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ABOSORBING IDEALS IN COMMUTATIVE RINGS: AN APPLICATION IN TOPOLOGICAL GROUP ACTION THEORY

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ABSTRACT. In this paper we present a certain theoretical construction with the so-called absorbing ideals in commutative rings, which were systematically defined and studied in [1] by D. F. Anderson and A. Badawi in 2011. We present a construction using tolls of the Absorbing Ideal Theory in Commutative Rings and an application in Topological Group Action Theory. More precisely, we are going to study certain primitive group actions on collections of proper ideals of commutative rings. Furthermore, we will classify geometrically all irreducible affine varieties for which the collection of radical ideals of its coordinate ring is under a certain primitive group action.

1. Introduction

Throughout this paper, we write R to be commutative ring with identity, $\mathscr{I}(R)$ the set of all proper ideals of R, $\sqrt{\mathscr{I}(R)}$ the set of proper radical ideals of R and $Aut(R)$ the group of automorphisms of R.

We will show how we can apply such construction in the *Topological Group Action Theory*, as well how to study the effects of this on algebraic varieties.

In what follows, we present the definitions and basic notions about the *Absorbing Ideal Theory* in commutative rings, some basic results of this theory and also some theoretical constructions.

Next, we provide basic definitions of *Topological Group Action Theory* and we study the effects of certain primitive group actions on collections of ideals of integral domains through the previously obtained constructions with the absorbing ideals.

In the last part, we will show what are the effects of these primitive group actions in relation to the irreducibility of algebraic varieties.

In order to facilitate the development of this work, we write $\mathbb{N}$ to denote the set of all non-negative integers, $Spec(R)$ to denote the set of all prime ideals of R, "$\subset$" and "$\subseteq$" to denote the strict inclusion and not necessarily strict inclusion of sets. Definitions that are not presented or explained here can be searched in [2], [3] and [5].

2. Basic definitions and some ring-theoretic constructions with absorbing ideals

Let R be a ring. We say that a proper ideal I of R is n-**absorbing** if, for all $x_1,\dots,x_{n+1} \in R$ such that $x_1,\dots,x_{n+1} \in I$, there are $1 \leq i_1 < \cdots < i_n \leq n$ such that $x_{i_1},\dots,x_{i_n} \in I$. The most trivial example is a proper ideal $P \subset R$ that is a 1-absorbing if and only if is prime.

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Let I be a proper ideal of R. We write $\omega_R(I)$ for denote the smallest positive integer for which I is an absorbing ideal. When $\omega(I)$ does not exist, we will write $\omega(I)=\infty$; hence either I is n-absorbing for all $n \in \mathbb{N}$, or I is not n-absorbing for every $n \in \mathbb{N}$. If $\omega_R(I) < \infty$, I is $\omega_R(I)$-absorbing, but it is not k-absorbing which is $1 \le k < \omega_R(I)$. We can establish a well defined correspondence $\omega_R : \mathscr{I}(R) \longrightarrow \mathbb{N} \cup \{\infty\}$ that associates each ideal $I \subset R$ to $\omega_R(I)$.

Let $\mathscr{I} \subseteq \mathscr{I}(R)$ be a non-empty set. We define

$$\Omega(R,\mathscr{I}) := \{\omega_R(I);\, I \in \mathscr{I}\}$$

Remark that $\Omega(R,\mathscr{I}) = \omega_R(\mathscr{I})$. If $\mathscr{I} = \mathscr{I}(R)$, we write $\Omega(R,\mathscr{I}) = \Omega(R)$. Hence, we have $\Omega(R,\mathscr{I}) \subseteq \Omega(R)$ always.

Let $\sim_{\mathscr{I}}$ an equivalence relation on $\mathscr{I}$. The equivalence class of an ideal $I \in \mathscr{I}$ is write as $[I]$.

Proposition 2.1. *There exist an one-to-one correspondence between the elements of* $\Omega(R,\mathscr{I})$ *and the equivalence class in* $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$.

Proof. In fact, for each $\omega_R(I) \in \Omega(R,\mathscr{I})$, consider the equivalence class $[I] \in \frac{\mathscr{I}}{\sim_{\mathscr{I}}}$. Of course, if $\omega_R(J) \in \Omega(R,\mathscr{I})$ and $\omega_R(J) \neq \omega_R(I)$, then $[J] \in \frac{\mathscr{I}}{\sim_{\mathscr{I}}}$ and $[J] \neq [I]$. Since every equivalence class in $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$ is represented by some ideal $I \in \mathscr{I}$, we concludes that exist an one-to-one correspondence between the elements of $\Omega(R,\mathscr{I})$ and the equivalence class in $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$, as desired. □

Let $I, J \in \mathscr{I}$. We define the relation $\sim_{\mathscr{I}}$ in $\mathscr{I}$ by

$$I \sim_{\mathscr{I}} J \Longleftrightarrow \omega_R(I) = \omega_R(J)$$

The relation $\sim_{\mathscr{I}}$ on $\mathscr{I}$ is an equivalence relation, since the equality relation $=$ it is also.

Theorem 2.1. *(Theorem 4.2, [1]). Let* $f : R \longrightarrow T$ *be a surjective ring homomorphism and* I *a* n*-absorbing ideal of* R *containing* $\ker(f)$*. Then* $f(I)$ *is an* n*-absorbing ideal of* T *if and only if* I *is an* n*-absorbing ideal of* R*. Moreover,* $\omega_R(f^{-1}(J)) = \omega_T(J)$*. In particular, this holds if* f *is an isomorphism.*

3. Topological group actions

For any non-empty set X, $Sym(X)$ will denote the group of permutations on X. If G is a group acting on X, we say that X is a G-**set**. For each $g \in G$ and $x \in X$, let is denote by x^g the image of the action of g on x whenever we want to omit it or when there are no ambiguities about the action in question. We say that G **acts transitively** on X if, for any $x, x' \in X$, there exist $g \in G$ such that $x' = x^g$.

Suppose that G acts transitively on X. A subset $B \subseteq X$ is called **block** of X when, for every $g \in G$, $B^g = B$ or $B^g \cap B = \emptyset$. Remark that X and all its singleton subsets are blocks of X, called **trivial blocks**. Also Remark that if $B \subseteq X$ is a block, then B^g is also a block of X, for every $g \in G$. In this case, the set $S(B) := \{B^g \,|\, g \in G\}$ is called **block system** of X. Thus, $S(X) = \{X\}$ and $S(\{x\}) = \{\{x^g\} \,|\, g \in G\}$, with $x \in X$, are the so-called **trivial block systems** of X. Remark that any block system $S(B)$ of X is a partition of X and, conversely, every partition of X is a block system of X. Hence, trivial blocks systems are trivial partitions of X and vice versa.

We say that G **acts primitively** on X if X admits only trivial blocks. According to this definition, we see that G acts primitively on X if and only if X admits only trivial blocks systems. Remark that if G acts primitively on X then G acts transitively on X. In fact, if G acts primitively on X then $x = x'$ or $x' \in S(\{x\})$ for every pair of elements $x, x' \in X$.

Let $\equiv$ be an equivalence relation on X. We say that $\equiv$ is a G-**congruence** if, for any $x, x' \in X$ and $g \in G$,

$$x \equiv x' \Leftrightarrow x^g \equiv x'^g$$

It is not difficult to see that the trivial equivalence relations given by

$$x \equiv_1 x' \Leftrightarrow x, x' \in X \;\text{ and }\; x \equiv_2 x' \Leftrightarrow x = x'$$

are the G-**trivial congruences**. Here, it should be Remark that a block system $S(B)$ of X is a partition of X that is obtained from a G-congruence $\equiv$, namely, $x \equiv x' \Leftrightarrow x, x' \in B^g$, for some $g \in G$. Conversely, every partition of X obtained by a G-congruence $\equiv$ is a block system of X, given by the collection of equivalence classes. Thus, a block system of X is trivial if and only if it is the partition obtained from one of the trivial G-congruences.

We say that the group G is a **topological group** if G is a topological space such that the maps $(g, g') \longmapsto gg'$ and $g \longmapsto g^{-1}$ are continuous. We say that X is a G-**space** if G is a topological group and X is a topological space and at the same time a G-set. Remark that if X is a G-set such that X and G are equipped with the discreet topology, then X is a G-space.

A non empty subset Y of a topological space X is **irreducible** if it can not be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y . The empty set is not considered to be irreducible.

If X is a topological space, we define the **dimension** of X, and denote it by $dim\,X$, to be the supremum of all integers n such that there exists a chain

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

of distinct irreducible closed subsets of X.

Proposition 3.1. *If X is a topological space and $Y \subseteq X$ is a subspace, then $dim\,Y \leq dim\,X$.*

Proof. Let

$$Y_0 \subset Y_1 \subset \cdots \subset Y_m$$

be a increasing chain of irreducible closed subsets of Y. Since each Y_i is closed and irreducible in Y, then each Y_i is closed and irreducible in X. Indeed, each Y_i can be written as $Y_i = Y_i \cap X$ and each Y_i can not be written as the union of two proper closed subsets, since Y_i is closed and irreducible in Y. Hence,

$$Y_0 \subset Y_1 \subset \cdots \subset Y_m$$

be a increasing chain of irreducible closed subsets of X and, therefore, $dim\,Y \leq dim\,X$. □

4. Main results

Let $\mathscr{I} \subseteq \mathscr{I}(R)$ be a non-empty set and $G(R)$ be a subgroup of $Aut(R)$. For each $\sigma \in G(R)$, define the correspondence

$$\widehat{\sigma} : \mathscr{I} \longrightarrow \mathscr{I}$$
$$I \longmapsto \widehat{\sigma} := \sigma(I)$$

If such correspondence is well defined for each $\sigma \in G(R)$, then we also define the set

$$G_R := \{\widehat{\sigma} \,|\, \sigma \in G(R)\}.$$

In this case, G_R is a subgroup of $Sym(\mathscr{I})$.

Consider the function

$$\Psi : G_R \times \mathscr{I} \longrightarrow \mathscr{I}$$
$$(\widehat{\sigma}, I) \longmapsto \Psi(\widehat{\sigma}, I) := \widehat{\sigma}(I)$$

Let $\widehat{\sigma}, \widehat{\tau} \in G_R$ and $I \in \mathscr{I}$. Then

$$\Psi\left(id_{\mathscr{I}}, I\right) = \widehat{id_R}(I) = I \text{ and } \Psi(\widehat{\sigma} \circ \widehat{\tau}, I) = (\widehat{\sigma} \circ \widehat{\tau})(I) = \widehat{\sigma}(\widehat{\tau}(I)) = \Psi(\widehat{\sigma}, \Psi(\tau, I))$$

Therefore, Ψ is a left group action of G_R on $\mathscr{I}$. Considering $\mathscr{I}$ and G_R equipped with the discreet topology, we see that $\mathscr{I}$ is a G_R-space.

Proposition 4.1. *$\sim_{\mathscr{I}}$ is a G_R-congruence.*

Proof. Now, Remark that, for all $I, J \in \mathscr{I}$, we have

$$I \sim_{\mathscr{I}} J \Leftrightarrow \omega_R(I) = \omega_R(J)$$

Let $I, J \in \mathscr{I}$ and $\widehat{\sigma} \in G_R$. By Theorem 2.1, since σ is an isomorphism, then

$$\omega_R(\widehat{\sigma}(I)) = \omega_R(\sigma(I)) = \omega_R(I) \text{ and } \omega_R(\widehat{\sigma}(J)) = \omega_R(\sigma(J)) = \omega_R(J).$$

Then for all $I, J \in \mathscr{I}$ and $\widehat{\sigma} \in G_R$, we have

$$I \sim_{\mathscr{I}} J \Leftrightarrow \omega_R(\widehat{\sigma}(I)) = \omega_R(\widehat{\sigma}(J)) \Leftrightarrow I^{\widehat{\sigma}} \sim_{\mathscr{I}} J^{\widehat{\sigma}}$$

Thus $\sim_{\mathscr{I}}$ is a G_R-congruence. □

Proposition 4.2. *Let R be a ring and $\mathscr{I} \subseteq \mathscr{I}(R)$. If G_R acts primitively on $\mathscr{I}$, then either $\Omega(R, \mathscr{I}) = \{\omega_R(I)\}$ for some absorbing ideal $I \in \mathscr{I}$, or there exists an one-to-one correspondence between $\Omega(R, \mathscr{I})$ and $\mathscr{I}$.*

Proof. By Proposition 4.1, $\sim_{\mathscr{I}}$ is a G_R-congruence. Hence, $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$ is a block system of $\mathscr{I}$. Assuming that G_R acts primitively on $\mathscr{I}$, then $\sim_{\mathscr{I}}$ is a G_R-trivial congruence, i.e, there exists only one equivalence class in $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$, or the equivalence classes in $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$ are singleton sets. In the first case, we concludes that all ideals in $\mathscr{I}$ are in the same equivalence class; so $\Omega(R, \mathscr{I}) = \{\omega_R(I)\}$ for some absorbing ideal $I \in \mathscr{I}$. In the second case, we see that there exist an one-to-one correspondence between $\frac{\mathscr{I}}{\sim_{\mathscr{I}}}$ and $\mathscr{I}$; so, by Proposition 2.1, there exist an one-to-one correspondence between $\Omega(R, \mathscr{I})$ and $\mathscr{I}$. □

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Corollary 4.1. *Let R be an integral domain and $\mathscr{I} \subseteq \mathscr{I}(R)$ such that $\mathbf{0} \in \mathscr{I}$. If G_R acts primitively on $\mathscr{I}$, then either all ideals in $\mathscr{I}$ are prime, or $\mathbf{0}$ is the only one prime ideal in $\mathscr{I}$.*

Proof. Suppose that G_R acts primitively on $\mathscr{I}$. By Proposition 4.2, either $\Omega(R,\mathscr{I}) = \{\omega_R(I)\}$ for some absorbing ideal $I \in \mathscr{I}$, or there exist an one-to-one correspondence between $\Omega(R,\mathscr{I})$ and $\mathscr{I}$. Since $\mathbf{0} \in \mathscr{I}$, then either $\omega_R(I) = \omega_R(\mathbf{0}) = 1$, or ω_R is an injective map on $\mathscr{I}$. Therefore, either all ideals in $\mathscr{I}$ are prime, or $\mathbf{0}$ is the only one prime ideal in $\mathscr{I}$, as desired. □

Corollary 4.2. *Let R be an integral domain. Then G_R acts primitively on $\mathscr{I}(R)$ if, and only if, R is a field.*

Proof. Suppose that G_R acts primitively on $\mathscr{I}(R)$. Since $\mathbf{0} \in \mathscr{I}(R)$ so, by Corollary, all ideals of R are prime or $\mathbf{0}$ is the only one prime ideal in R. In the first case, assume that all proper ideals of R are prime and let any $x \in R$, $x \neq 0$. Since R is an integral domain, then $x^2 \neq 0$. Take the non-zero ideal $\langle x^2 \rangle$. Since $\langle x^2 \rangle$ is prime (by hypothesis) and $x^2 \in \langle x^2 \rangle$, then $x \in \langle x^2 \rangle$. Hence, $x = x^2 y$, for some $y \in R$. Thus, we have

$$x(xy-1) = 0$$

Since R is an integral domain and $x \neq 0$, we obtain $xy = 1$. Thus, x is invertible. Therefore, R is a field. In the second case, assume that $\mathbf{0}$ is the only one prime ideal in R. Then $\mathbf{0}$ is the only one maximal ideal in R and hence R is also a field.

Reciprocally, if R is a field, then $\mathbf{0}$ is the only one proper ideal of R. Hence, G_R acts trivially and so primitively on $\mathscr{I}(R)$. □

Recall that an automorphism $\sigma \in Aut(R)$ is an **involution** if $\sigma^2 = id_R$, or equivalently, if $\sigma = \sigma^{-1}$. For example, R always admit the *trivial involution* $id_R : x \longmapsto x$ and the *involution sign change* $\sigma : x \longmapsto -x$. For each involution $\sigma \in Aut(R)$, consider the correspondence $\widehat{\sigma} : \sqrt{\mathscr{I}(R)} \longrightarrow \sqrt{\mathscr{I}(R)}$ given by $\widehat{\sigma}(I) := \sigma(I)$ for every $I \in \sqrt{\mathscr{I}(R)}$.

Proposition 4.3. *For each involution $\sigma \in Aut(R)$, $\widehat{\sigma}$ is well defined.*

Proof. Let $I \in \sqrt{\mathscr{I}(R)}$. Given $x \in \sqrt{\sigma(I)}$, we have $x^n \in \sigma(I)$ for some $n > 0$. Thus, we get $x^n = \sigma(i)$ for some $i \in I$. Therefore, we have

$$(\sigma(x))^n = \sigma(x^n) = \sigma(\sigma(i)) = \sigma^2(i) = i \in I$$

and $\sigma(x) \in \sqrt{I}$. Since I is a radical ideal, then $\sigma(x) \in I$. So, exists $i' \in I$ such that $\sigma(x) = i'$. Therefore, we have

$$x = \sigma^{-1}(i') = \sigma(i') \in \sigma(I).$$

This shows that $\sqrt{\sigma(I)} \subseteq \sigma(I)$ and, then, $\sigma(I)$ is a radical ideal. □

Hence, the set

$$G_R = \{\widehat{\sigma} \mid \sigma \in Aut(R) \text{ is an involution}\}$$

is a subgroup of $Sym(\sqrt{\mathscr{I}(R)})$ and $\sqrt{\mathscr{I}(R)}$ is a G_R-space.

Proposition 4.4. *Let R be an integral domain. If G_R acts primitively on $\sqrt{\mathscr{I}(R)}$, then either every radical ideal of R is prime (primary), or $\mathbf{0}$ is the only one prime (primary) radical ideal of R.*

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Proof. Recall that a proper ideal of R is primary if and only if it is radical is prime. Since $\mathbf{0} = \sqrt{\mathbf{0}}$, so it follows from the Corollary 4.1. □

5. Affine varieties and irreducibility

Let F an algebraically closed field, $F[T_1, \ldots, T_n]$ the ring of polynomials in n independent variables and $\mathbb{A}_F^n := F^n$ the n-dimensional affine space. Let $\mathbf{X} \subseteq \mathbb{A}_F^n$ be an affine variety and $I(\mathbf{X})$, i.e,

$$I(\mathbf{X}) := \{p \in F[T_1, \ldots, T_n] \,|\, p(x) = 0, \forall x \in \mathbf{X}\}$$

the ideal generated by $\mathbf{X}$.

Remember that there is an one-to-one correspondence between the closed sets in $\mathbf{X}$ and the radical ideals of the ring of coordinates $F[\mathbf{X}] := \frac{F[T_1,\ldots,T_n]}{I(\mathbf{X})}$ (see in [4], Theorem 1.13), i.e, if

$$\mathscr{Z}(\mathbf{X}) := \{\mathbf{Y} \subseteq \mathbf{X} \,|\, \mathbf{Y} \text{ is closed in the induced Zariski topology on } \mathbf{X}\}$$

then there exist an one-to-one map $I_{\mathbf{X}} : \mathscr{Z}(\mathbf{X}) \longmapsto \sqrt{F[\mathbf{X}]}$ given by

$$I_{\mathbf{X}}(\mathbf{Y}) := \{\overline{p} \in F[\mathbf{X}] \,|\, \overline{p}(y) = 0, \forall y \in \mathbf{Y}\}$$

Recall also that an affine variety $\mathbf{X} \subseteq \mathbb{A}_F^n$ is irreducible if $\mathbf{X}$ cannot be written as the union of two proper algebraic subsets. In this case, $\mathbf{X}$ is irreducible if, and only if, $I(\mathbf{X})$ is a prime ideal of $F[T_1, \ldots, T_n]$. A closed subset $\mathbf{Y} \subseteq \mathbf{X}$ with this same property is irreducible in $\mathbf{X}$. Hence, $\mathbf{Y}$ irreducible in $\mathbf{X}$ if, and only if, $I_{\mathbf{X}}(\mathbf{Y})$ is a prime ideal of $F[\mathbf{X}]$. We define the dimension of an affine variety to be its dimension as a topological space.

Proposition 5.1. *(Corollary 3.2.9, [5]). The dimension of $\mathbb{A}_F^n$ is n.*

Corollary 5.1. *Let $\mathbf{X} \subseteq \mathbb{A}_F^n$ an affine variety. Then $\dim \mathbf{X} \leq n$.*

Proof. Follow from Propositions 3.1 and 5.1. □

Consider $\sqrt{F[\mathbf{X}]}$ as a $G_{F[\mathbf{X}]}$-space, were

$$G_{F[\mathbf{X}]} = \{\widehat{\sigma} \,|\, \sigma \in Aut(F[\mathbf{X}]) \text{ is an involution}\}$$

Theorem 5.1. *Let $\mathbf{X} \subset \mathbb{A}_F^n$ an irreducible affine variety. Then $G_{F[\mathbf{X}]}$ acts primitively on $\sqrt{F[\mathbf{X}]}$ if, and only if, $\mathbf{X}$ is a point.*

Proof. Suppose that $G_{F[X]}$ acts primitively on $\sqrt{F[\mathbf{X}]}$. Then, by Proposition, either all radical ideals of R are prime, or $\mathbf{0}$ is the only one prime radical ideal of R. Hence, either all closed subsets of $\mathbf{X}$ are irreducible, or $\mathbf{X}$ does not have irreducible proper closed subsets. Remark that the second case occur if, and only if, $\mathbf{X}$ is a point, since points in any non singleton algebraic set are trivial irreducible proper closed subsets. In the first case, remark that if $\mathbf{X}$ it were infinite, then we could get an ascending chain of strict inclusions of finite closed subsets that, by hypothesis, are irreducible. In fact, fix $y_0 \in \mathbf{X}$ and consider the singleton closed subset $\mathbf{Y}_0 = \{y_0\}$, after $y_1 \in \mathbf{X} \setminus \mathbf{Y}_0$ and the closed subset $\mathbf{Y}_1 = \mathbf{Y}_0 \cup \{y_1\}$, after $y_2 \in \mathbf{X} \setminus \mathbf{Y}_1$ and the closed subset $\mathbf{Y}_2 = \mathbf{Y}_1 \cup \{y_2\}$ and so on, obtaining the ascending chain of strict inclusions of irreducible finite closed subsets

$$\mathbf{Y}_0 \subset \mathbf{Y}_1 \subset \mathbf{Y}_2 \subset \cdots \subset \mathbf{Y}_n \subset \mathbf{Y}_{n+1} \subset \cdots \subset \mathbf{X}$$

Well, this contradicts the Corollary 5.1. So, $\mathbf{X}$ is finite and, since is irreducible, $\mathbf{X}$ a point.

Reciprocally, if $\mathbf{X}$ is a point, then $I(\mathbf{X})$ is a maximal ideal in $F[T_1,\ldots,T_n]$ and hence, $F[\mathbf{X}]$ is a field. In particular, the only one proper radical ideal in $F[\mathbf{X}]$ is the zero ideal $\mathbf{0}$. Thus, $G_{F[\mathbf{X}]}$ acts primitively on $\sqrt{F[\mathbf{X}]} = \{\mathbf{0}\}$ trivially. □

References

[1] Anderson, D. F and Badawi. A, *On n-absorbing ideals of commutative rings*, Comm. Algebra, **39**, pp. 1646-1672, 2011.

[2] Grove, L. C, *Classical Groups and Geometric Algebra*, Vol. **39**, American Mathematical Soc, pp.1-3, 2002.

[3] Bredon, G. E, *Introduction to compact transformation group*, Academic press, Vol. **46**, p.32, 1972.

[4] Coelho. J, *Introdução à Geometria Algébrica*, Class handout, Federal University Fluminense, 2009.

[5] Singh, U. *Affine Varieties*, Department of Mathematics, Indian Institute of Technology Guwahati, 2015.

[6] Atiyah, M. F and Macdonald, I. G, *Introduction to Commutative Algebra*, CRC Press, 2018.

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