Convergence theory of bipolar fuzzy soft nets and its applications

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Abstract. In this paper, firstly, in a different way than in the literature, we define the concept of a quasi-coincident using the bipolar fuzzy soft points proposed in [10] and investigate its basic properties. Then, we introduce the notion of a bipolar fuzzy soft net (for short BFS-net) and give convergence of the BFS-nets in a bipolar fuzzy soft topological space with useful results. Also, we show how a BFS-net is derived from a BFS-filter and obtain a characterization about bipolar fuzzy soft Hausdorff spaces. Moreover, based on the idea of quasi-coincident, we give a new kind of bipolar fuzzy soft continuity and analyze its relationship with the BFS-nets. Next, we put forward the idea of compactness in the setting of bipolar fuzzy soft sets and characterize it through the contribution of the BFS-subnets. Finally, we present some examples to better understand the defined concepts.

Keywords: Bipolar fuzzy soft set, BFS-net, convergence, bipolar fuzzy soft continuity, bipolar fuzzy soft Hausdorff space, bipolar fuzzy soft compactness.

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1 Introduction

Classical methods are inadequate due to various uncertainties in solving complex problems in the fields of economics, engineering and environment. In order to overcome with this uncertainty, many theories have been presented. The most known theories are fuzzy set theory introduced by Zadeh [33] in 1965 and rough set theory which was introduced in 1982 by Pawlak [23]. Both of these theories are useful tools to deal with uncertainties. However, as pointed by Molodtsov [21], these theories have their own difficulties and inadequacies because of parameterization tool not being enough. So, Molodtsov [21] invented a new notion named soft set, which handles ambiguities and imprecisions in parametric manners. Then, a lot of researchers have utilized this theory as a powerful tool to define uncertainties. For instance, Maji et al. [20] investigated the terms such as subset, union, intersection and complement for soft sets. Moreover, Maji et al. [19] introduced a more general concept, which is a combination of fuzzy set and soft set; the fuzzy soft set. Ali et al. [3] proposed new operations of the algebraic nature on soft sets and studied their properties. Shabir and Naz [30] investigated soft topological spaces. Later, Al-shami [4] applied soft compactness on ordered settings to expect the missing values on the information systems. Kharal and Ahmad [16] introduced a mapping on classes of fuzzy soft sets and also studied the properties of fuzzy soft images. Demir et al. [9] investigated convergence of fuzzy soft filters in a fuzzy soft topological space. Afterwards, with the help of the Q-neighborhoods, Gao and Wu [15] redefined convergence of the fuzzy soft filters. Recently, many papers concerning soft set theory and fuzzy soft set theory have been published

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[2, 5, 6, 32].

Fuzzy sets are unable to represent the satisfaction degree to counter-property although they are able to represent uncertainties in membership degree assignments. In order to get over this problem, Lee [17] introduced bipolar valued fuzzy set which the membership degree range is [-1,1], making the coexistence of negativity and positivity. In a bipolar valued fuzzy set, the membership value 0 of an element shows that the element is irrelevant to the corresponding property, the membership degree (0, 1] of an element means that the element somewhat satisfies the property, and the membership degree [-1, 0) of an element shows that the element somewhat satisfies the implicit counter-property. Afterwards, Abdullah et al. [1] and Naz and Shabir [22] defined independently bipolar fuzzy soft sets (henceforth, BFS-sets), combining both the bipolar fuzzy sets and the soft sets. Riaz and Tehrim [27] initiated bipolar fuzzy soft topology (BFS-topology) and discussed certain aspects of BFS-topology. Moreover, Riaz and Tehrim [26] indicated the concept of mappings between BFS-sets and applied this concept to the problem of medical diagnosis. Afterwards, Riaz and Tehrim [25], in bipolar fuzzy soft setting, presented the concept of a quasi-coincident, and with the use of Q-neighborhoods, they discussed certain properties of BFS-topology. In recent years, there has been a considerable literature on BFS-set and its applications [7, 11, 12, 18, 29, 31, 34].

In topology, a subfield of mathematics, the nets are used to study the basic topological concepts such as convergent, continuity, compactness, and more. Therefore, the problem of extensions of nets have been tackled by many authors. With the requirement of fuzzy setting, Pu and Liu [24] introduced the notions of fuzzy nets and Q-neighborhoods, and established the Moore Smith convergence theory in fuzzy topology by the Q-neighborhood structure. Also, Sarma and Ajmal [28] gave the notion of fuzzy nets of fuzzy sets based on Q-neighborhood structure. On the basis of the concept of soft neighborhoods, Demir and Özbakır [8] established the convergence theory of soft nets. Later, by using the Q-neighborhood theory, Gao and Wu [13] defined the convergence of a fuzzy soft net and characterized the continuity of fuzzy soft mappings by the fuzzy soft nets. Moreover, they [14] obtained some important results about the closure, separation and compactness by means of fuzzy soft nets.

Inspired by these works we introduce the concept of a quasi-coincident and give some of its basic properties under bipolar fuzzy soft environment. Then, we obtain the notion of a BFS-net and study its convergence properties in the light of the bipolar fuzzy soft quasi-neighborhoods (BFS-q-neighborhoods) of bipolar fuzzy soft points (BFS-points) due to Demir and Saldamlı [10]. Also, we prove that a BFS-net converges to unique BFS-point in a bipolar fuzzy soft Hausdorff space. Moreover, we show that there is a relation between the convergence of bipolar fuzzy soft filters (BFS-filters) and the convergence of BFS-nets similar to the one which exists between the convergence of filters and the convergence of nets in topological spaces. Afterwards, we study on the applications of BFS-nets to bipolar fuzzy soft quasi-continuity and bipolar fuzzy soft quasi-compactness by means of the convergence theory. Finally, we provide suitable examples to illustrate the effectiveness of the proposed results.

2 Preliminaries

In this section, we review some basic notions of BFS-sets that we will use in the subsequent sections.

Throughout this paper, U be a universe of alternatives (objects) and E be a set of specified parameters (criteria or attributes) unless otherwise explicit.

Definition 2.1 ([17]). *Consider a universal set U. A set having form*

$$\eta = \{(u, \delta^+_\eta(u), \delta^-_\eta(u)) : u \in U\}$$

denotes a bipolar fuzzy set on U, where $\delta^+_\eta(u)$ denotes the positive memberships ranges over $[0, 1]$ *and $\delta^-_\eta(u)$ denotes the negative memberships ranges over* $[-1, 0]$.

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Definition 2.2 ([17]). *Let η_1 and η_2 be two bipolar fuzzy sets on U. Then, their intersection and union are defined as follows:*

(i) $\eta_1 \wedge \eta_2 = \{(u, \min\{\delta^+_{\eta_1}(u),\ \delta^+_{\eta_2}(u)\}, \max\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}) : u \in U\}$.

(ii) $\eta_1 \vee \eta_2 = \{(u, \max\{\delta^+_{\eta_1}(u), \delta^+_{\eta_2}(u)\}, \min\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}) : u \in U\}$.

Definition 2.3 ([1, 22]). *Consider a universal set U and a set of parameters E. Let $A \subseteq E$ and define a mapping $\Omega : E \to BF^U$, where BF^U represents the family of all bipolar fuzzy subsets of U. Then, Ω_A is called a BFS-set on U, where*

$$\Omega_A = \{\langle e, \Omega(e)\rangle : e \in E\}$$

such that $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ for all $e \notin A$ and all $u \in U$.

Note that the set of all bipolar fuzzy soft sets on U with attributes from E is denoted by $(BF^U)^E$.

Example 2.4. *Suppose that Mrs.X wishes to purchase a mobile phone and let $E = \{e_1 = display\ resolution,\ e_2 = CPU\ performance,\ e_3 = main\ camera\ resolution,\ e_4 = memory\ capacity\}$ be the set of decision variables. Then, consider the set of three types of model mobile phones $U = \{u_1, u_2, u_3\}$ by keeping in view the requirements of Mrs.X. After a research, we see that a website has assigned the numerical values for each decision variable to three model mobile phones, taking into account the positive and negative feedbacks based on customers. The tabular representation of these numerical values is as follows:*

Table 1
Tabular reprentation of positive feedbacks

	e_1	e_2	e_3	e_4
u_1	0.4	0.3	0.6	0.7
u_2	0.2	0.7	0.8	0.2
u_3	0.7	0.3	0.2	0.4

Table 2
Tabular reprentation of negative feedbacks

	e_1	e_2	e_3	e_4
u_1	−0.4	−0.6	−0.4	−0.4
u_2	−0.4	−0.5	−0.4	−0.5
u_3	−0.1	−0.3	−0.8	−0.5

Therefore, the following bipolar fuzzy soft set on U with the set E of decision variables reporting the positive-negative informations is obtained:

$$\Omega_A = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.4, -0.4), (u_2, 0.2, -0.4), (u_3, 0.7, -0.1)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.3, -0.6), (u_2, 0.7, -0.5), (u_3, 0.3, -0.3)\}\rangle, \\ \langle e_3, \Omega(e_3) = \{(u_1, 0.6, -0.4), (u_2, 0.8, -0.4), (u_3, 0.2, -0.8)\}\rangle, \\ \langle e_4, \Omega(e_4) = \{(u_1, 0.7, -0.4), (u_2, 0.2, -0.5), (u_3, 0.4, -0.5)\}\rangle \end{array} \right\}.$$

Definition 2.5 ([34]). *(i) The BFS-set $\Omega_E \in (BF^U)^E$ is called an absolute BFS-set, denoted by U_E, if $\delta^+_{\Omega(e)}(u) = 1$ and $\delta^-_{\Omega(e)}(u) = -1$ for all $u \in U$ and all $e \in E$.*

(ii) The BFS-set $\Omega_A \in (BF^U)^E$ is called a null BFS-set, denoted by ϕ_A, if $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ for all $u \in U$ and all $e \in A$.

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Definition 2.6 ([1, 22]). *Let $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. Then,*

(i) The union of $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ is a bipolar fuzzy soft set $\Omega^3_{A_3}$ over U such that for all $e \in E$, $\Omega^3(e) = \Omega^1(e) \vee \Omega^2(e)$ and denoted by $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2}$.

(ii) The intersection of $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ is a bipolar fuzzy soft set $\Omega^3_{A_3}$ over U such that for all $e \in E$, $\Omega^3(e) = \Omega^1(e) \wedge \Omega^2(e)$ and denoted by $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2}$.

Definition 2.7 ([1, 22]). *The complement of a BFS-set $\Omega_A \in (BF^U)^E$ is shown by $(\Omega_A)^c = \Omega^c_{A_1}$ where $\Omega^c : E \to BF^U$ is a mapping defined by $\delta^+_{\Omega^c(e)}(u) = 1 - \delta^+_{\Omega(e)}(u)$ and $\delta^-_{\Omega^c(e)}(u) = -1 - \delta^-_{\Omega(e)}(u)$ for all $e \in E$ and all $u \in U$.*

Definition 2.8 ([34]). *Let $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in (BF^U)^E$. Then, $\Omega^1_{A_1}$ is a BFS-subset of $\Omega^2_{A_2}$ if $\delta^+_{\Omega^1(e)}(u) \leq \delta^+_{\Omega^2(e)}(u)$ and $\delta^-_{\Omega^1(e)}(u) \geq \delta^-_{\Omega^2(e)}(u)$, which is shown by $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$.*

Theorem 2.9 ([22]). *Let $\Omega^1_{A_1}$, $\Omega^2_{A_2}$ be two BFS-sets over U.*

(i) $((\Omega^1_{A_1})^c)^c = \Omega^1_{A_1}$.

(ii) If $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, then $(\Omega^2_{A_2})^c \tilde{\subseteq} (\Omega^2_{A_2})^c$.

(iii) $(\Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cup}\, (\Omega^2_{A_2})^c$.

(iv) $(\Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cap}\, (\Omega^2_{A_2})^c$.

Definition 2.10 ([27]). *Let $\Gamma_B \in (BF^U)^E$ with $B = \{e\} \subseteq E$. If $\delta^+_{\Gamma(e)}(u) \neq 0$ and $\delta^-_{\Gamma(e)}(u) \neq 0$ for all $u \in U$, then Γ_B is called a BFS-point. It is denoted by $\beta(\Gamma_B)$.*

Demir and Saldamlı [10] redefined the concept of BFS-point as follows.

Definition 2.11 ([10]). *Let $\Omega_A \in (BF^U)^E$ with $A = \{e\}$. If there is a $u \in U$ such that $\delta^+_{\Omega(e)}(u) \neq 0$ or $\delta^-_{\Omega(e)}(u) \neq 0$ and $\delta^+_{\Omega(e)}(u') = \delta^-_{\Omega(e)}(u') = 0$ for all $u' \in U \backslash \{u\}$, then Ω_A is called a BFS-point in U. It is denoted by $e_u^{(p,n)}$.*

The following example reveals that there is no relationship between these two BFS-points.

Example 2.12. *Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Consider $\Gamma_B \in (BF^U)^E$ with*

$$\Gamma_B = \left\{ \begin{array}{l} \langle e_1, \Gamma(e_1) = \{(u_1, 0.35, -0.63), (u_2, 0.45, -0.51), (u_3, 0.64, -0.51)\}\rangle, \\ \langle e_2, \Gamma(e_2) = \{(u_1, 0, 0), (u_2, 0, 0), (u_3, 0, 0)\}\rangle \end{array} \right\},$$

where $B = \{e_1\}$. Then, $\beta(\Gamma_B)$ is a BFS-point with respect to Definition 2.10 but not with respect to Definition 2.11. On the other hand, let $A = \{e_1\}$. Then, to the BFS-set Ω_A given by

$$\Omega_A = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.4, 0), (u_2, 0, 0), (u_3, 0, 0)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0, 0), (u_2, 0, 0), (u_3, 0, 0)\}\rangle \end{array} \right\},$$

$(e_1)_{u_1}^{(0,4,0)}$ is a BFS-point with respect to Definition 2.11 but not with respect to Definition 2.10.

Throughout this paper, we adopt the BFS-point in the sense of Demir and Saldamlı [10] for the convenience of obtaining the desired important results in bipolar fuzzy soft settings.

Let $\mathcal{P}(U, E)$ be the family of all BFS-points on U.

Definition 2.13 ([11]). *Let $(e_1)_{u_1}^{(p_1,n_1)}, (e_2)_{u_2}^{(p_2,n_2)} \in \mathcal{P}(U, E)$. These two BFS-points are called equal if $e_1 = e_2$, $u_1 = u_2$ and $(p_1, n_1) = (p_2, n_2)$. Moreover, $(e_1)_{u_1}^{(p_1,n_1)} \neq (e_2)_{u_2}^{(p_2,n_2)} \Leftrightarrow u_1 \neq u_2$ or $e_1 \neq e_2$ or $(p_1, n_1) \neq (p_2, n_2)$.*

Definition 2.14 ([10]). *The BFS-point $e_u^{(p,n)}$ is said to belongs to a BFS-set Ω_A, denoted by $e_u^{(p,n)} \tilde{\in} \Omega_A$, if $p \leq \delta^+_{\Omega(e)}(u)$ and $n \geq \delta^-_{\Omega(e)}(u)$.*

Definition 2.15 ([26]). *Let $(BF^U)^E$ and $(BF^V)^D$ be two the families of all bipolar fuzzy soft sets on U and V with parameters from E and D, respectively. Assume that $\mathfrak{u} : U \to V$ and $\mathfrak{g} : E \to D$ be two mappings. Then, the mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ is called a BFS-mapping from U to V, defined as the following :*

(i) Let $\Omega_A \in (BF^U)^E$. Then, $\mathfrak{f}(\Omega_A) = (\mathfrak{f}(\Omega))_{A_1}$ is the BFS-set over V with parameters from D given by $\mathfrak{f}(\Omega_A) = \{\langle d, \mathfrak{f}(\Omega)(d)\rangle : d \in D\}$ such that $\mathfrak{f}(\Omega)(d) = \{(v, \delta^+_{\mathfrak{f}(\Omega)(d)}(v), \delta^-_{\mathfrak{f}(\Omega)(d)}(v)) : v \in V\}$, where

$$\delta^+_{\mathfrak{f}(\Omega)(d)}(v) = \begin{cases} \sup\{\delta^+_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(d) \cap A \neq \emptyset, \\ 0, & \text{if otherwise,} \end{cases}$$

$$\delta^-_{\mathfrak{f}(\Omega)(d)}(v) = \begin{cases} \inf\{\delta^-_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(d) \cap A \neq \emptyset, \\ 0, & \text{if otherwise.} \end{cases}$$

Then, $\mathfrak{f}(\Omega_A)$ is called BFS-image of BFS-set Ω_A under $\mathfrak{f}$.

(ii) Let $\Omega^1_{A_1} \in (BF^V)^D$. Then, $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = (\mathfrak{f}^{-1}(\Omega^1))_A$ is the BFS-set over U with parameters from E given by $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = \{\langle e, \mathfrak{f}^{-1}(\Omega^1)(e)\rangle : e \in E\}$ such that $\mathfrak{f}^{-1}(\Omega^1)(e) = \{(u, \delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u), \delta^-_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u)) : u \in U\}$, where

$$\delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^+_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{if } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if otherwise,} \end{cases}$$

$$\delta^-_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^-_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{if } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if otherwise.} \end{cases}$$

Then, $\mathfrak{f}^{-1}(\Omega^1_{A_1})$ is called BFS inverse image of BFS-set $\Omega^1_{A_1}$.

Theorem 2.16 ([26]). *Let $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ be a BFS-mapping. Then, for $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in (BF^U)^E$ and $\Gamma^1_{B_1}, \Gamma^2_{B_2} \in (BF^V)^D$, the following properties are satisfied.*
(i) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cup} \Omega^2_{A_2}) = \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cup} \mathfrak{f}(\Omega^2_{A_2})$.
(ii) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cup} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cup} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.
(iii) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}) \tilde{\subseteq} \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap} \mathfrak{f}(\Omega^2_{A_2})$.
(iv) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cap} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cap} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.
(v) $\Omega^1_{A_1} \tilde{\subseteq} \mathfrak{f}^{-1}(\mathfrak{f}(\Omega^1_{A_1}))$, $\mathfrak{f}(\mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\subseteq} \Gamma^1_{B_1}$.
(vi) If $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, then $\mathfrak{f}(\Omega^1_{A_1}) \tilde{\subseteq} \mathfrak{f}(\Omega^2_{A_2})$.
(vii) If $\Gamma^1_{B_1} \tilde{\subseteq} \Gamma^2_{B_2}$, then $\mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

Definition 2.17 ([27]). *The family τ of BFS-sets over U is said to be a BFS-topology on U if it satisfies the following properties:*
(BFST1) U_E and ϕ_A are members of τ.
(BFST2) If $\Omega^i_{A_i} \in \tau$ for all $i \in J$, an index set, then $\tilde{\bigcup}_{i \in J} \Omega^i_{A_i} \in \tau$.
(BFTS3) If $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in \tau$, then $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \tau$.

We say (U, τ, E) is a BFS-topological space. A member in τ is called a BFS-open set and its complement is called a BFS-closed set.

Definition 2.18 ([10]). *Let (U, τ, E) be a BFS-topological space. The BFS-set Ω_A is called a BFS-neighborhood of a BFS-point $e_u^{(p,n)}$ if there is a BFS-open set $\Omega^1_{A_1}$ such that $e_u^{(p,n)} \tilde{\in} \Omega^1_{A_1} \tilde{\subseteq} \Omega_A$.*

The collection of all BFS-neighborhoods of $e_u^{(p,n)}$ is called the neighborhood system of $e_u^{(p,n)}$ and denoted by $\mathcal{N}(e_u^{(p,n)})$.

Definition 2.19 ([27]). *Let (U,τ,E) be a BFS-topological space and $\Omega_A \in (BF^U)^E$. The BFS-interior of Ω_A is the union of all BFS-open sets contained in Ω_A, denoted by $(\Omega_A)^o$. From (BFST2) it is clear that $(\Omega_A)^o$ is a BFS-open set. This set is largest BFS-open set contained in Ω_A.*

Definition 2.20 ([27]). *Let (U,τ,E) be a BFS-topological space and $\Omega_A \in (BF^U)^E$. The closure of Ω_A is the intersection of all BFS-closed sets containing Ω_A; this set is denoted $\overline{\Omega_A}$. It is easily seen that $\overline{\Omega_A}$ is the smallest closed set containing Ω_A.*

Theorem 2.21 ([27]). *Let (U,τ,E) be a BFS-topological space and $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. Then,*

i) $\Omega^1_{A_1}$ is BFS-open set if and only if $(\Omega^1_{A_1})^o = \Omega^1_{A_1}$.

ii) If $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, then $(\Omega^1_{A_1})^o \tilde{\subseteq} (\Omega^2_{A_2})^o$.

iii) $\Omega^1_{A_1}$ is BFS-closed set if and only if $\overline{\Omega^1_{A_1}} = \Omega^1_{A_1}$.

iv) If $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, then $\overline{\Omega^1_{A_1}} \tilde{\subseteq} \overline{\Omega^2_{A_2}}$.

Definition 2.22 ([11]). *The BFS-filter $\mathcal{F}$ on U is a nonempty collection of subsets of $(BF^U)^E$ if it satisfies the following conditions:*

$(BFSF1)$ $\phi_A \notin \mathcal{F}$,

$(BFSF2)$ If $\Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{F}$, then $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \mathcal{F}$,

$(BFSF3)$ If $\Omega^1_{A_1} \in \mathcal{F}$ and $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$ then $\Omega^2_{A_2} \in \mathcal{F}$.

Definition 2.23 ([11]). *Let (U,τ,E) be a BFS-topological space, $\mathcal{F}$ be a BFS-filter on U and $e_u^{(p,n)} \in \mathcal{P}(U,E)$. The BFS-filter $\mathcal{F}$ is said to converge to $e_u^{(p,n)}$ if $\mathcal{N}(e_u^{(p,n)}) \subseteq \mathcal{F}$ and denoted by $\mathcal{F} \to e_u^{(p,n)}$.*

3 Bipolar fuzzy soft quasi-coincident

In this section, we give a new notion of bipolar fuzzy soft quasi-coincident (BFS-q-coincident) as different from the one given by Riaz and Tehrim in [25], and discuss its related properties. Then, we compare the properties obtained for this notion with those of existing model. Moreover, we establish the concept of a bipolar fuzzy soft quasi-neighborhood (BFS-q-neighborhood) through the assistance of BFS-points. Note that these ideas seem to be extremely suitable for the bipolar fuzzy soft situation.

Definition 3.1 ([25]). *Let $\Omega_A \in (BF^U)^E$ and $\beta(\Gamma_B)$ be a BFS-point in the sense of Riaz et al., where $B = \{e\}$. The BFS-point $\beta(\Gamma_B)$ is called a BFS-q-coincident with the BFS-set Ω_A, denoted by $\beta(\Gamma_B)\ q\ \Omega_A$, if $\delta^+_{\Gamma(e)}(u) + \delta^+_{\Omega(e)}(u) > 1$ and $\delta^-_{\Gamma(e)}(u) + \delta^-_{\Omega(e)}(u) < -1$ for some $u \in U$.*

In classical set theory, if an element is in the union of two sets, then it is in the first set, the second set, or both. However, this basic property not valid in the setting of bipolar fuzzy soft theory. So, we obtain a new model of being the BFS-q-coincident of a BFS-point with a BFS-set, in a different way than in Definition 3.1. This new model make up for the lack of the original model that does not satisfy the above basic property.

Definition 3.2. *Let $\Omega_A \in (BF^U)^E$ and $e_u^{(p,n)} \in \mathcal{P}(U,E)$. The BFS-point $e_u^{(p,n)}$ is called a BFS-q-coincident with the BFS-set Ω_A, denoted by $e_u^{(p,n)} q\ \Omega_A$, if $p + \delta^+_{\Omega(e)}(u) > 1$ or $n + \delta^-_{\Omega(e)}(u) < -1$. If $e_u^{(p,n)}$ is not BFS-q-coincident with Ω_A, then it is denoted by $e_u^{(p,n)}\ \overline{q}\ \Omega_A$.*

Example 3.3. *Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$. Let Ω_E be a BFS-set in $(BF^U)^E$ with*

$$\Omega_E = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.5, -0.2), (u_2, 0.3, -0.6)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.3, -0.8), (u_2, 0.5, -0.1)\}\rangle \end{array} \right\}.$$

We can easily see that $(e_1)_{u_1}^{(0.4,-0.9)} q\ \Omega_E$ since $-0,9 + \delta^-_{\Omega(e_1)}(u_1) < -1$. However, we verify that $(e_2)_{u_2}^{(0.1,-0.8)}\ \overline{q}\ \Omega_E$ because $0.1 + \delta^+_{\Omega(e_2)}(u_2) \leq 1$ and $-0.8 + \delta^-_{\Omega(e_2)}(u_2) \geq -1$.

Remark 3.4. *By Example 3.3, observe that the above given concept of BFS-q-coincident is different from the notion of BFS-q-coincident introduced in Definition 3.1. These new approach, strictly related with the notion of BFS-point established by Demir and Saldamlı [10], enable us to obtain a natural (similar to fuzzy-quasi-coincident) behavior of BFS-q-coincident. Furthermore, this type of BFS-q-coincident will play an important role in obtaining the fundamental theorems in the next sections.*

Theorem 3.5. *Let $\Omega_A, \Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$ and $\{\Omega^i_{A_i} : i \in J\}$ be a family of BFS-sets over U Then,*
(i) If $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, then $e_u^{(p,n)} q\ \Omega^2_{A_2}$ for each $e_u^{(p,n)} q\ \Omega^1_{A_1}$.
(ii) $e_u^{(p,n)} q\ \Omega_A$ if and only if $e_u^{(p,n)} \tilde{\notin} (\Omega_A)^c$.
(iii) If $e_u^{(p,n)} q\ \widetilde{\bigcap}_{i\in J}\Omega^i_{A_i}$, then $e_u^{(p,n)} q\ \Omega^i_{A_i}$ for all $i \in J$.
(iv) $e_u^{(p,n)} q\ \widetilde{\bigcup}_{i\in J}\Omega^i_{A_i}$ if and only if there exists an $i_0 \in J$ such that $e_u^{(p,n)} q\ \Omega^{i_0}_{A_{i_0}}$.

Proof. (i) Let $e_u^{(p,n)} q\ \Omega^1_{A_1}$. Then, we have $p+\delta^+_{\Omega^1(e)}(u) > 1$ or $n+\delta^-_{\Omega^1(e)}(u) < -1$. Due to $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, we know that $\delta^+_{\Omega^1(e)}(u) \le \delta^+_{\Omega^2(e)}(u)$ and $\delta^-_{\Omega^1(e)}(u) \ge \delta^-_{\Omega^2(e)}(u)$. Therefore, we get $p + \delta^+_{\Omega^2(e)}(u) > 1$ or $n + \delta^-_{\Omega^2(e)}(u) < -1$, which implies that $e_u^{(p,n)} q\ \Omega^2_{A_2}$.

(ii) Let $e_u^{(p,n)} q\ \Omega_A$. Then, we have $p + \delta^+_{\Omega(e)}(u) > 1$ or $n + \delta^-_{\Omega(e)}(u) < -1$ and from here we get $p > 1 - \delta^+_{\Omega(e)}(u)$ or $n < -1 - \delta^-_{\Omega(e)}(u)$. By Definition 2.7, it follows that $p > \delta^+_{\Omega^c(e)}(u)$ or $n < \delta^-_{\Omega^c(e)}(u)$. This shows that $e_u^{(p,n)} \tilde{\notin} \Omega^c_A$. For sufficiency, let $e_u^{(p,n)} \tilde{\notin} \Omega^c_A$. From Definition 2.7, we obtain $p > 1 - \delta^+_{\Omega(e)}(u)$ or $n < -1 - \delta^-_{\Omega(e)}(u)$ and so that $e_u^{(p,n)} q\ \Omega_A$.

(iii) Consider $e_u^{(p,n)} q\ \widetilde{\bigcap}_{i\in J}\Omega^i_{A_i}$. Then, $p + \delta^+_{\bigwedge_{i\in J}\ \Omega^i(e)}(u) > 1$ or $n + \delta^-_{\bigvee_{i\in J}\ \Omega^i(e)}(u) < -1$. Since $\delta^+_{\bigwedge_{i\in J}\ \Omega^i(e)}(u) \le \delta^+_{\Omega^i(e)}(u)$ and $\delta^-_{\bigvee_{i\in I}\ \Omega^i(e)}(u) \ge \delta^-_{\Omega^i(e)}(u)$ for all $i \in J$, we have $p + \delta^+_{\Omega^i(e)}(u) > 1$ or $n + \delta^-_{\Omega^i(e)}(u) < -1$ for all $i \in J$. Thus, $e_u^{(p,n)} q\ \Omega^i_{A_i}$ for all $i \in J$.

(iv) The sufficiency part is obvious and we only prove the necessary. Let $e_u^{(p,n)} q\ \widetilde{\bigcup}_{i\in J}\Omega^i_{A_i}$. Suppose that $e_u^{(p,n)}\ \overline{q}\ \Omega^i_{A_i}$ for all $i \in J$. Then, we get $p + \delta^+_{\Omega^i(e)}(u) \le 1$ and $n + \delta^-_{\Omega^i(e)}(u) \ge -1$ for all $i \in J$. Therefore, we obtain $p + \delta^+_{\bigvee_{i\in J}\ \Omega^i(e)}(u) \le 1$ and $n + \delta^-_{\bigwedge_{i\in J}\ \Omega^i(e)}(u) \ge -1$. So, it follows that $e_u^{(p,n)}\ \overline{q}\ \widetilde{\bigcup}_{i\in J}\Omega^i_{A_i}$, which leads to a contradiction. □

To see that the the converse of the Theorem 3.5(iii) need not be true, we give the following example.

Example 3.6. *Let $U = \{u_1, u_2\}$ and $E = \{e_1, e_2\}$. Let us take a BFS-point $(e_1)_{u_1}^{(0.6,-0.3)}$ and consider the BFS-sets Ω^i_E such that*

$$\begin{array}{llll} \delta^+_{\Omega^i(e_1)}(u_1) > 0.4, & \delta^-_{\Omega^i(e_1)}(u_1) < -0.7, & \delta^+_{\Omega^i(e_1)}(u_2) = 0.8, & \delta^-_{\Omega^i(e_1)}(u_2) = -0.1, \\ \delta^+_{\Omega^i(e_2)}(u_1) = 0.7, & \delta^-_{\Omega^i(e_2)}(u_1) = -0.4, & \delta^+_{\Omega^i(e_2)}(u_2) = 0, & \delta^-_{\Omega^i(e_2)}(u_2) = 0, \end{array}$$

for all $i \in J$. Therefore, one easily observes that $(e_1)_{u_1}^{(0.6,-0.3)} q\ \Omega^i_E$ for all $i \in J$. However, it follows from $0,6 + \delta^+_{\bigwedge_{i\in J}\ \Omega^i(e_1)}(u_1) \le 1$ and $-0.3 + \delta^-_{\bigvee_{i\in J}\ \Omega^i(e_1)}(u_1) \ge -1$ that $(e_1)_{u_1}^{(0.6,-0.3)}\ \overline{q}\ \widetilde{\bigcap}_{i\in J}\Omega^i_{A_i}$.

The necessarity condition of Theorem 3.5 (iv) is not true in general, however, if we take the definition of the BFS-q-coincident in the sense of Riaz et al. [25], as the next example elucidates.

Example 3.7. *Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Take Ω^1_E, $\Omega^2_E \in (BF^U)^E$, where*

$$\Omega^1_E = \left\{ \begin{array}{l} \langle e_1, \Omega^1(e_1) = \{(u_1, 0.91, -0.13), (u_2, 0.32, -0.61), (u_3, 0.22, -0.31)\}\rangle, \\ \langle e_2, \Omega^1(e_2) = \{(u_1, 0.32, -0.43), (u_2, 0.51, -0.33), (u_3, 0.23, -0.51)\}\rangle \end{array} \right\},$$

$$\Omega^2_E = \left\{ \begin{array}{l} \langle e_1, \Omega^2(e_1) = \{(u_1, 0.11, -0.92), (u_2, 0.21, -0.35), (u_3, 0.33, -0.34)\}\rangle, \\ \langle e_2, \Omega^2(e_2) = \{(u_1, 0.22, -0.20), (u_2, 0.21, -0.23), (u_3, 0.15, -0.36)\}\rangle \end{array} \right\}.$$

Therefore, we have

$$\Omega^1_E \,\tilde{\cup}\, \Omega^2_E = \Omega^3_E = \left\{ \begin{array}{l} \langle e_1, \Omega^3(e_1) = \{(u_1, 0.91, -0.92), (u_2, 0.32, -0.61), (u_3, 0.33, -0.34)\}\rangle, \\ \langle e_2, \Omega^3(e_2) = \{(u_1, 0.32, -0.43), (u_2, 0.51, -0.33), (u_3, 0.23, -0.51)\}\rangle \end{array} \right\}.$$

Now, choose a BFS-point as in Definition 2.10

$$\beta(\Gamma_B) = \left\{ \begin{array}{l} \langle e_1, \Gamma(e_1) = \{(u_1, 0.35, -0.63), (u_2, 0.45, -0.51), (u_3, 0.64, -0.51)\}\rangle, \\ \langle e_2, \Gamma(e_2) = \{(u_1, 0, 0), (u_2, 0, 0), (u_3, 0, 0)\}\rangle \end{array} \right\},$$

where $B = \{e_1\}$. *It is clear that* $\beta(\Gamma_B)\; q\; \Omega^1_E \,\tilde{\cup}\, \Omega^2_E$ *but* $\beta(\Gamma_B)\; \overline{q}\; \Omega^1_E$ *and* $\beta(\Gamma_B)\; \overline{q}\; \Omega^2_E$ *with respect to Definition 3.1.*

Definition 3.8. *Let* $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. $\Omega^1_{A_1}$ *is called BFS-q-coincident with* $\Omega^2_{A_2}$, *which is denoted by* $\Omega^1_{A_1}\, q\, \Omega^2_{A_2}$, *if* $\delta^+_{\Omega^1(e)}(u) + \delta^+_{\Omega^2(e)}(u) > 1$ *or* $\delta^-_{\Omega^1(e)}(u) + \delta^-_{\Omega^2(e)}(u) < -1$ *for some* $u \in U$ *and an* $e \in E$. *If* $\Omega^1_{A_1}$ *is not BFS-q-coincident with* $\Omega^2_{A_2}$, *then it is denoted by* $\Omega^1_{A_1}\, \overline{q}\, \Omega^2_{A_2}$.

Theorem 3.9. *Let* $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. *If* $\Omega^1_{A_1}\, q\, \Omega^2_{A_2}$, *then* $\Omega^1_{A_1} \,\tilde{\cap}\, \Omega^2_{A_2} \neq \phi_A$.

Proof. Let $\Omega^1_{A_1}\, q\, \Omega^2_{A_2}$. Then, we get $\delta^+_{\Omega^1(e)}(u) + \delta^+_{\Omega^2(e)}(u) > 1$ or $\delta^-_{\Omega^1(e)}(u) + \delta^-_{\Omega^2(e)}(u) < -1$ for some $u \in U$ and an $e \in E$. This implies that $\delta^+_{\Omega^1(e)}(u)$, $\delta^+_{\Omega^2(e)}(u) \neq 0$ or $\delta^-_{\Omega^1(e)}(u)$, $\delta^-_{\Omega^2(e)}(u) \neq 0$. So, we have $\delta^+_{\Omega^1(e)\wedge\Omega^2(e)}(u) \neq 0$ or $\delta^-_{\Omega^1(e)\vee\Omega^2(e)}(u) \neq 0$. This show that $\Omega^1_{A_1} \,\tilde{\cap}\, \Omega^2_{A_2} \neq \phi_A$. □

As explained in the following example, the converse of above theorem is not necessarily true.

Example 3.10. *Let* $U = \{u_1, u_2\}$ *and* $E = \{e_1, e_2\}$. *Let* $\Omega^1_E, \Omega^2_E \in (BF^U)^E$ *be defined by*

$$\Omega^1_E = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.4, -0.1), (u_2, 0.5, -0.2)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.3, -0.4), (u_2, 0, -0.2)\}\rangle \end{array} \right\},$$

$$\Omega^2_E = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.5, -0.8), (u_2, 0.5, -0.3)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.4, -0.6), (u_2, 0.8, -0.3)\}\rangle \end{array} \right\}.$$

Then, we observe that $\Omega^1_E \,\tilde{\cap}\, \Omega^2_E \neq \phi_A$ *but* $\Omega^1_E\, \overline{q}\, \Omega^2_E$.

Definition 3.11. *Let* (U, τ, E) *be a BFS-topological space. The BFS-set* $\Omega^1_{A_1}$ *is called a BFS-q-neighborhood of a BFS-point* $e_u^{(p,n)}$ *if there exists an* $\Omega^2_{A_2} \,\tilde{\in}\, \tau$ *such that* $e_u^{(p,n)} q\, \Omega^2_{A_2} \,\tilde{\subseteq}\, \Omega^1_{A_1}$.

Theorem 3.12. *Let* $\mathcal{N}_q(e_u^{(p,n)})$ *be a collection of all BFS-q-neighborhoods of a BFS-point* $e_u^{(p,n)}$ *in a BFS-topological space* (U, τ, E). *Then, the following properties hold:*
$(BFSN1)$ *If* $\Omega_A \in \mathcal{N}_q(e_u^{(p,n)})$, *then* $e_u^{(p,n)}\; q\; \Omega_A$.
$(BFSN2)$ *If* $\Omega^1_{A_1} \in \mathcal{N}_q(e_u^{(p,n)})$ *and* $\Omega^1_{A_1} \,\tilde{\subseteq}\, \Omega^2_{A_2}$, *then* $\Omega^2_{A_2} \in \mathcal{N}_q(e_u^{(p,n)})$.
$(BFSN3)$ *If* $\Omega_A \in \mathcal{N}_q(e_u^{(p,n)})$, *then there exists a* $\Gamma_B \in \mathcal{N}_q(e_u^{(p,n)})$ *with* $\Gamma_B \,\tilde{\subseteq}\, \Omega_A$ *and* $\Gamma_B \in \mathcal{N}_q(d_v^{(p',n')})$ *for all* $d_v^{(p',n')}\; q\; \Gamma_B$.

Proof. Since the other properties are easily verified, it suffices to show that $\mathcal{N}_q(e_u^{(p,n)})$ satisfies $(BFSN3)$. Let $\Omega_A \in \mathcal{N}_q(e_u^{(p,n)})$. In this case, there exists a $\Gamma_B \in \tau$ such that $e_u^{(p,n)}\; q\; \Gamma_B \,\tilde{\subseteq}\, \Omega_A$. Therefore, we obtain $\Gamma_B \in \mathcal{N}_q(e_u^{(p,n)})$ with $\Gamma_B \tilde{\subseteq}\, \Omega_A$. Moreover, for all $d_v^{(p',n')} q\, \Gamma_B$, from $\Gamma_B \in \tau$ it follows that $\Gamma_B \in \mathcal{N}_q(d_v^{(p',n')})$ and thus the desired result is obtained. □

We point out by the next example that the following property fails.

$$If\;\; \Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{N}_q(e_u^{(p,n)}),\;\; then\; \Omega^1_{A_1} \,\tilde{\cap}\, \Omega^2_{A_2} \in \mathcal{N}_q(e_u^{(p,n)}). \qquad (*)$$

Example 3.13. *Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Let Ω_E^1 and Ω_E^2 be two BFS-sets in $(BF^U)^E$ with*

$$\Omega_E^1 = \left\{ \begin{array}{l} \langle e_1, \Omega^1(e_1) = \{(u_1, 0.6, -0.2), (u_2, 0.3, -0.2), \{(u_3, 0.2, -0.3)\}\rangle, \\ \langle e_2, \Omega^1(e_2) = \{(u_1, 0.3, -0.4), (u_2, 0.5, -0.3), \{(u_3, 0.2, -0.5)\}\rangle \end{array} \right\},$$

$$\Omega_E^2 = \left\{ \begin{array}{l} \langle e_1, \Omega^2(e_1) = \{(u_1, 0.1, -0.7), (u_2, 0.2, -0.3), \{(u_3, 0.3, -0.3)\}\rangle, \\ \langle e_2, \Omega^2(e_2) = \{(u_1, 0.5, -0.3), (u_2, 0.3, -0.5), \{(u_3, 0.5, -0.3)\}\rangle \end{array} \right\}.$$

Then, $\tau_1 = \{\phi_A, U_E, \Omega_E^1, \Omega_E^2, \Omega_E^1 \tilde{\cup} \Omega_E^2, \Omega_E^1 \tilde{\cap} \Omega_E^2\}$ is a BFS-topology over U. Let us consider two BFS-sets Ω_E^3 and Ω_E^4 in $(BF^U)^E$ satisfying

$$\Omega_E^3 = \left\{ \begin{array}{l} \langle e_1, \Omega^3(e_1) = \{(u_1, 0.7, -0.3), (u_2, 0.6, -0.3), \{(u_3, 0.5, -0.6)\}\rangle, \\ \langle e_2, \Omega^3(e_2) = \{(u_1, 0.4, -0.5), (u_2, 0.6, -0.9), \{(u_3, 0, 6, -0.6)\}\rangle \end{array} \right\},$$

$$\Omega_E^4 = \left\{ \begin{array}{l} \langle e_1, \Omega^4(e_1) = \{(u_1, 0.2, -0.8), (u_2, 0.6, -0.7), \{(u_3, 0.5, -0.4)\}\rangle, \\ \langle e_2, \Omega^4(e_2) = \{(u_1, 0.6, -0.4), (u_2, 0.5, -0.6), \{(u_3, 0.7, -0.8)\}\rangle \end{array} \right\}$$

and choose a BFS-point $(e_1)_{u_1}^{(0,5,-0.4)}$. It can be seen that $(e_1)_{u_1}^{(0,5,-0.4)} q\, \Omega_E^1$ and $\Omega_E^1 \tilde{\subseteq} \Omega_E^3$. For this reason, Ω_E^3 is a BFS-q-neighborhood of $(e_1)_{u_1}^{(0,5,-0.4)}$. In a similar way, we can see that Ω_E^4 is also a BFS-q-neighborhood of $(e_1)_{u_1}^{(0,5,-0.4)}$. However, $\Omega_E^3 \tilde{\cap} \Omega_E^4$ is not a BFS-q-neighborhood of $(e_1)_{u_1}^{(0,5,-0.4)}$ since $(e_1)_{u_1}^{(0,5,-0.4)}$ is not BFS-q-coincident with $\Omega_E^3 \tilde{\cap} \Omega_E^4$.

In the remainder of this paper, if the family $\mathcal{N}_q(e_u^{(p,n)})$ also satisfies the property $(*)$, then this will be presented by $\mathcal{N}_q^*(e_u^{(p,n)})$. In general, unless otherwise specified, we will consider this family as the BFS-q-neighborhoods of a BFS-point to move forward effectively.

4 BFS-nets

In this section, we introduce and study the notion of convergence for BFS-nets in the BFS-topological spaces by means of the concept of a BFS-q-neighborhood of a BFS-point given by Demir and Saldamlı [10]. This will enable us to give some results about bipolar fuzzy soft Hausdorff spaces. Moreover, like in convergence theory in general topology, we can associate with each BFS-filter on U a BFS-net over U.

Throughout this paper, Ξ is a directed set with the partial order $\preceq$ such that for each pair ξ_1, ξ_2 of elements of Ξ, there exists an element ξ of Ξ having the property that $\xi_1 \preceq \xi$ and $\xi_2 \preceq \xi$. Also, for $\xi_1, \xi_2 \in \Xi$, we shall often write $\xi_2 \succeq \xi_1$ instead of $\xi_1 \preceq \xi_2$.

Definition 4.1. *A mapping $\mathbb{F} : \Xi \to \mathcal{P}(U, E)$ is called a BFS-net over U and we denote this by $\{\mathbb{F}(\xi) : \xi \in \Xi\}$, or $\mathbb{F}$ for sake of simplicity.*

Example 4.2. *The set $\mathcal{N}_q^*(e_u^{(p,n)})$ with the relation $\preceq$ defined by*

$$\Omega_{A_1}^1 \preceq \Omega_{A_2}^2 \text{ if and only if } \Omega_{A_2}^2 \tilde{\subseteq} \Omega_{A_1}^1$$

forms a directed set. Therefore, $\mathbb{F} = \{\mathbb{F}(\Omega_A) : \Omega_A \in \mathcal{N}_q^*(e_u^{(p,n)})\}$ is a BFS-net over U.

Definition 4.3. *Let $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ be a BFS-net over U and $\Omega_A \in (BF^U)^E$.*

(i) The BFS-net $\mathbb{F}$ is called in Ω_A if $\mathbb{F}(\xi) \tilde{\in} \Omega_A$ for all $\xi \in \Xi$.

(ii) The BFS-net $\mathbb{F}$ is called eventually BFS-q-coincident with Ω_A if there exists a $\xi_0 \in \Xi$ such that $\mathbb{F}(\xi)\, q\, \Omega_A$ for all $\xi \succeq \xi_0$ with $\xi \in \Xi$.

Definition 4.4. *The BFS-net $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ in a BFS-topological space (U, τ, E) is said to converge to a BFS point $e_u^{(p,n)}$, and we write $\lim \mathbb{F}(\xi) = e_u^{(p,n)}$ or $\mathbb{F} \to e_u^{(p,n)}$, if it is eventually BFS-q coincident with each BFS-set in $\mathcal{N}_q^*(e_u^{(p,n)})$. In this case, $e_u^{(p,n)}$ is called the limit of $\mathbb{F}$.*

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Example 4.5. *Consider Example 4.2. For all $\Omega_A \in \mathcal{N}_q^*(e_u^{(p,n)})$, pick a BFS-point $\mathbb{F}(\Omega_A)\ q\ \Omega_A$. Then, we have $\mathbb{F} \to e_u^{(p,n)}$. Indeed, for each $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$, there exists a $\Lambda_C \in \mathcal{N}_q^*(e_u^{(p,n)})$ satisfying $\Lambda_C \tilde{\subseteq} \Gamma_B$. From the fact that for all $\Omega_A \geq \Lambda_C$, we have $\Omega_A \tilde{\subseteq} \Lambda_C$ it follows that $\mathbb{F}(\Omega_A)\ q\ \Omega_A \tilde{\subseteq} \Gamma_B$. Thus, we get the desired result.*

Theorem 4.6. *Let (U, τ, E) be a BFS-topological space, $\Omega_A \in (BF^U)^E$ and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. $e_u^{(p,n)} q\ (\Omega_A)^o$ if and only if there exists a $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$ such that $\Gamma_B \tilde{\subseteq} \Omega_A$.*

Proof. To prove the necessary part, let $e_u^{(p,n)} q\ (\Omega_A)^o$. From Theorem 3.5(iv) it follows that there exists a BFS-open set Γ_B such that $\Gamma_B \tilde{\subseteq} \Omega_A$ and $e_u^{(p,n)} q\ \Gamma_B$. Thus, we have that $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$.

To prove the sufficient part, let $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$ with $\Gamma_B \tilde{\subseteq} \Omega_A$. Then, we obtain a BFS-open set $\Gamma^1_{B_1}$ satisfying $e_u^{(p,n)}\ q\ \Gamma^1_{B_1} \tilde{\subseteq} \Gamma_B$. So, $\Gamma^1_{B_1} \tilde{\subseteq} (\Gamma_B)^o \tilde{\subseteq} (\Omega_A)^o$, which means that $e_u^{(p,n)} q\ (\Omega_A)^o$. □

Theorem 4.7. *Let (U, τ, E) be a BFS-topological space, $\Omega_A \in (BF^U)^E$ and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. $e_u^{(p,n)} \tilde{\in} \overline{\Omega_A}$ if and only if each set in $\mathcal{N}_q^*(e_u^{(p,n)})$ is BFS-q-coincident with Ω_A.*

Proof. Let $e_u^{(p,n)} \tilde{\in} \overline{\Omega_A}$ and $\Omega^1_{A_1} \in \mathcal{N}_q^*(e_u^{(p,n)})$. Then, there exists an $\Omega^2_{A_2} \tilde{\in} \tau$ such that $e_u^{(p,n)} q\ \Omega^2_{A_2} \tilde{\subseteq} \Omega^1_{A_1}$. Suppose that $\Omega_A\ \overline{q}\ \Omega^2_{A_2}$. Therefore, for all $d \in E$ and all $v \in U$, we have $\delta^+_{\Omega(d)}(v) + \delta^+_{\Omega^2(d)}(v) \leq 1$ and $\delta^-_{\Omega(d)}(v) + \delta^-_{\Omega^2(d)}(v) \geq -1$ and so that $\Omega_A \tilde{\subseteq} (\Omega^2_{A_2})^c$. Since $(\Omega^2_{A_2})^c$ is BFS-closed set, we get $\overline{\Omega_A} \tilde{\subseteq} (\Omega^2_{A_2})^c$ and by hypothesis, we get $e_u^{(p,n)} \tilde{\in} (\Omega^2_{A_2})^c$. But this contradicts $(e_u^{(p,n)})\ q\ \Omega^2_{A_2}$ by considering Teorem 3.5(ii). Thus, $\Omega_A\ q\ \Omega^2_{A_2}$, which proves that $\Omega_A\ q\ \Omega^1_{A_1}$.
For the converse, suppose that $e_u^{(p,n)} \tilde{\notin} \overline{\Omega_A}$. By Theorem 3.5(ii), we have $e_u^{(p,n)}\ q\ (\overline{\Omega_A})^c$. Because $(\overline{\Omega_A})^c$ is a BFS-open set, we get $(\overline{\Omega_A})^c \in \mathcal{N}_q^*(e_u^{(p,n)})$. From the hypothesis it follows that $(\overline{\Omega_A})^c\ q\ \Omega_A$. Put $\overline{\Omega_A} = \Omega^1_{A_1}$. Therefore, there exist a $v \in U$ and a $d \in E$ such that $\delta^+_{\Omega(d)}(v) + \delta^+_{(\Omega^1)^c(d)}(v) > 1$ or $\delta^-_{\Omega(d)}(v) + \delta^-_{(\Omega^1)^c(d)}(v) < -1$. Hence, $\delta^+_{\Omega^1(d)}(v) < \delta^+_{\Omega(d)}(v)$ or $\delta^-_{\Omega^1(d)}(v) > \delta^-_{\Omega(d)}(v)$ and so we get a contradiction. □

Theorem 4.8. *Let (U, τ, E) be a BFS-topological space, $\Omega_A \in (BF^U)^E$ and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. $e_u^{(p,n)} \tilde{\in} \overline{\Omega_A}$ if and only if there is a BFS-net $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ in Ω_A such that $\lim \mathbb{F}(\xi) = e_u^{(p,n)}$.*

Proof. Let $e_u^{(p,n)} \tilde{\in} \overline{\Omega_A}$ and choose the directed set $\mathcal{N}_q^*(e_u^{(p,n)})$ with the relation $\preceq$ defined as $\Omega^1_{A_1} \preceq \Omega^2_{A_2}$ if and only if $\Omega^2_{A_2} \tilde{\subseteq} \Omega^1_{A_1}$. By Theorem 4.7, for every $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$, we obtain $\Omega_A\ q\ \Gamma_B$. Then, there exist an $e_{\Gamma_B} \in E$ and a $u_{\Gamma_B} \in U$ such that

$$\delta^+_{\Omega(e_{\Gamma_B})}(u_{\Gamma_B}) + \delta^+_{\Gamma(e_{\Gamma_B})}(u_{\Gamma_B}) > 1 \ \ or \ \ \delta^-_{\Omega(e_{\Gamma_B})}(u_{\Gamma_B}) + \delta^-_{\Gamma(e_{\Gamma_B})}(u_{\Gamma_B}) < -1. \tag{1}$$

Put $p_{\Gamma_B} = \delta^+_{\Omega(e_{\Gamma_B})}(u_{\Gamma_B})$ and $n_{\Gamma_B} = \delta^-_{\Omega(e_{\Gamma_B})}(u_{\Gamma_B})$. Therefore, we have $(e_{\Gamma_B})_{u_{\Gamma_B}}^{(p_{\Gamma_B}, n_{\Gamma_B})} \tilde{\in} \Omega_A$ and $(e_{\Gamma_B})_{u_{\Gamma_B}}^{(p_{\Gamma_B}, n_{\Gamma_B})}\ q\ \Gamma_B$. So, $\mathbb{F} = \{\mathbb{F}(\Gamma_B) = (e_{\Gamma_B})_{u_{\Gamma_B}}^{(p_{\Gamma_B}, n_{\Gamma_B})} : \Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})\}$ is a BFS-net in Ω_A. Now, we will prove that this BFS-net converges to $e_u^{(p,n)}$. Take any $\Lambda_C \in \mathcal{N}_q^*(e_u^{(p,n)})$. Then, for all $\Delta_D \geq \Lambda_C$ with $\Delta_D \in \mathcal{N}_q^*(e_u^{(p,n)})$, we know that $\Delta_D \tilde{\subseteq} \Lambda_C$. Therefore, utilizing (1), we have at least one from the following inequalities

$$p_{\Delta_D} + \delta^+_{\Lambda(e_{\Delta_D})}(u_{\Delta_D}) \geq \delta^+_{\Omega(e_{\Delta_D})}(u_{\Delta_D}) + \delta^+_{\Delta(e_{\Delta_D})}(u_{\Delta_D}) > 1,$$
$$n_{\Delta_D} + \delta^-_{\Lambda(e_{\Delta_D})}(u_{\Delta_D}) \leq \delta^-_{\Omega(e_{\Delta_D})}(u_{\Delta_D}) + \delta^-_{\Delta(e_{\Delta_D})}(u_{\Delta_D}) < -1,$$

which give $\mathbb{F}(\Delta_D) = (e_{\Delta_D})_{u_{\Delta_D}}^{(p_{\Delta_D}, n_{\Delta_D})}\ q\ \Lambda_C$. Thus, the BFS-net $\mathbb{F}$ converges to $e_u^{(p,n)}$.
Conversely, let $\mathbb{F}$ be a BFS-net in Ω_A satisfying $\mathbb{F} \to e_u^{(p,n)}$. Then, for every $\Gamma_B \in \mathcal{N}_q^*(e_u^{(p,n)})$, we get a $\xi_0 \in \Xi$ such that $\mathbb{F}(\xi_0) \tilde{\in} \Omega_A$ and $\mathbb{F}(\xi_0)\ q\ \Gamma_B$. Define $\mathbb{F}(\xi_0) = d_v^{(p',n')} \in \mathcal{P}(U, E)$, where $d \in E, v \in U$ and ($p' \neq 0$ or $n' \neq 0$). Therefore, we obtain

$$\delta^+_{\Omega(d)}(v) + \delta^+_{\Gamma(d)}(v) \geq p' + \delta^+_{\Gamma(d)}(v) > 1 \ \ or \ \ \delta^-_{\Omega(d)}(v) + \delta^-_{\Gamma(d)}(v) \leq n' + \delta^-_{\Gamma(d)}(v) < -1,$$

which yield $\Omega_A\ q\ \Gamma_B$. Thus, from Theorem 4.7 it follows that $e_u^{(p,n)} \tilde{\in} \overline{\Omega_A}$. □

Now, we shall generate a BFS-net by defining any BFS-filter.

Let $\mathcal{F}$ be a BFS-filter in a BFS-topological space (U,τ,E) and let us denote by $\Xi_{\mathcal{F}}$ the set of all pairs $(e_u^{(p,n)},\Omega_A)$ such that $e_u^{(p,n)}\ q\ \Omega_A$ and $\Omega_A\in\mathcal{F}$. Then, the set $\Xi_{\mathcal{F}}$ forms a directed set with the relation $\preceq$ defined as $((e_1)_{u_1}^{(p_1,n_1)},\Omega_{A_1}^1)\preceq((e_2)_{u_2}^{(p_2,n_2)},\Omega_{A_2}^2)$ if and only if $\Omega_{A_2}^2\,\tilde{\subseteq}\,\Omega_{A_1}^1$. Therefore,

$$\mathbb{F}_{\mathcal{F}}=\{\mathbb{F}_{\mathcal{F}}((e_u^{(p,n)},\Omega_A)):(e_u^{(p,n)},\Omega_A)\in\Xi_{\mathcal{F}}\} \tag{2}$$

is a BFS-net over U, where $\mathbb{F}_{\mathcal{F}}((e_u^{(p,n)},\Omega_A))=e_u^{(p,n)}$.

Definition 4.9. *The BFS-net $\mathbb{F}_{\mathcal{F}}$ in* (2) *is called the BFS-net based on the BFS-filter $\mathcal{F}$.*

Theorem 4.10. *Let (U,τ,E) be a BFS-topological space. If a BFS-filter $\mathcal{F}$ converges to a BFS-point $e_u^{(p,n)}$ in U, then the BFS-net based on $\mathcal{F}$ converges to $e_u^{(p,n)}$.*

Proof. Consider the BFS-filter $\mathcal{F}$ converges to $e_u^{(p,n)}$ and let $\Omega_A\in\mathcal{N}_q^*(e_u^{(p,n)})$. Then, we have $\Omega_A\in\mathcal{F}$. Pick $(e_0)_{u_0}^{(p_0,n_0)}\ q\ \Omega_A$. Therefore, $((e_0)_{u_0}^{(p_0,n_0)},\Omega_A)\in\Xi_{\mathcal{F}}$ and so that for all $(d_v^{(p',n')},\Gamma_B)\succeq((e_0)_{u_0}^{(p_0,n_0)},\Omega_A)$ with $(d_v^{(p',n')},\Gamma_B)\in\Xi_{\mathcal{F}}$, we get $\mathbb{F}_{\mathcal{F}}((d_v^{(p',n')},\Gamma_B))=d_v^{(p',n')}\ q\ \Gamma_B\,\tilde{\subseteq}\,\Omega_A$. Thus, we obtain $\mathbb{F}_{\mathcal{F}}\to e_u^{(p,n)}$. □

Definition 4.11. *The BFS-topological space (U,τ,E) is called a bipolar fuzzy soft Hausdorff space (BFS-Hausdorff space) if for any two distinct BFS-soft points $(e_1)_{u_1}^{(p_1,n_1)},(e_2)_{u_2}^{(p_2,n_2)}\in\mathcal{P}(U,E)$, there exist an $\Omega_{A_1}^1\in\mathcal{N}_q^*((e_1)_{u_1}^{(p_1,n_1)})$ and an $\Omega_{A_2}^2\in\mathcal{N}_q^*((e_2)_{u_2}^{(p_2,n_2)})$ such that $e_u^{(p,n)}\ \overline{q}\ \Omega_{A_1}^1$ or $e_u^{(p,n)}\ \overline{q}\ \Omega_{A_2}^2$ for all $e_u^{(p,n)}\in\mathcal{P}(U,E)$.*

Theorem 4.12. *The BFS-topological space (U,τ,E) is a BFS-Hausdorff space if and only if every BFS-net over U converges to at most one BFS-point.*

Proof. Necessity: Let (U,τ,E) be a BFS-Hausdorff space and let us suppose that a BFS-net $\mathbb{F}=\{\mathbb{F}(\xi):\xi\in\Xi\}$ converges to two distinct $(e_1)_{u_1}^{(p_1,n_1)}$ and $(e_2)_{u_2}^{(p_2,n_2)}$. By the BFS-Hausdorff property, there exist an $\Omega_{A_1}^1\in\mathcal{N}_q^*((e_1)_{u_1}^{(p_1,n_1)})$ and an $\Omega_{A_2}^2\in\mathcal{N}_q^*((e_2)_{u_2}^{(p_2,n_2)})$ such that $e_u^{(p,n)}\ \overline{q}\ \Omega_{A_1}^1$ or $e_u^{(p,n)}\ \overline{q}\ \Omega_{A_2}^2$ for all $e_u^{(p,n)}\in\mathcal{P}(U,E)$. As $\mathbb{F}$ converges to $(e_1)_{u_1}^{(p_1,n_1)}$ and $(e_2)_{u_2}^{(p_2,n_2)}$, there exists an index $\xi\in\Xi$ such that $\mathbb{F}(\xi)\ q\ \Omega_{A_1}^1$ and $\mathbb{F}(\xi)\ q\ \Omega_{A_2}^2$, a contradiction. This means that every BFS-net over U converges to at most one BFS-point.
Sufficiency: Suppose that (U,τ,E) is not a BFS-Hausdorff space. This implies that there exist two distinct BFS-points $(e_1)_{u_1}^{(p_1,n_1)},(e_2)_{u_2}^{(p_2,n_2)}\in\mathcal{P}(U,E)$ such that for any BFS-q-neighborhood $\Omega_{A_1}^1$ of $(e_1)_{u_1}^{(p_1,n_1)}$ and for any BFS-q-neighborhood $\Omega_{A_2}^2$ of $(e_2)_{u_2}^{(p_2,n_2)}$, we have $e_u^{(p,n)}\ q\ \Omega_{A_1}^1$ and $e_u^{(p,n)}\ q\ \Omega_{A_2}^2$ for some $e_u^{(p,n)}\in\mathcal{P}(U,E)$. Then, the set

$$\Xi=\{(\Omega_{A_1}^1,\Omega_{A_2}^2):\Omega_{A_1}^1\in\mathcal{N}_q^*((e_1)_{u_1}^{(p_1,n_1)}),\Omega_{A_2}^2\in\mathcal{N}_q^*((e_2)_{u_2}^{(p_2,n_2)})\}$$

is a directed set with the relation $\preceq$ defined as $(\Omega_{A_1}^1,\Omega_{A_2}^2)\preceq(\Gamma_{B_1}^1,\Gamma_{B_2}^2)$ if and only if $\Gamma_{B_1}^1\,\tilde{\subseteq}\,\Omega_{A_1}^1$, $\Gamma_{B_2}^2\,\tilde{\subseteq}\,\Omega_{A_2}^2$. On taking $\mathbb{F}((\Omega_{A_1}^1,\Omega_{A_2}^2))\in\mathcal{P}(U,E)$ satisfying $\mathbb{F}((\Omega_{A_1}^1,\Omega_{A_2}^2))\ q\ \Omega_{A_1}^1$ and $\mathbb{F}((\Omega_{A_1}^1,\Omega_{A_2}^2))\ q\ \Omega_{A_2}^2$, we obtain a BFS-net

$$\mathbb{F}=\{\mathbb{F}((\Omega_{A_1}^1,\Omega_{A_2}^2)):(\Omega_{A_1}^1,\Omega_{A_2}^2)\in\Xi\}$$

over U. Now, we shall prove that this BFS-net converges to both $(e_1)_{u_1}^{(p_1,n_1)}$ and $(e_2)_{u_2}^{(p_2,n_2)}$. Let $\Omega_{A_1}^1\in\mathcal{N}_q^*((e_1)_{u_1}^{(p_1,n_1)})$ and $\Omega_{A_2}^2\in\mathcal{N}_q^*((e_2)_{u_2}^{(p_2,n_2)})$. Therefore, for all $(\Gamma_{B_1}^1,\Gamma_{B_2}^2)\succeq(\Omega_{A_1}^1,\Omega_{A_2}^2)$ with $(\Gamma_{B_1}^1,\Gamma_{B_2}^2)\in\Xi$, we get $\mathbb{F}((\Gamma_{B_1}^1,\Gamma_{B_2}^2))\ q\ \Gamma_{B_1}^1$ and $\mathbb{F}((\Gamma_{B_1}^1,\Gamma_{B_2}^2))\ q\ \Gamma_{B_2}^2$. From the fact that $\Gamma_{B_1}^1\,\tilde{\subseteq}\,\Omega_{A_1}^1$ and $\Gamma_{B_2}^2\,\tilde{\subseteq}\,\Omega_{A_2}^2$ it follows that $\mathbb{F}((\Gamma_{B_1}^1,\Gamma_{B_2}^2))\ q\ \Omega_{A_1}^1$ and $\mathbb{F}((\Gamma_{B_1}^1,\Gamma_{B_2}^2))\ q\ \Omega_{A_2}^2$. Thus, we conclude that $\mathbb{F}\to(e_1)_{u_1}^{(p_1,n_1)}$ and $\mathbb{F}\to(e_2)_{u_2}^{(p_2,n_2)}$, which leads to a contradiction. □

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5 BFS-q-continuity and BFS-q-compactness

In this section, we construct some applications of the BFS-nets in the bipolar fuzzy soft topological spaces. Our aim is to study the characterizations of bipolar fuzzy soft quasi-continuity (BFS-q-continuity) and bipolar fuzzy soft quasi-compactness (BFS-q-compactness) with the aid of the convergence of BFS-nets.

Firstly, we provide some important characterizations with regard to the BFS-mappings, as shown below.

Proposition 5.1. *Let $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ be a BFS-mapping, $\Omega_A \in (BF^U)^E$ and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. Then, the following results hold:*
i) $\mathfrak{f}(e_u^{(p,n)}) = \mathfrak{g}(e)_{\mathfrak{u}(u)}^{(p,n)}$.
ii) If $e_u^{(p,n)} q\ \Omega_A$, then $\mathfrak{f}(e_u^{(p,n)})\ q\ \mathfrak{f}(\Omega_A)$.
iii) If $e_u^{(p,n)} \tilde{\in}\ \Omega_A$, then $\mathfrak{f}(e_u^{(p,n)})\ \tilde{\in}\ \mathfrak{f}(\Omega_A)$.

Proof. It follows easily from the definition of BFS-mapping. □

Proposition 5.2. *Let $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ be a BFS-mapping and $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. If $\Omega^1_{A_1}\ q\ \Omega^2_{A_2}$, then $\mathfrak{f}(\Omega^1_{A_1})\ q\ \mathfrak{f}(\Omega^2_{A_2})$.*

Proof. Let $\Omega^1_{A_1}\ q\ \Omega^2_{A_2}$. Then, there exist a $u \in U$ and an $e \in E$ such that $\delta^+_{\Omega^1(e)}(u) + \delta^+_{\Omega^2(e)}(u) > 1$ or $\delta^-_{\Omega^1(e)}(u) + \delta^-_{\Omega^2(e)}(u) < -1$. Taking $\delta^+_{\Omega^1(e)}(u) = p_1$ and $\delta^-_{\Omega^1(e)}(u) = n_1$, we have $e_u^{(p_1,n_1)}\ \tilde{\in}\ \Omega^1_{A_1}$ and $e_u^{(p_1,n_1)}\ q\ \Omega^2_{A_2}$. According to Proposition 5.1, we find that $\mathfrak{f}(e_u^{(p_1,n_1)})\ \tilde{\in}\ \mathfrak{f}(\Omega^1_{A_1})$ and $\mathfrak{f}(e_u^{(p_1,n_1)})\ q\ \mathfrak{f}(\Omega^2_{A_2})$. Therefore, we reach to the following inequalities, respectively:

$$\delta^+_{\mathfrak{f}(\Omega^1)(\mathfrak{g}(e))}\mathfrak{u}(u) \geq p_1 \quad and \quad \delta^-_{\mathfrak{f}(\Omega^1)(\mathfrak{g}(e))}\mathfrak{u}(u) \leq n_1,$$
$$p_1 + \delta^+_{\mathfrak{f}(\Omega^2)(\mathfrak{g}(e))}\mathfrak{u}(u) > 1 \ \ or \ \ \delta^-_{\mathfrak{f}(\Omega^2)(\mathfrak{g}(e))}\mathfrak{u}(u) + n_1 < -1.$$

Thus, $\delta^+_{\mathfrak{f}(\Omega^1)(\mathfrak{g}(e))}\mathfrak{u}(u) + \delta^+_{\mathfrak{f}(\Omega^2)(\mathfrak{g}(e))}\mathfrak{u}(u) > 1$ or $\delta^-_{\mathfrak{f}(\Omega^1)(\mathfrak{g}(e))}\mathfrak{u}(u) + \delta^-_{\mathfrak{f}(\Omega^2)(\mathfrak{g}(e))}\mathfrak{u}(u) < -1$, which confirms that $\mathfrak{f}(\Omega^1_{A_1})\ q\ \mathfrak{f}(\Omega^2_{A_2})$. □

Definition 5.3. *Let $\Omega^1_{A_1}$,$\Omega^2_{A_2} \in (BF^U)^E$. If $e_u^{(p,n)} q\ \Omega^2_{A_2}$ for each $e_u^{(p,n)} q\ \Omega^1_{A_1}$, then $\Omega^1_{A_1}$ is called a bipolar fuzzy soft quasi-subset (BFS-q-subset) of $\Omega^2_{A_2}$ and denoted as $\Omega^1_{A_1}\ \tilde{\subseteq}_q\ \Omega^2_{A_2}$.*

Definition 5.4. *Let (U, τ_1, E), (V, τ_2, D) be two BFS-topological spaces and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. The BFS-mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ is called BFS-q-continuous at $e_u^{(p,n)}$ provided that for every BFS-open set Γ_B which is BFS-q-coincident with $\mathfrak{f}(e_u^{(p,n)})$, there exists a BFS-open set Ω_A which is BFS-q-coincident with $e_u^{(p,n)}$ such that $\Omega_A\ \tilde{\subseteq}_q\ \mathfrak{f}^{-1}(\Gamma_B)$. We say $\mathfrak{f}$ is BFS-q-continuous on U if $\mathfrak{f}$ is BFS-q-continuous at each $e_u^{(p,n)}$.*

Example 5.5. *Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Let us consider the following BFS-sets Ω^1_E and Ω^2_E on U with the set E of parameters:*

$$\Omega^1_E = \left\{ \begin{array}{l} \langle e_1, \Omega^1(e_1) = \{(u_1, 0.5, -0.2), (u_2, 0.3, -0.2), (u_3, 0.2, -0.3)\}\rangle, \\ \langle e_2, \Omega^1(e_2) = \{(u_1, 0.2, -0.3), (u_2, 0.2, -0.3), (u_3, 0.2, -0.5)\}\rangle \end{array} \right\},$$

$$\Omega^2_E = \left\{ \begin{array}{l} \langle e_1, \Omega^2(e_1) = \{(u_1, 0.3, -0.5), (u_2, 0.2, -0.3), (u_3, 0.3, -0.3)\}\rangle, \\ \langle e_2, \Omega^2(e_2) = \{(u_1, 0.5, -0.3), (u_2, 0.5, -0.5), (u_3, 0.5, -0.3)\}\rangle \end{array} \right\}.$$

Then, $\tau_1 = \{\phi_A, U_E, \Omega^1_E, \Omega^2_E, \Omega^1_E\ \tilde{\cup}\ \Omega^2_E, \Omega^1_E\ \tilde{\cap}\ \Omega^2_E\}$ is a BFS-topology over U. On the other hand, let $V = U$ and $D = \{d_1, d_2\}$. Choose the BFS-sets Ω^3_D and Ω^4_D on V with the set D of parameters such that

$$\Omega^3_D = \left\{ \begin{array}{l} \langle d_1, \Omega^3(d_1) = \{(u_1, 0.6, -0.4), (u_2, 0.5, -0.7), (u_3, 0.6, -0.6)\}\rangle, \\ \langle d_2, \Omega^3(d_2) = \{(u_1, 0.3, -0.5), (u_2, 0.4, -0.9), (u_3, 0.7, -0.8)\}\rangle \end{array} \right\},$$

$$\Omega_D^4 = \left\{ \begin{array}{l} \langle d_1, \Omega^4(d_1) = \{(u_1, 0.5, -0.3), (u_2, 0.5, -0.3), (u_3, 0.5, -0.4)\}\rangle, \\ \langle d_2, \Omega^4(d_2) = \{(u_1, 0.3, -0.4), (u_2, 0.3, -0.6), (u_3, 0.4, -0.6)\}\rangle \end{array} \right\}.$$

Therefore, $\tau_2 = \{\phi_A, V_D, \Omega_D^3, \Omega_D^4,\}$ is a BFS-topology over V.
Now, we define a BFS-mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ by

$$\mathfrak{u}(u_1) = u_1, \quad \mathfrak{u}(u_2) = u_2, \quad \mathfrak{u}(u_3) = u_3,$$
$$\mathfrak{g}(e_1) = d_1, \quad \mathfrak{g}(e_2) = d_2$$

and let us take a BFS-point $(e_1)_{u_1}^{(0.7,-0.3)}$. Then, we have $\mathfrak{f}((e_1)_{u_1}^{(0.7,-0.3)}) = (d_1)_{u_1}^{(0.7,-0.3)}$. Thus, the BFS-mapping $\mathfrak{f}$ is BFS-q-continuous at $(e_1)_{u_1}^{(0.7,-0.3)}$ because for the BFS-open sets Ω_D^3 and Ω_D^4 which is BFS-q-coincident with $(d_1)_{u_1}^{(0.7,-0.3)}$, there exists a BFS-open set Ω_E^1 which is BFS-q-coincident with $(e_1)_{u_1}^{(0.7,-0.3)}$ such that $\Omega_E^1 \tilde{\subseteq}_q \mathfrak{f}^{-1}(\Omega_D^3)$ and $\Omega_E^1 \tilde{\subseteq}_q \mathfrak{f}^{-1}(\Omega_D^4)$, where

$$\mathfrak{f}^{-1}(\Omega_D^3) = \left\{ \begin{array}{l} \langle e_1, \mathfrak{f}^{-1}(\Omega^3)(e_1) = \{(u_1, 0.6, -0.4), (u_2, 0.5, -0.7), (u_3, 0.6, -0.6)\}\rangle, \\ \langle e_2, \mathfrak{f}^{-1}(\Omega^3)(e_2) = \{(u_1, 0.3, -0.5), (u_2, 0.4, -0.9), (u_3, 0.7, -0.8)\}\rangle \end{array} \right\},$$

$$\mathfrak{f}^{-1}(\Omega_D^4) = \left\{ \begin{array}{l} \langle e_1, \mathfrak{f}^{-1}(\Omega^4)(e_1) = \{(u_1, 0.5, -0.3), (u_2, 0.5, -0.3), (u_3, 0.5, -0.4)\}\rangle, \\ \langle e_2, \mathfrak{f}^{-1}(\Omega^4)(e_2) = \{(u_1, 0.3, -0.4), (u_2, 0.3, -0.6), (u_3, 0.4, -0.6)\}\rangle \end{array} \right\}.$$

Theorem 5.6. *Let $(U, \tau_1, E), (V, \tau_2, D)$ be two BFS-topological spaces and $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ be a BFS-mapping. Then, the following statements are equivalent:*

(i) $(\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ is BFS-q-continuous.

ii) For every BFS-net $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ over U which converges to $e_u^{(p,n)}$,

$$\mathfrak{f}(\mathbb{F}) = \{\mathfrak{f}(\mathbb{F})(\xi) : \xi \in \Xi\}, \quad \textit{where} \quad \mathfrak{f}(\mathbb{F})(\xi) = \mathfrak{f}(\mathbb{F}(\xi)),$$

is a BFS-net over V converging to $\mathfrak{f}(e_u^{(p,n)})$.

Proof. We shall prove that (i) $\Rightarrow$ (ii). Consider $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\} \to e_u^{(p,n)}$ and let $\Gamma_B \in \mathcal{N}_q^*(\mathfrak{f}(e_u^{(p,n)}))$. Then, we have a $\Gamma_{B_1}^1 \in \tau_2$ such that $\mathfrak{f}(e_u^{(p,n)})\ q\ \Gamma_{B_1}^1 \tilde{\subseteq} \Gamma_B$. By (i), we get an $\Omega_A \in \tau_1$ such that $e_u^{(p,n)}\ q\ \Omega_A \tilde{\subseteq}_q \mathfrak{f}^{-1}(\Gamma_{B_1}^1)$. As $\mathbb{F} \to e_u^{(p,n)}$, there exists a $\xi_0 \in \Xi$ such that $\mathbb{F}(\xi)\ q\ \Omega_A$ for all $\xi \geq \xi_0$ with $\xi \in \Xi$. Therefore, Proposition 5.1 yields $\mathfrak{f}(\mathbb{F}(\xi))\ q\ \mathfrak{f}(\Omega_A)$, which means that $\mathfrak{f}(\mathbb{F}(\xi)) \to \mathfrak{f}(e_u^{(p,n)})$.

To prove that (ii) $\Rightarrow$ (i) suppose that $\mathfrak{f}$ is not BFS-q-continuous. Then, for some $e_u^{(p,n)} \in \mathcal{P}(U, E)$, there exists a $\Gamma_B \in \tau_2$ such that $\mathfrak{f}(e_u^{(p,n)})\ q\ \Gamma_B$ and we have $\Omega_A \tilde{\nsubseteq}_q \mathfrak{f}^{-1}(\Gamma_B)$ for every $\Omega_A \in \tau_1$ with $e_u^{(p,n)}\ q\ \Omega_A$. On account of Definition 5.3, we can choose an $(e_{\Omega_A})_{u_{\Omega_A}}^{(p_{\Omega_A}, n_{\Omega_A})} \in \mathcal{P}(U, E)$ satisfying $(e_{\Omega_A})_{u_{\Omega_A}}^{(p_{\Omega_A}, n_{\Omega_A})}\ q\ \Omega_A$ but $(e_{\Omega_A})_{u_{\Omega_A}}^{(p_{\Omega_A}, n_{\Omega_A})}\ \overline{q}\ \mathfrak{f}^{-1}(\Gamma_B)$. We know that the family $\mathcal{U}_q^*(e_u^{(p,n)}) = \{\Omega_A : e_u^{(p,n)}\ q\ \Omega_A \textit{ and } \Omega_A \in \tau_1\}$ forms a directed set by $\tilde{\supseteq}$ and so, by taking $\mathbb{F}(\Omega_A) = (e_{\Omega_A})_{u_{\Omega_A}}^{(p_{\Omega_A}, n_{\Omega_A})}$, we obtain that $\mathbb{F} = \{\mathbb{F}(\Omega_A) : \Omega_A \in \mathcal{U}_q^*(e_u^{(p,n)})\}$ is a BFS-net over U. It is easy to verify that $\mathbb{F} \to e_u^{(p,n)}$. Hence, from the hypothesis it follows that $\mathfrak{f}(\mathbb{F}) \to \mathfrak{f}(e_u^{(p,n)})$. Since $\Gamma_B \in \tau_2$ and $\mathfrak{f}(e_u^{(p,n)})\ q\ \Gamma_B$, $\mathfrak{f}(\mathbb{F})$ is eventually BFS-q-coincident with Γ_B. This brings a contradiction to the fact that $\mathbb{F}(\Omega_A)\ \overline{q}\ \mathfrak{f}^{-1}(\Gamma_B)$, that is, $\mathfrak{f}(\mathbb{F}(\Omega_A))\ \overline{q}\ \Gamma_B$ for every $\Omega_A \in \mathcal{U}_q^*(e_u^{(p,n)})$. Thus, the BFS-mapping $\mathfrak{f}$ is BFS-q-continuous. □

Definition 5.7. *The BFS-net $\mathbb{E} = \{\mathbb{E}(\rho) : \rho \in \Theta\}$ is called a BFS-subnet of a BFS-net $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ if there is a mapping $\psi : \Theta \to \Xi$ that satisfies the following properties:*

i) For all $\xi \in \Xi$, there is a $\rho \in \Theta$ such that $\xi \leq \psi(\rho)$,

ii) $\psi(\rho_1) \leq \psi(\rho_2)$ whenever $\rho_1 \leq \rho_2$,

iii) $\mathbb{E}(\rho) = \mathbb{F}(\psi(\rho))$.

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Definition 5.8. *Let (U,τ,E) be a BFS-topological space and $\Omega_A \in (BF^U)^E$.*

(i) The family $\Psi = \{\Omega^i_{A_i} : i \in J\}$ of BFS-sets over U is called a BFS-q-cover of Ω_A if $\Omega_A \tilde{\subseteq}_q \tilde{\bigcup}_{i\in J} \Omega^i_{A_i}$. Moreover, it is called a BFS-q-open cover of Ω_A if each member of Ψ is a BFS-open set over U. The BFS-q-subcover of Ψ is a subfamily of Ψ which is also a BFS-q-cover.

(ii) The BFS-set Ω_A is called a BFS-q-compact if each BFS-q-open cover of Ω_A has a finite BFS-q-subcover.

Example 5.9. *Let $U = \{u_1, u_2, ..., u_n, ...\}$ be the universal set and $E = \{e_1, e_2, ..., e_n, ...\}$ be the set of parameters. Define the BFS-sets $\Omega^i_{A_i} \in (BF^U)^E$, where $i \in \{1, 2, 3, ...\}$, as the following: for all $u \in U$,*

$$\delta^+_{\Omega^i(e_j)}(u) = \begin{cases} \frac{1}{j}, & \text{if } i \geq j; \\ 0, & \text{if } i < j, \end{cases} \qquad \delta^-_{\Omega^i(e_j)}(u) = \begin{cases} -\frac{1}{j}, & \text{if } i \geq j; \\ 0, & \text{if } i < j. \end{cases}$$

Then, $\tau = \{\Omega^i_{A_i} : i \in \{1, 2, 3, ...\}\} \cup \{U_E, \phi_A\}$ is a BFS-topology on U. Now, let us consider $\Gamma_B \in (BF^U)^E$ such that

$$\delta^+_{\Gamma(e)}(u) = \begin{cases} 0.7, & \text{if } e = e_1; \\ 0, & \text{if } e \neq e_1, \end{cases} \qquad \delta^-_{\Gamma(e)}(u) = \begin{cases} -0.2, & \text{if } e = e_1; \\ 0, & \text{if } e \neq e_1, \end{cases}$$

for all $u \in U$. Hence, one can easily verify that the BFS-set Γ_B is a BFS-q-compact.

Lemma 5.10. *Let $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ be a BFS-net and $e_u^{(p,n)} \in \mathcal{P}(U,E)$. If $\mathbb{F}$ has no BFS-subnet converging to the BFS-point $e_u^{(p,n)}$, then there exist an $N_{e_u^{(p,n)}} \in \mathcal{N}^*_q(e_u^{(p,n)})$ and a $\xi_{e_u^{(p,n)}} \in \Xi$ such that $\mathbb{F}(\xi)\ \overline{q}\ N_{e_u^{(p,n)}}$ for all $\xi \geq \xi_{e_u^{(p,n)}}$ with $\xi \in \Xi$.*

Proof. Suppose that for all $\xi \in \Xi$ and for all $\Omega_A \in \mathcal{N}^*_q(e_u^{(p,n)})$, we have a $\xi_{\Omega_A} \geq \xi$ with $\xi_{\Omega_A} \in \Xi$ such that $\mathbb{F}(\xi_{\Omega_A})\ q\ \Omega_A$. Set

$$\widehat{\Xi} = \{\xi_{\Omega_A} \in \Xi : \xi \leq \xi_{\Omega_A} \text{ and } \mathbb{F}(\xi_{\Omega_A})\ q\ \Omega_A, \text{ where } \Omega_A \in \mathcal{N}^*_q(e_u^{(p,n)}),\ \xi \in \Xi\}. \tag{3}$$

Then, $\widehat{\Xi}$ is a directed set with the same relation $\leq$ in Ξ. Now, we shall show that a subnet of $\mathbb{F}$ converges to $e_u^{(p,n)}$. Since the sets $\mathcal{N}^*_q(e_u^{(p,n)})$ and $\widehat{\Xi}$ are directed sets, the product set $\mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}$ is also a directed set with the relation $\preceq_p$ as follows:

$$(\Omega^1_{A_1}, \xi_{\Omega^1_{A_1}}) \preceq_p (\Omega^2_{A_2}, \xi_{\Omega^2_{A_2}}) \text{ if and only if } \Omega^2_{A_2} \tilde{\subseteq} \Omega^1_{A_1} \text{ and } \xi_{\Omega^1_{A_1}} \leq \xi_{\Omega^2_{A_2}}.$$

Define a $\psi : \mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi} \to \Xi$ given by $\psi(\Omega_A, \xi_{\Omega_A}) = \xi_{\Omega_A}$. Then, a BFS-net

$$\mathbb{E} = \{\mathbb{E}((\Omega_A, \xi_{\Omega_A})) = \mathbb{F}(\xi_A) : (\Omega_A, \xi_{\Omega_A}) \in \mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}\}$$

is a BFS-subnet of $\mathcal{F}$. Indeed, taking into account (3), for all $\delta \in \Xi$, we have a $((\Omega_A)^\delta, \xi_{(\Omega_A)^\delta}) \in \mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}$ satisfying $\delta \leq \psi(((\Omega_A)^\delta, \xi_{(\Omega_A)^\delta})) = \xi_{(\Omega_A)^\delta}$ and $\mathbb{F}(\xi_{(\Omega_A)^\delta})\ q\ (\Omega_A)^\delta$. Therefore, $\mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}$ satisfy the condition (i) of Definition 5.7. To investigate the condition (ii), take $(\Omega^1_{A_1}, \xi_{\Omega^1_{A_1}}) \preceq_p (\Omega^2_{A_2}, \xi_{\Omega^2_{A_2}})$. From $\xi_{\Omega^1_{A_1}} \leq \xi_{\Omega^2_{A_2}}$ it follows that $\psi((\Omega^1_{A_1}, \xi_{\Omega^1_{A_1}})) \leq \psi((\Omega^2_{A_2}, \xi_{\Omega^2_{A_2}}))$. So, $\mathbb{E}$ is a BFS-subnet of $\mathbb{F}$. For any $\Omega_A \in \mathcal{N}^*_q(e_u^{(p,n)})$, let us choose a $(\Omega_A, \xi_{\Omega_A}) \in \mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}$. Hence, for all $(\Gamma_B, \xi_{\Gamma_B}) \succeq_p (\Omega_A, \xi_{\Omega_A})$ with $(\Gamma_B, \xi_{\Gamma_B}) \in \mathcal{N}^*_q(e_u^{(p,n)}) \times \widehat{\Xi}$, we obtain $\Gamma_B \tilde{\subseteq} \Omega_A$, $\xi_{\Omega_A} \leq \xi_{\Gamma_B}$. Due to $\mathbb{E}((\Gamma_B, \xi_{\Gamma_B})) = \mathbb{F}(\xi_{\Gamma_B})\ q\ \Gamma_B$, we get $\mathbb{E}((\Gamma_B, \xi_{\Gamma_B}))\ q\ \Omega_A$. This implies that $\mathbb{E} \to e_u^{(p,n)}$, which contradicts the hypothesis that any subnet of $\mathbb{F}$ does not take $e_u^{(p,n)}$ as its limit. □

Theorem 5.11. *The BFS-set Ω_A is a BFS-q-compact if and only if every BFS-net whose the elements are BFS-q-coincident with Ω_A has a BFS-subnet whose the limit is BFS-q-coincident with Ω_A.*

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Proof. Let Ω_A be a BFS-q-compact and consider a BFS-net $\mathbb{F} = \{\mathbb{F}(\xi) : \xi \in \Xi\}$ such that $\mathbb{F}(\xi)\ q\ \Omega_A$ for all $\xi \in \Xi$. Suppose that for all BFS-points $e_u^{(p,n)}$ being BFS-q-coincident with Ω_A, any subnet of $\mathbb{F}$ does not take $e_u^{(p,n)}$ as its limit. By utilizing Lemma 5.10, there is an $N_{e_u^{(p,n)}} \in \mathcal{N}_q^*(e_u^{(p,n)})$ and a $\xi_{e_u^{(p,n)}} \in \Xi$ such that $\mathbb{F}(\xi)\ \overline{q}\ N_{e_u^{(p,n)}}$ for all $\xi \geq \xi_{e_u^{(p,n)}}$ with $\xi \in \Xi$. Then, we obtain

$$\Omega_A \subseteq_q \widetilde{\bigcup}\{N_{e_u^{(p,n)}} \in (BF^U)^E : N_{e_u^{(p,n)}} \in \mathcal{N}_q^*(e_u^{(p,n)})\ and\ \ e_u^{(p,n)} q\ \Omega_A\}.$$

Since Ω_A is BFS-q-compact, it has a finite BFS-q-subcover, say $\{N_{(e_1)_{u_1}^{(p_1,n_1)}}, ..., N_{(e_k)_{u_k}^{(p_k,n_k)}}\}$. Therefore, we have that for all $i \in \{1, 2, ..., k\}$, there exists a $\xi_{(e_i)_{u_i}^{(p_i,n_i)}} \in \Xi$ such that $\mathbb{F}(\xi)\ \overline{q}\ N_{(e_i)_{u_i}^{(p_i,n_i)}}$ for all $\xi \geq \xi_{(e_i)_{u_i}^{(p_i,n_i)}}$ with $\xi \in \Xi$. As Ξ is a directed set, we get a $\xi_0 \in \Xi$ satisfying $\mathbb{F}(\xi_0)\ \overline{q}\ N_{(e_i)_{u_i}^{(p_i,n_i)}}$ and $\xi_0 \geq \xi_{(e_i)_{u_i}^{(p_i,n_i)}}$ for all $i \in \{1, 2, ..., k\}$. Thus, by Theorem 3.5(iv), from the fact that $\Omega_A \tilde{\subseteq}_q \widetilde{\bigcup}_{i=1}^k N_{(e_i)_{u_i}^{(p_i,n_i)}}$ it follows that $\mathbb{F}(\xi_0)\ \overline{q}\ \Omega_A$, which leads to a contradicition.
For the converse, let $\Psi = \{\Omega_{A_i}^i : i \in J\}$ be a BFS-q-open cover of Ω_A and let us note that the family $\mathbb{I} = \{I : I\ is\ a\ finite\ subset\ of\ J\}$ forms a directed set by inclusion $\subseteq$. For the proof by contradiction suppose that Ω_A is not a BFS-q-compact. This means that $\Omega_A \tilde{\nsubseteq}_q \widetilde{\bigcup}_{i\in I}\Omega_{A_i}^i$ for all $I \in \mathbb{I}$. Then, from Definition 5.3, there exists a BFS-point $(e_I)_{u_I}^{(p_I,n_I)}\ q\ \Omega_A$ such that

$$(e_I)_{u_I}^{(p_I,n_I)}\ \overline{q}\ \widetilde{\bigcup}_{i\in I}\Omega_{A_i}^i \quad for\ all\ I \in \mathbb{I}. \tag{4}$$

Choosing $\mathbb{F}(I) = (e_I)_{u_I}^{(p_I,n_I)}$, we obtain a BFS-net $\mathbb{F} = \{\mathbb{F}(I) : I \in \mathbb{I}\}$. Since $\mathbb{F}(I)\ q\ \Omega_A$ for all $I \in \mathbb{I}$, by hypothesis, there exists a BFS-subnet $\mathbb{E} = \{\mathbb{E}(\rho) : \rho \in \Theta\}$ of $\mathbb{F}$ such that $\lim \mathbb{E}(\rho) = e_u^{(p,n)}$ with $e_u^{(p,n)}\ q\ \Omega_A$. Due to $\Omega_A \tilde{\subseteq}_q \widetilde{\bigcup}_{i\in J}\Omega_{A_i}^i$, from Theorem 3.5(iv), there exists an $i_0 \in J$ such that $e_u^{(p,n)} q\ \Omega_{A_{i_0}}^{i_0}$. Because $\Omega_{A_{i_0}}^{i_0}$ is BFS-open set and $\mathbb{E} \to e_u^{(p,n)}$, we get a $\rho_0 \in \Theta$ such that

$$\mathbb{E}(\rho)\ q\ \Omega_{A_{i_0}}^{i_0} \quad for\ all\ \rho \geq \rho_0\ with\ \rho \in \Theta. \tag{5}$$

Now, by the BFS-subnet property, let us take a mapping $\psi : \Theta \to \mathbb{I}$. For $\{i_0\} \in \mathbb{I}$, there is a $\rho_1 \in \Theta$ such that $\{i_0\} \subseteq \psi(\rho_1)$. Therefore, we observe that for all $\rho \geq \rho_1$ with $\rho \in \Theta$, we have $\{i_0\} \subseteq \psi(\rho)$ and so that $i_0 \in \psi(\rho)$. From (4) it follows that

$$\mathbb{F}(\psi(\rho)) = \mathbb{E}(\rho)\ \overline{q}\ \Omega_{A_{i_0}}^{i_0}. \tag{6}$$

Hence, for a $\rho_3 \in \Theta$ satisfying $\rho_0 \leq \rho_3$ and $\rho_1 \leq \rho_3$, by utilizing (5) and (6), we obtain $\mathbb{E}(\rho_3)\ q\ \Omega_{A_{i_0}}^{i_0}$ and $\mathbb{E}(\rho_3)\ \overline{q}\ \Omega_{A_{i_0}}^{i_0}$, respectively. Thus, we arrive at a contradiction and so the BFS-set Ω_A is BFS-q-compact. □

6 Conclusion

The convergence theory not only is a basic theory of topology but also has wide applications in other fields including information technology, economics and computer science. Besides, the convergence of nets is one of the most important tools used in topology to characterize certain concepts such as the closure of a set, the continuous mappings, Hausdorff spaces, compact spaces and so on. In this research article, firstly, we introduce and study the concept of a BFS-q-coincidence and also show some of its properties. Then, we discuss the notion of a net in the bipolar fuzzy soft inference, and obtain some results on convergence by making use of the concept of a BFS-q-neighborhood of a BFS-point in a BFS-topological space. This enable us to give some applications about the closure of a BFS-set, BFS-Hausdorff spaces, BFS-q-continuity and BFS-q-compactness. Moreover, we present the concept of a BFS-net generated by a BFS-filter, and in the following we investigate the connection between BFS-net convergence and BFS-filter convergence.

Therefore, it is hoped that these theoretical works will play a guiding role in further investigation of new approaches for BFS-net and BFS-topology as well as in many areas of application. As a piece of future work, one can analyze further topological concepts via the BFS-nets such as product spaces, uniform spaces, ultranets and cluster points. Also, one can investigate the possibility of applying some of the notions explored here to real-life problems. Next, one can reformulate our studies to some frames such as bipolar fuzzy N-soft sets, bipolar fuzzy soft expert sets, bipolar complex fuzzy soft sets and rough fuzzy bipolar soft sets to study the net and its applications on these models.

Declaration

Conflict of interest Authors do not have any conflict of interest with any other person or organization.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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