)$ for almost every $t \in]0,T[$ is a solution to **Problem (PV)**. □

In the second step, let $z \in L^2(0,T;W^*)$, and use the displacement field u_β obtained in Lemma 4.1 to derive the following variational formulation of the electric potential.
Problem (PV2): Find an electric potential $\varphi_z : \Omega \times]0,T[\to \mathbb{R}$ a.e. $t \in]0,T[$ for all $\xi \in W$ such that

$$c(\varphi_z(t),\xi) - e(u_z(t),\xi) + (z(t),\xi)_W = (q(t),\xi)_W\,. \tag{79}$$

The well-posedness of **Problem (PV2)** follows

Lemma 4.4. *For $\xi \in W$ and for a.e. $t \in]0,T[$,* ***Problem (PV2)*** *has a unique solution $\varphi_z \in L^2(0,T;W)$.*

Proof. Let $t \in]0,T[$.
Using Riesz's representation theorem, we define the operator $\mathscr{A} : W \to W$ as follows:

$$\mathscr{A}(\varphi_z(t),\xi) := c(\varphi_z(t),\xi) - e(u_z(t),\xi)\,, \tag{80}$$

and

$$(q_z(t),\xi)_W = (q(t),\xi)_W - (z(t),\xi)_W\,, \text{ for all } \xi \in W. \tag{81}$$

By (28) and in accordance with (H3), we obtain that $q_z \in L^2(0,T;W)$
Then equation (79) can be written

$$\mathscr{A}(\varphi_z(t),\xi) = (q_z(t),\xi)_W\,, \text{ for all } \xi \in W. \tag{82}$$

Let $\varphi_{z_1}, \varphi_{z_2} \in W$. Then, assumption (H1) implies

$$\mathscr{A}(\varphi_{z_1}(t) - \varphi_{z_2}(t), \varphi_{z_1}(t) - \varphi_{z_2}(t)) \geq m_c \|\varphi_{z_1}(t) - \varphi_{z_2}(t)\|_W^2\,, \tag{83}$$

and by hypothesis (H2), we have that

$$\|\mathscr{A}(\varphi_{z_1}(t) - \varphi_{z_2}(t),\xi)\|_W \leq \sup(M_a,M_e)\|\varphi_{z_1}(t) - \varphi_{z_2}(t)\|_W \|\xi\|_W. \tag{84}$$

Thus

$$\|\mathscr{A}(\varphi_{z_1}(t) - \varphi_{z_2}(t))\|_W \leq \sup(M_a,M_e)\|\varphi_{z_1}(t) - \varphi_{z_2}(t)\|_W\,. \tag{85}$$

Inequalities (73) and (75) demonstrate that the operator $\mathscr{A}$ is strongly monotone and Lipschitz continuous on W. Therefore, we deduce the existence of a unique element $\varphi_z \in L^2(0,T;W)$. □

In the last step, for all $t \in (0,T)$, we define the operator

$$\Lambda(z,\beta)(t) := (\Lambda_1(z,\beta)(t), \Lambda_2(z,\beta)(t))\,, \tag{86}$$

given by

$$\begin{aligned} (\Lambda_1(z,\beta)(t),v) &= e(v,\varphi_z(t))\,, \text{ for all } v \in V, && (87)\\ (\Lambda_2(z,\beta)(t),\xi) &= \ell\left(u_\beta(t),\varphi_z(t),\xi\right)\,, \text{ for all } \xi \in W. && (88)\end{aligned}$$

We have the following result

Lemma 4.5. *For* $(z,\beta)\in L^2(0,T;W^*)\times L^2(0,T;V^*)$ *the operator* Λ *is continuous and has a unique fixed point* (z^*,β^*).

Proof. Let $(z,\beta)\in L^2(0,T;W^*\times V^*)$ and $t_1,t_2\in[0,T]$. From (87) and (H2), we have

$$\|\Lambda_1(z,\beta)(t_1)-\Lambda_1(z,\beta)(t_2)\|_{W\times V}\le M_e\|\varphi_z(t_1)-\varphi_z(t_2)\|_W. \tag{89}$$

By (88), in combination with (24) and (30), there exists a positive constant c depending on L_ψ, M_ψ, and c_d such that

$$\begin{aligned}&\|\Lambda_2(z,\beta)(t_1)-\Lambda_2(z,\beta)(t_2)\|_{W\times V}\\&\le c\left(\left\|u_\beta(t_1)-u_\beta(t_2)\right\|_V+\|\varphi_z(t_1)-\varphi_z(t_2)\|_W\right).\end{aligned} \tag{90}$$

Considering the regularity of $\varphi_{z\beta}$ and $u_{z\beta}$, we find that

$$(\Lambda_1(z,\beta)\times\Lambda_2(z,\beta))\in C([0,T],W\times V). \tag{91}$$

Hence, the operator Λ is continuous.
Let $(z_1,\beta_1),\ (z_2,\beta_2)\in L^2(0,T;W^*\times V^*)$ and $t\in]0,T[$.
Similar to (90) and utilizing (86), we have

$$\begin{aligned}&\|\Lambda(z_1,\beta_1)(t)-\Lambda(z_2,\beta_2)(t)\|_{W\times V}\\&\le c\left(\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V+\|\varphi_{z_1}(t)-\varphi_{z_2}(t)\|_W\right).\end{aligned} \tag{92}$$

Therefore, from (35), we obtain

$$\begin{aligned}&b\left(u_{\beta_1}(t)-u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)\\&\le -a\left({}_0^CD_t^\alpha u_{\beta_1}(t)-{}_0^CD_t^\alpha u_{\beta_1}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)\\&-\left(\beta_1(t)-\beta_2(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)_{V^*\times V}.\end{aligned} \tag{93}$$

By Definition 4.2, we deduce that

$$\left\|{}_0^CD_t^\alpha u_{\beta_1}(t)-{}_0^CD_t^\alpha u_{\beta_2}(t)\right\|_V\le\frac{T^{1-\alpha}}{\Gamma(\alpha)}\left\|\dot u_{\beta_1}(t)-\dot u_{\beta_2}(t)\right\|_V. \tag{94}$$

Now, from (H1) and (H2), we have

$$\begin{aligned}&m_b\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2\\&\le M_a\left\|{}_0^CD_t^\alpha u_{\beta_1}(t)-{}_0^CD_t^\alpha u_{\beta_2}(t)\right\|_V\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V\\&+\|\beta_1(t)-\beta_2(t)\|_V\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V.\end{aligned} \tag{95}$$

Also for $u_{\beta_i}(t)=\int_0^t\dot u_{\beta_i}(s)ds+u_0$ for $i=1,2$, we deduce that

$$\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V\le c\int_0^t\left\|\dot u_{\beta_1}(s)-\dot u_{\beta_2}(s)\right\|_V ds. \tag{96}$$

Combining equations (94) through (96), integrating from 0 to t and applying Gronwall's inequality, we can conclude that there exists $c>0$ such that

$$\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_{L^2(0,T;V)} \leq c\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(0,T;V)}. \tag{97}$$

On the other hand, using (79), we can express

$$\begin{aligned} & c\left(\varphi_{z_1}(t)-\varphi_{z_2}(t)\,,\ \varphi_{z_1}(t)-\varphi_{z_2}(t)\right) \\ & -e\left(u_{z_1}(t)-u_{z_2}(t)\,,\ \varphi_{z_1}(t)-\varphi_{z_2}(t)\right) \\ & \left(z_1(t)-z_2(t)\,,\ \varphi_{z_1}(t)-\varphi_{z_2}(t)\right)_W=0. \end{aligned} \tag{98}$$

Similarly to (97), we find that

$$\left\|\varphi_{z_1}(t)-\varphi_{z_2}(t)\right\|_{L^2(0,T;W)} \leq c\left\|z_1(t)-z_2(t)\right\|_{L^2(0,T;W)}. \tag{99}$$

We substitute (97) and (99) into (92) to obtain

$$\left\|\Lambda(z_1,\beta_1)(t)-\Lambda(z_2,\beta_2)(t)\right\|_{L^2(0,T;W\times V)} \leq c\left\|(z_1,\beta_1)-(z_2,\beta_2)\right\|_{L^2(0,T;W\times V)}. \tag{100}$$

Reiterating this inequality for n times

$$\left\|\Lambda^n(z_1,\beta_1)(t)-\Lambda^n(z_2,\beta_2)(t)\right\|_{L^2(0,T;W\times V)} \leq \frac{(cT)^n}{n!}\left\|(z_1,\beta_1)-(z_2,\beta_2)\right\|_{L^2(0,T;W\times V)}. \tag{101}$$

Thus, for sufficiently large values of n, Λ^n becomes a contraction mapping in the Banach space $L^2(0,T;W\times V)$, and as a result, Λ possesses a unique fixed point □

We are now prepared to prove Theorem (3.1)

Proof of Theorem (3.1). Let $(z^*,\beta^*)\in L^2(0,T;W^*\times V^*)$ be the fixed point of the operator Λ and denote $\varphi^*_{z^*\beta^*}$, $u^*_{z^*\beta^*}$ be the solutions of **Problem (PV1)** and **Problem (PV2)**, respectively. For $(z,\beta)=(z^*,\beta^*)$, the definition of Λ implies that $\left(u^*_{z^*\beta^*},\varphi^*_{z^*\beta^*}\right)$ is a solution of **Problem (PV)**. □

Appendix

Now, we recall the well-established definition in the field of fractional calculus theory and nonlinear analysis, as presented in references [4, 7, 11, 14].

Definition 4.1 (Riemann-Liouville fractional integral)**.** *Let X be a Banach space and $(0,T)$ be a finite time interval. The Riemann-Liouville fractional integral of order $\alpha>0$ for a given function $f\in L^1(0,T;X)$ is defined by*

$${}_0I_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\ \ \forall t\in(0,T),$$

where $\Gamma(.)$ stands for the Gamma function defined by

$$\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}dt.$$

To complement the definition, we set ${}_0I_t^0=I$, where I is the identity operator, which means that ${}_0I_t^0f(t)=f(t)$ for a.e. $t\in(0,T)$.

Definition 4.2 (Caputo derivative of order. $0<\alpha\leq 1$)**.** *Let X be a Banach space, $0<\alpha\leq 1$ and $(0,T)$ be a finite time interval. For a given function $f\in AC(0,T;W)$, the Caputo fractional derivative of f is defined by*

$$ {}_0^C D_t^{\alpha} f(t) = {}_0 I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \ \ \forall t\in(0,T). $$

The notation $AC(0,T;X)$ refers to the space of all absolutely continuous functions from $(0,T)$ into X.

It is obvious that if $\alpha=1$, the Caputo derivative reduces to the classical first-order derivative, that is, we have

$$ {}_0^C D_t^{1} f(t) = I f'(t) = f'(t), \ \ \text{for a.e. } t\in(0,T). $$

Proposition 4.1. *Let X be a reflexive Banach space, and assume that $T: X\to 2^{X^*}$ satisfies the following conditions*

(a) *for every $v\in X$, Tv is a nonempty, closed and convex subset of X^*,*
(b) *the operator T is bounded,*
(c) *if $v_n\to v$ weakly in X and $v_n^*\to v^*$ weakly in X^* with $v_n^*\in Tv_n$ and if*

$$ \limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle \leq 0, $$

then $v^*\in Tv$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$.

Then the operator A is pseudomonotone.

Lemma 4.6. *Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be nonnegative sequences satisfying*

$$ u_n \leq v_n + \sum_{k=1}^{n-1} w_k u_k \ \ \textit{for all} \ \ n\geq 1. $$

Then, we have

$$ u_n \leq v_n + \sum_{k=1}^{n-1} v_k w_k \exp\left(\sum_{j=k+1}^{n-1} w_j\right) \ \ \textit{for all} \ \ n\geq 1. $$

Moreover, if $\{u_n\}$ and $\{w_n\}$ are such that

$$ u_n \leq \alpha = \sum_{k=1}^{n-1} w_k u_k \ \ \textit{for all} \ \ n\geq 1, $$

where $\alpha>0$ is a constant, then for all $n\geq 1$, it holds

$$ u_n \leq \alpha \exp\left(\sum_{k=1}^{n-1} w_k\right). $$

References

[1] M. Bouallala, E. H. Essoufi, *A Thermo-Viscoelastic Fractional Contact Problem with Normal Compliance and Coulomb's Friction*. Journal of Mathematical Physics, Analysis, Geometry, **17** (2021), no. 3, 280-294.

[2] C. Carstensen, J. Gwinner, *A theory of discretization for nonlinear evolution inequalities applied to parabolic Signorini problems*. Annali di Matematica Pura ed Applicata, **177** (1999), no. 1, 363-394.

[3] W. Chen, H, Sun, X. Li, X. *Fractional derivative modeling in mechanics and engineering*. Beijing, China: Springer., 2022.

[4] F.H. Clarke, *Optimization and nonsmooth analysis*. Society for Industrial and Applied Mathematics, 1990.

[5] J. Han, S. Migórski, H.Zeng, *Weak solvability of a fractional viscoelastic frictionless contact problem*. Applied Mathematics and Computation Elsevier. **303** (2017), no. 170, 1-18.

[6] W. Han, M. Sofonea, *Quasistatic contact problems in viscoelasticity and viscoplasticity*. Studies in Advanced Mathematics, American Mathematical Society, 2001.

[7] A.A Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier Science Limited.**204**, 2006.

[8] Z. Lerguet, M. Shillor, M. Sofonea, *A frictional contact problem for an electro-viscoelastic body*. Electronic Journal of Differential Equations. **2007** (2007).

[9] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. World Scientific, 2022.

[10] S. Migórski, A. Ochal, *Quasi-static hemivariational inequality via vanishing acceleration approach*. SIAM Journal on Mathematical Analysis, **41** (2009), 1415-1435.

[11] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear inclusions and hemivariational inequalities: models and analysis of contact problems*. Springer Science and Business Media, **26**, 2012.

[12] S. Müller, M. Kästner, J. Brummund, V. Ulbricht, *A Nonlinear Fractional Viscoelastic Material Model for Polymers*. Computational Materials Science, **50** (2011), no. 10, 2938-2949.

[13] D. Paola, R. Heuer, A. Pirrotta, *Fractional visco-elastic EulerBernoulli beam*. International Journal of Solids and Structures. **50** (2013), no. 22-23, 3505-3510.

[14] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[15] M. Rochdi M. Shillor, M. Sofonea, *Analysis of a quasistatic viscoelastic problem with friction and damage*. Advances in Mathematical Sciences and Applications. **10** (2000), no. 1, 173-189.

[16] M. Rochdi, M. Shillor, M. Sofonea, *Quasistatic viscoelastic contact with normal compliance and friction*. Journal of Elasticity. **51** (1998), no. 2, 105-126.

[17] S. Zeng, Z. Liu, S. Migórski, *A class of fractional differential hemivariational inequalities with application to contact problem*. Zeitschrift für angewandte Mathematik und Physik, **69** (2018), no. 2, 1-23.

[18] F. Zeng, C. Li, F. Liu, I. Turner, *The use of finite difference/element approaches for solving the time-fractional subdiffusion equation*. SIAM Journal on Scientific Computing, **35** (2013), no. 6, 2976-3000.

[19] S. Zeng, S. Migórski, *A class of time-fractional hemivariational inequalities with application to frictional contact problem*. Communications in Nonlinear Science and Numerical Simulation Elsevier. **56** (2018), 34-48.

[20] M. Bouallala, EL-H. Essoufi, V. T. Nguyen, W. Pang, *A time-fractional of a viscoelastic frictionless contact problem with normal compliance*. European Physical Journal Special Topics, **232** (2023), 2549-2558.

[21] Z. Lerguet, M. Shillor, M. Sofonea, *A frictional contact problem for an electro-viscoelastic body*. Electronic Journal of Differential Equations, (2007), no. 170, 16.

[22] M. Sofonea, K. Kazmi, M. Barboteu, W. Han,s *Analysis and numerical solution of a piezoelectric frictional contact problem*. Applied mathematical modelling, **36** (2012), no. 9, 4483-4501.

[23] M. Barboteu, M. Sofonea, *Modeling and analysis of the unilateral contact of a piezoelectric body with a conductive support*. Journal of mathematical analysis and applications, **358** (2009), no. 1, 110-124.

[24] Y. Ouafik, *A piezoelectric body in frictional contact*. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, (2005), 233-242.

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