THE LEVINSON FORMULA FOR INVERSE SCATTERING PROBLEM OF QUADRATIC EIGENPARAMETER DEPENDENT DISCRETE STURM-LIOUVILLE EQUATION

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Abstract. In this paper, the uniqueness of the kernel and the continuity of the scattering function for inverse discrete Sturm-Liouville problem with quadratic eigenparameter dependent boundary condition is studied by finding the main equation. Also, on the basis of the continuity of the scattering function an appropriate formula of the Levinson type has been derived.

1. Introduction

In Quantum Mechanics, Geo-Physics and Engineering ([1-8]), there has been considerable study of inverse scattering theory for differential equations. To study inverse problems, there has been considerable use of Fourier Analysis ([9-15]).

The nonhomogeneous boundary value problem (BVP)

$$\begin{cases} -y^{''} + q(x)y - \lambda^2 y = f(x) \ , \quad 0 \leq x < \infty \ , \\ \quad y^{'}(0) - hy(0) = 0 \end{cases} \tag{1.1}$$

has been considered in [16], where q and f are complex valued functions, $h \in \mathbb{C}$ and λ is a spectral parameter. The authors of [16] have investigated the eigenvalues and spectral singularities of the BVP (1.1) using the boundary uniqueness theorems of analytic functions, and they have also proved that the finiteness of numbers and multiplicities of the eigenvalues and spectral singularities of the BVP (1.1). Furthermore, spectral and scattering analysis of some difference equations with principal functions have been investigated in [17-26]. Examining the properties of scattering data by using potential function is called a direct problem for scattering theory. On the other hand, the inverse scattering problem deals with obtaining the potential function according to the scattering data and kernel function. In this sense, the inverse scattering theory has been considered in [27]. Also, some inverse Sturm-Liouville scattering problems with uniqueness of the solution and continuity of scattering function have been studied in [28-37].

Spectral theory of Sturm-Liouville problems with a spectral parameter occurs not only in differential equation but also in boundary conditions. The study of spectral theory plays an important role in many physical and technical studies such as heat conduction problems, vibrating string problems, and some problems in mechanical

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engineering ([38-40]). Many researchers have studied the inverse scattering theory for the Sturm-Liouville operators with eigenparameter in boundary condition, or the Sturm-Liouville difference equations with eigenparameter independent boundary condition. Nevertheless the absence of a study for an inverse discrete Sturm-Liouville problem involving a quadratic spectral parameter dependent boundary condition stands out as a deficiency that motivated us to do this study. We begin description of our problem and its solution from the next paragraph.

Let L_λ denote the discrete operator of the BVP in $\ell_2(\mathbb{N})$ given by

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.2}$$

$$(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2)y_1 + (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y_0 = 0 \tag{1.3}$$

for $\lambda = 2\cos z$ where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences, $a_n \neq 0$ for all $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ with $a_0 > 0$. Also, $\gamma_i,\ \beta_i \in \mathbb{R}^+\cup\{0\}$ for $i = 0, 1, 2$ with $\gamma_m\beta_n - \beta_m\gamma_n > 0$ where $(m, n) \in \mathbb{N}\times\mathbb{N}_0$ and $n < m \leq 2$. Eq. (1.2) can be expressed in the Sturm-Liouville form

$$\nabla(a_n\Delta y_n) + h_n y_n = \lambda y_n,\ n \in \mathbb{N},$$

where Δ and ∇ are respectively forward and backward difference operators with $h_n = a_{n-1} + a_n + b_n$.

This paper is designed to study the inverse scattering theory of discrete operator L_λ which has quadratic spectral parameter in the boundary condition according to the continuity of the scattering function, main equation, and Levinson type formula under the condition

$$\sum_{n=1}^{\infty} n\left(|1 - a_n| + |b_n|\right) < \infty. \tag{1.4}$$

The contribution of our study to the inverse scattering theory of difference equations is in finding the solution of the inverse problem uniquely when the spectral parameter in the discrete Dirac system is also in the boundary conditions at quadratic form.

In this paper we will use many of the notations and concepts used in our paper [41], but for the sake of completeness, we will mention them for our readers.

2. Preliminaries

Let (1.4) be satisfied for a_n,$b_n \in \mathbb{R}$. The Eq. (1.2) has the Jost solution $e(z) = \{e_n(z)\}$ as

$$e_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{nm}e^{imz}\right),\ n \in \mathbb{N}_0,$$

for $\lambda = 2\cos z$ in $\overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$ where α_n, A_{nm} are expressed in terms of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $\alpha_1 = a_0\alpha_0$ ([23,24]). We also assume the following conditions:

$$\lim_{n\to\infty} e_n(z)e^{-inz} = 1,\ \ z \in \overline{\mathbb{C}}_+ \tag{2.1}$$

and

$$|A_{nm}| \leq c_1 \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1 - a_k| + |b_k|\right), \tag{2.2}$$

where $c_1 > 0$ is constant and $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$. Thus, $e_n(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z > 0\}$ and continuous on $\mathbb{R}$.

If $\widehat{\psi}(\lambda) = \left\{\widehat{\psi}_n(\lambda)\right\}$, $n \in \mathbb{N}_0$ be another solution of (1.2) which has the initial conditions

$$\widehat{\psi}_0(\lambda) = -\left(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2\right)\ ,\ \widehat{\psi}_1(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2,$$

then

$$\psi(z) = \{\psi_n(z)\} := \widehat{\psi}(2\cos z) = \left\{\widehat{\psi}_n(2\cos z)\right\}$$

is a 2π periodic entire function ([26]).

Let the semi-strips be defined by $B_0 := \{z \in \mathbb{C} : z = x + iy, -\pi \leq x \leq \pi,\ y > 0\}$ and $B := B_0 \cup [-\pi, \pi]$. In $(-\pi, \pi) \setminus \{0\}$, the wronskian of $e(z)$ and $e(-z)$ is

$$\begin{aligned} W\left[e(z), e(-z)\right] &= a_n\left[e_n(z)e_{n+1}(-z) - e_{n+1}(z)e_n(-z)\right] \\ &= \lim_{n\to\infty}\left[a_n\left(e^{inz}e^{-i(n+1)z} - e^{i(n+1)z}e^{-inz}\right)\right] \\ &= -2i\sin z \neq 0. \end{aligned} \tag{2.3}$$

So, $e(z)$ and $e(-z)$ are fundamental solutions of L_λ where $e(-z)$ is analytic in $\mathbb{C}_-$ $:= \{z : z \in \mathbb{C},\ \operatorname{Im} z < 0\}$ and continuous on the real axis. Furthermore, the Jost function of L_λ can be defined by

$$\begin{aligned} F(z) &= W\left[e(z), \psi(z)\right] \\ &= a_n\left[\widehat{\psi}_{n+1}(2\cos z)e_n(z) - e_{n+1}(z)\widehat{\psi}_n(2\cos z)\right] \\ &= a_0\left[\widehat{\psi}_1(2\cos z)e_0(z) - e_1(z)\widehat{\psi}_0(2\cos z)\right] \end{aligned} \tag{2.4}$$

where F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$ and $F(z) = F(z + 2\pi)$. The open form of $F(z)$ is

$$\begin{aligned} F(z) = {} & a_0\left\{\left[\gamma_0 + \gamma_1\left(e^{iz} + e^{-iz}\right) + \gamma_2(2 + e^{2iz} + e^{-2iz})\right]\right. \\ & \times\left[\alpha_1 e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1m}e^{imz}\right)\right] \\ & + \left[\beta_0 + \beta_1\left(e^{iz} + e^{-iz}\right) + \beta_2(2 + e^{2iz} + e^{-2iz})\right] \\ & \left.\times\left[\alpha_0\left(1 + \sum_{m=1}^{\infty} A_{0m}e^{imz}\right)\right]\right\} \\ = {} & a_0\left\{\alpha_0\beta_2 e^{-2iz} + \left(\alpha_1\gamma_2 + \alpha_0\beta_1\right)e^{-iz} + \alpha_1\gamma_1 + \alpha_0\left(\beta_0 + 2\beta_2\right)\right. \\ & + \left[\alpha_1\left(\gamma_0 + 2\gamma_2\right) + \alpha_0\beta_1\right]e^{iz} + (\alpha_1\gamma_1 + \alpha_0\beta_2)e^{2iz} + \alpha_1\gamma_2 e^{3iz} \\ & + \sum_{m=1}^{\infty}\alpha_0\beta_2 A_{0m}e^{i(m-2)z} + \sum_{m=1}^{\infty}\left(\alpha_1\gamma_2 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{i(m-1)z} \\ & + \sum_{m=1}^{\infty}\left[\alpha_1\gamma_1 A_{1m} + \alpha_0\left(\beta_0 + 2\beta_2\right)A_{0m}\right]e^{imz} \\ & + \sum_{m=1}^{\infty}\left[\alpha_1\left(\gamma_0 + 2\gamma_2\right)A_{1m} + \alpha_0\beta_1 A_{0m}\right]e^{i(m+1)z} \\ & \left.+ \sum_{m=1}^{\infty}\left(\alpha_1\gamma_1 A_{1m} + \alpha_0\beta_2 A_{0m}\right)e^{i(m+2)z} + \sum_{m=1}^{\infty}\alpha_1\gamma_2 A_{1m}e^{i(m+3)z}\right\}, \end{aligned}$$

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and clearly $\overline{F(z)} = F(-z)$ in $(-\pi,\pi)\setminus\{0\}$ because of $\overline{e(z)} = e(-z)$. When $|z|\to\infty$, $F(z)$ has the following asymptotic behavior:

$$F(z) \approx \begin{cases} \alpha_1\beta_2 e^{-2iz} & , \ \beta_2 \neq 0, \ z\in B, \\ \alpha_1\left(a_0\gamma_2+\beta_1\right) & , \ \beta_2 = 0, \ z\in B, \end{cases} , \tag{2.5}$$

and so $F(z)$ has finite number of zeros in B for

$$\sup_{n\in\mathbb{N}} e^{\varepsilon n^{\delta}}\left(|1-a_n|+|b_n|\right) < \infty \tag{2.6}$$

where $\varepsilon > 0$ and $\delta \in \left[\frac{1}{2},1\right]$ ([26]).

3. The Scattering Function and Main Equation of L_λ

Definition 3.1. *For $z\in(-\pi,\pi)\setminus\{0\}$, the scattering function of L_λ is defined by*

$$S(z) = \frac{\overline{F(z)}}{F(z)}. \tag{3.1}$$

Using (2.4) and (3.1), it can be easily found that

$$S(z) = 1 + O(1)\ ;\ z\in(-\pi,\pi)\setminus\{0\}\,,\ \ |z|\to\pi$$

and

$$\overline{S(z)} = [S(z)]^{-1}\ \text{, i.e. } |S(z)| = 1.$$

The next theorem uses a different form of $F(z)$ and we show the results that one can get for this choice of F.

Theorem 3.1. *Under the condition (1.4),*

$F(z) = a_0\left[\left(\gamma_0 + 2\gamma_1\cos z + 4\gamma_2\cos^2 z\right)e_1(z) + \left(\beta_0 + 2\beta_1\cos z + 4\beta_2\cos^2 z\right)e_0(z)\right]$
is nonzero in $(-\pi,\pi)\setminus\{0\}$.

Proof. Let $F(z_1)=0$ for any $z_1\in(-\pi,\pi)\setminus\{0\}$. Since $a_0\neq 0$,

$$e_1(z_1) = -\frac{\beta_0+2\beta_1\cos z_1+4\beta_2\cos^2 z_1}{\gamma_0+2\gamma_1\cos z_1+4\gamma_2\cos^2 z_1}e_0(z_1). \tag{3.2}$$

By using (2.3), (3.2), $\overline{\cos z_1} = \cos z_1$ and $\overline{e(z_1)} = e(-z_1)$ for $z_1\in(-\pi,\pi)\setminus\{0\}$, it is obtained that $2i\sin z_1 = 0$ but this is a contradiction. □

We use the two lemmas which are mentioned below for to construce the main equation for Lemma 3.4.

Lemma 3.2. *The equation*

$$\frac{2i\sin z\psi(z)}{F(z)} = S(z)e(z) - e(-z) \tag{3.3}$$

is satisfied in $(-\pi,\pi)\setminus\{0\}$.

Proof. First of all, it can be written that

$$\psi(z) = c_2 e(z) + c_3 e(-z) \tag{3.4}$$

for constants c_2 and c_3. Then,

$$\begin{aligned} \widehat{\psi}_0(2\cos z) &= c_2 e_0(z) + c_3 e_0(-z), \\ \widehat{\psi}_1(2\cos z) &= c_2 e_1(z) + c_3 e_1(-z) \end{aligned}$$

and

$$c_2 = \frac{\overline{F(z)}}{2i\sin z},\ c_3 = -\frac{F(z)}{2i\sin z}.$$

So, (3.1) and (3.4) indicates the Eq. (3.3). □

Lemma 3.3. *All of the zeros of the Jost function $F(z)$ are simple on the positive imaginary axis and they lie in $B_1 := \{z \in \mathbb{C} : z = i\tau, \tau \geq 0\}$.*

Proof. Assume that $F(z_1) = 0$ for an arbitrary $z_1 \in B$. If $\lambda_1 = 2\cos z_1$, then

$$\begin{cases} a_{n-1}e_{n-1}(z_1) + b_n e_n(z_1) + a_n e_{n+1}(z_1) = \lambda_1 e_n(z_1) \\ a_{n-1}\overline{e_{n-1}(z_1)} + b_n\overline{e_n(z_1)} + a_n\overline{e_{n+1}(z_1)} = \overline{\lambda_1}\overline{e_n(z_1)} \end{cases}.$$

So,

$$\begin{aligned} \left(\lambda_1 - \overline{\lambda_1}\right)\left|e_n(z_1)\right|^2 &= a_{n-1}\left[e_{n-1}(z_1)\overline{e_n(z_1)} - \overline{e_{n-1}(z_1)}e_n(z_1)\right] \\ &\quad + a_n\left[e_{n+1}(z_1)\overline{e_n(z_1)} - \overline{e_{n+1}(z_1)}e_n(z_1)\right] \\ &= W\left[e_{n-1}(z_1), \overline{e_{n-1}(z_1)}\right] - W\left[e_n(z_1), \overline{e_n(z_1)}\right] \end{aligned}$$

using the usual definition of the wronskian. Also, (2.1) and (3.2) says that

$$\left(\lambda_1 - \overline{\lambda_1}\right)\left[\sum_{n=1}^{\infty}\left|f_n(z_1)\right|^2 + a_0\left|f_0(z_1)\right|^2\zeta(\lambda_1)\right] = 0$$

where $\zeta(\lambda_1) = \frac{(\gamma_1\beta_0-\gamma_0\beta_1)+2\operatorname{Re}\lambda_1(\gamma_2\beta_0-\gamma_0\beta_2)+|\lambda_1|^2(\gamma_2\beta_1-\gamma_1\beta_2)}{\gamma_0^2+2\gamma_0\gamma_1\operatorname{Re}\lambda_1+\gamma_1^2|\lambda_1|^2+2\gamma_0\gamma_2\left[(\operatorname{Re}\lambda_1)^2-(\operatorname{Im}\lambda_1)^2\right]+\gamma_1\gamma_2|\lambda_1|^2\lambda_1+\gamma_2^2|\lambda_1|^4}$. Thus, it must be $\lambda_1 = 0$ and $\lambda_1 = \overline{\lambda_1}$, i.e. $\lambda_1 = 2\cos z_1 \in \mathbb{R}$. Moreover, for $z_1 = \operatorname{Re} z_1 + i\operatorname{Im} z_1$,

$$2\cos z_1 = \cos(\operatorname{Re} z_1)\left(e^{-\operatorname{Im} z_1} + e^{\operatorname{Im} z_1}\right) + i\sin(\operatorname{Re} z_1)\left(e^{-\operatorname{Im} z_1} - e^{\operatorname{Im} z_1}\right)$$

and hence $\sin(\operatorname{Re} z_1)\left(e^{-\operatorname{Im} z_1} - e^{\operatorname{Im} z_1}\right) = 0$. It means that, $\sin(\operatorname{Re} z_1) = 0$ or $e^{-\operatorname{Im} z_1} = e^{\operatorname{Im} z_1}$. From this result and Theorem 3.1, it can be found that $\operatorname{Re} z_1 = 0$ or $z_1 = 0$, i.e. $z_1 \in B_1$.

Additionally, the proof will be completed if it is shown that z_1 is simple on the positive imaginary axis. From (1.2), it is obtained that

$$\begin{cases} a_{n-1}e_{n-1}(z_1) + b_n e_n(z_1) + a_n e_{n+1}(z_1) = 2\cos z_1 e_n(z_1), \\ a_{n-1}e'_{n-1}(z_1) + b_n e'_n(z_1) + a_n e'_{n+1}(z_1) = -2\sin z_1 e_n(z_1) + 2\cos z_1 e'_n(z_1). \end{cases}$$

Then, the last system reveals that

$$\begin{aligned} 2\sin z_1 e_n^2(z_1) &= a_{n-1}\left[e_{n-1}(z_1)e'_n(z_1) - e'_{n-1}(z_1)e_n(z_1)\right] \\ &\quad + a_n\left[e_{n+1}(z_1)e'_n(z_1) - e'_{n+1}(z_1)e_n(z_1)\right] \end{aligned}$$

and

$$\begin{aligned} 2\sin z_1\sum_{n=1}^{k}e_n^2(z_1) &= a_0\left[e_0(z_1)e'_1(z_1) - e'_0(z_1)e_1(z_1)\right] \\ &\quad + a_k\left[e_{k+1}(z_1)e'_k(z_1) - e'_{k+1}(z_1)e_k(z_1)\right]. \end{aligned} \tag{3.5}$$

From (2.1) and (3.5),

$$2\sin z_1\sum_{n=1}^{\infty}e_n^2(z_1) = a_0\left[e_0(z_1)e'_1(z_1) - e'_0(z_1)e_1(z_1)\right] \tag{3.6}$$

Since

$$e_1(z) = \frac{1}{a_0\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right)} F(z) - \frac{\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z}{\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z} e_0(z)$$

and $F(z_1) = 0$,

$$\begin{aligned} 2\sin z_1 \sum_{n=1}^{\infty} e_n^2(z_1) \quad &= \quad \frac{e_0(z_1)F'(z_1)}{\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z} \\ &- \frac{\left[a_0 e_0^2(z_1) 2\sin z_1\right]}{\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right)^2} \\ &\times \left\{(\gamma_1\beta_0 - \beta_1\gamma_0) + 4\cos z_1\right. \\ &\times \left.\left[(\gamma_2\beta_1 - \gamma_1\beta_2)\cos z_1 + (\gamma_2\beta_0 - \beta_2\gamma_0)\right]\right\} \end{aligned} \tag{3.7}$$

can be found by using (3.6). Therefore,

$$\sum_{n=1}^{\infty} e_n^2(z_1) + a_0 e_0^2(z_1) \frac{(\gamma_1\beta_0 - \beta_1\gamma_0) + 4\cos z_1\left[(\gamma_2\beta_1 - \gamma_1\beta_2)\cos z_1 + (\gamma_2\beta_0 - \beta_2\gamma_0)\right]}{\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right)^2} = \frac{e_0(z_1)F'(z_1)}{\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) 2\sin z_1} \tag{3.8}$$

from (3.7). In here, the function $e_n(z_1)$ is real and $\cos z_1 > 0$ since z_1 is on the positive imaginary axis. In addition, if $e_0(z_1) = 0$ in (3.8), then $e_n(z_1) \equiv 0$ but this is not possible. Hence, $F'(z_1) \neq 0$ because $\sin z_1 \neq 0$ and the left side of (3.8) is positive. □

Remark 3.1. *It can be easily shown that*

$$1 - S(z) \approx O(1)\ ,\ \ |z| \to \pi,$$

and for this reason, the Fourier transform of $1 - S(z)$ *is given by*

$$F_S(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 - S(z)\right] e^{inz} dz. \tag{3.9}$$

$F_S(n)$ *belongs to* $L_2(-\pi, \pi)$ *for all* $n \in \mathbb{N}_0$. *For the real numbers*

$$\begin{aligned} m_k^{-1} \quad &= \quad \sum_{n=1}^{\infty} e_n^2(z_k) + \frac{a_0 e_0^2(z_k)}{\left(\gamma_0 + 2\gamma_1 \cos z_k + 4\gamma_2 \cos^2 z_k\right)^2} \\ &\times \left\{(\gamma_1\beta_0 - \beta_1\gamma_0) + 4\cos z_k\right. \\ &\times \left.\left[(\gamma_2\beta_1 - \gamma_1\beta_2)\cos z_k + (\gamma_2\beta_0 - \beta_2\gamma_0)\right]\right\}, \end{aligned} \tag{3.10}$$

we have the following lemma, where z_k $(k = 1, 2, ..., p)$ *are zeros of the Jost function* $F(z)$ *on the positive imaginary axis.*

Lemma 3.4. *The main equation of* L_λ

$$T(n+N) + A_{n(n+N)} + \sum_{m=1}^{\infty} A_{nm} T(m+n+N) = 0,\ \ n < N \tag{3.11}$$

is satisfied for the kernel A_{nm} *where*

$$T(n) = F_S(n) + \sum_{k=1}^{p} m_k e^{inz_k}. \tag{3.12}$$

Proof. From $e(z)$ and (3.3),

$$\frac{-2i\sin z\psi(z)}{\alpha_n F(z)}+2i\sin(nz) = [1-S(z)]\,e^{inz}+\sum_{m=1}^{\infty}[1-S(z)]\,A_{nm}e^{i(m+n)z} + \sum_{m=1}^{\infty}A_{nm}e^{-i(m+n)z}-\sum_{m=1}^{\infty}A_{nm}e^{i(m+n)z},$$

and then

$$\frac{-1}{\alpha_n 2\pi}\int_{-\pi}^{\pi}\frac{2i\sin z\psi(z)}{F(z)}e^{iNz}dz+\frac{1}{2\pi}\int_{-\pi}^{\pi}2i\sin(nz)\,e^{iNz}dz$$

$$=$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[1-S(z)]\,e^{i(n+N)z}dz+\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}[1-S(z)]\,A_{nm}e^{i(m+n+N)z}dz$$

$$-\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}A_{nm}\left[e^{i(m+n+N)z}-e^{-i(m+n+N)z}\right]dz.$$

Taking into account the Riemann-Lebesgue Theorem and properties of Fourier series related to Delta function, we can say that

$$\frac{-i}{\alpha_n\pi}\int_{-\pi}^{\pi}\frac{\sin z\psi(z)e^{iNz}}{F(z)}dz=F_S(n+N)+A_{n(n+N)}+\sum_{m=1}^{\infty}A_{nm}F_S(m+n+N). \quad (3.13)$$

So, from the Residue Theorem and (3.10), the left side of (3.13) is

$$-\sum_{k=1}^{p}m_k\left[e^{i(n+N)z_k}\left(1+\sum_{m=1}^{\infty}A_{nm}e^{imz_k}\right)\right]$$

because $\psi(z_k)$ and $e(z_k)$ are linearly dependent with

$$\psi_n(z_k)=-\left(\gamma_0+2\gamma_1\cos z_k+4\gamma_2\cos^2 z_k\right)\frac{e_n(z_k)}{e_0(z_k)}$$

since $F(z_k)=0$. Therefore, the main equation (3.11) is obtained from (3.12). □

The main equation can be formed when the function $T(n)$ is known. Conversely, the function $T(n)$ and the unknown kernel A_{nm} can be found through the scattering data set $\{S(z),\ (-\pi\le z\le\pi);\ z_k;\ m_k,\ k=1,2,...,p\}$ and (3.12). The uniqueness of the main equation is considered in the next theorem.

Theorem 3.5. *The main equation (3.11) has a unique solution A_{nm} in $\ell_1(\mathbb{N})$.*

Proof. It needs to be proven that the homogeneous equation

$$V_{n+N}+\sum_{m=1}^{\infty}V_mT(m+n+N)=0 \quad (3.14)$$

has only zero solution in $\ell_1(\mathbb{N})$ for $n<N$.

Let (3.14) has a nonzero solution. So,

$$\sum_{N=1}^{\infty}V_N^2+\sum_{N=1}^{\infty}V_N\sum_{m=1}^{\infty}V_mT(m+N)=0$$

for $n=0$ in (3.14), and

$$0=\sum_{N=1}^{\infty}V_N^2+\sum_{k=1}^{p}m_k\left[\sum_{m=1}^{\infty}V_m e^{imz_k}\right]^2+\frac{1}{2\pi}\int_{-\pi}^{\pi}[1-S(z)]\left[\sum_{m=1}^{\infty}V_m e^{imz}\right]^2 dz$$

is satisfied from (3.9) and (3.12). Also, using Parseval equation of Fourier transformation

$$0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Phi(z)|^2\,dz+\sum_{k=1}^{p}m_k\Phi^2(z_k)+\frac{1}{2\pi}\int_{-\pi}^{\pi}[1-S(z)]\,\Phi^2(z)dz \tag{3.15}$$

can be attained where $\sum_{m=1}^{\infty}V_m^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Phi(z)|^2\,dz$ is Parseval equation of $\Phi(z)=\sum_{m=1}^{\infty}V_m e^{imz}$. Then the real part of the polar form of (3.15) yields

$$\begin{aligned}0 &= \sum_{k=1}^{p}|m_k|\,|\Phi(z_k)|^2\cos[\theta_1(z_k)+2\theta_2(z_k)]\\ &\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Phi(z)|^2+\{1+|1-S(z)|\cos[2\theta_2(z)+\theta_3(z)]\}\,dz,\end{aligned}\tag{3.16}$$

since $\arg m_k=\theta_1(z_k)$, $\arg\Phi(z)=\theta_2(z)$ and $\arg|1-S(z)|=\theta_3(z)$. However, (3.16) is obtained only

$$\Phi(z)=0,\text{ and so } V_m=0.$$

Therefore, the main equation (3.11) has a unique solution. □

4. The Levinson Type Formula of L_λ

Since $\alpha_1=a_0\alpha_0\neq 0$, the equation

$$(\gamma_0+2\gamma_1+4\gamma_2)\,a_0\left(1+\sum_{m=1}^{\infty}A_{1m}\right)+(\beta_0+2\beta_1+4\beta_2)\left(1+\sum_{m=1}^{\infty}A_{0m}\right)=0 \tag{4.1}$$

is satisfied by using (2.4) when $F(0)=0$.

Lemma 4.1. *Let $F(0)=0$. If*

$$\begin{aligned}D_z(q) &= \sum_{N=q}^{\infty}\left[e^{iz}\left(\gamma_0+2\gamma_1\cos z+4\gamma_2\cos^2 z\right)a_0A_{1N}\right.\\ &\quad\left.+\left(\beta_0+2\beta_1\cos z+4\beta_2\cos^2 z\right)A_{0N}\right],\end{aligned}\tag{4.2}$$

then $D_0(q)$ belongs to $\ell_1(\mathbb{N})$ space and it is bounded.

Proof. $D_0(q)$ is in $\ell_1(\mathbb{N})$ providing from (1.4) and (2.2).

If $n=0$ and $n=1$ in the main equation (3.11), then

$$T(1+N)+A_{1(N+1)}+\sum_{m=1}^{\infty}A_{1m}T(1+m+N) = 0, \tag{4.3}$$

$$T(N)+A_{0N}+\sum_{m=1}^{\infty}A_{0m}T(m+N) = 0. \tag{4.4}$$

Also, the equations

$$\begin{aligned} 0 &= e^{iz}\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) a_0 \\ &\times \sum_{N=q-1}^{\infty} \left[T(1+N) + A_{1(N+1)} + \sum_{m=1}^{\infty} A_{1m} T(1+m+N) \right] \end{aligned}$$

and

$$0 = \left(\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z\right) \sum_{N=q}^{\infty} \left[T(N) + A_{0N} + \sum_{m=1}^{\infty} A_{0m} T(m+N) \right]$$

can be written from (4.3) and (4.4). By summing the last two equation,

$$\begin{aligned} 0 &= D_z(q) + \left[e^{iz}\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) a_0 \right. \\ &\left. + \left(\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z\right)\right] \sum_{N=q}^{\infty} T(N) \\ &+ \sum_{m=1}^{\infty} \sum_{N=q}^{\infty} \left[e^{iz}\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) a_0 A_{1m} \right. \\ &\left. + \left(\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z\right) A_{0m} \right] T(m+N). \end{aligned}$$

Additionally, if

$$G(z) = e^{iz}\left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) a_0 + \left(\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z\right) \tag{4.5}$$

where $G(0) = -D_0(1)$, then

$$\begin{aligned} 0 &= D_z(q) + G(z)T(q) + [G(z) + D_z(1)] \sum_{N=1}^{\infty} T(q+N) \\ &- \sum_{N=1}^{\infty} D_z(N+1) T(q+N), \end{aligned}$$

and so.

$$D_0(q) - \sum_{m=1}^{\infty} D_0(m+1) T(q+m) = -G(0) T(q)$$

by using (4.1) and (4.2). Hence, $D_0(q)$ is bounded. □

Theorem 4.2. *The scattering function $S(z)$ is continuous on $(-\pi, \pi)$.*

Proof. It can be easily said that $S(z)$ is continuous for $z \in (-\pi, \pi) \setminus \{0\}$ from Theorem 3.1. Moreover,

$$F(z) = a_0 \alpha_0 K(z)$$

by using (2.4) where

$$\begin{aligned} K(z) &= \left(\beta_0 + 2\beta_1 \cos z + 4\beta_2 \cos^2 z\right) \left(1 + \sum_{m=1}^{\infty} A_{0m} e^{imz}\right) \\ &+ a_0 \left(\gamma_0 + 2\gamma_1 \cos z + 4\gamma_2 \cos^2 z\right) e^{iz} \left(1 + \sum_{m=1}^{\infty} A_{0m} e^{imz}\right) \end{aligned}$$

and $\overline{K(0)} = K(0)$.

Let $F(0) \neq 0$. Then $K(0) \neq 0$ and

$$S(0) = \frac{\overline{F(0)}}{F(0)} = \frac{\alpha_1 \overline{K(0)}}{\alpha_1 K(0)} = 1.$$

Now, suppose that $F(0) = 0$. Considering the definition of $F(z)$, (4.2) and (4.5), it is obtained that

$$F(z) = \alpha_1 \left[G(z) + J_1 \right] \tag{4.6}$$

where

$$\begin{aligned} J_1 &= D_z(1) \sum_{m=1}^{\infty} \left[e^{imz} - e^{i(m-1)z} \right] - \sum_{m=1}^{\infty} D_z(m+1) \left[e^{imz} - e^{i(m-1)z} \right] \\ &= D_z(1) \lim_{k\to\infty} \sum_{m=1}^{k} \left[e^{imz} - e^{i(m-1)z} \right] - \sum_{m=1}^{\infty} D_z(m+1) \left[e^{imz} - e^{i(m-1)z} \right] \\ &= D_z(1) \lim_{k\to\infty} (-1 - e^{ikz}) - \sum_{m=1}^{\infty} D_z(m+1) \left[e^{imz} - e^{i(m-1)z} \right]. \end{aligned} \tag{4.7}$$

Because $G(0) = (\gamma_0 + 2\gamma_1 + 4\gamma_2)\, a_0 + (\beta_0 + 2\beta_1 + 4\beta_2) \neq 0$ and $G(0) = -D_0(1) \in \mathbb{R}$, it can be found that

$$\begin{aligned} S(0) &= \frac{\overline{F(0)}}{F(0)} = \frac{\alpha_1 \left[\overline{G(0)} - 2\overline{D_0(1)} \right]}{\alpha_1 \left[G(0) - 2D_0(1) \right]} \\ &= \frac{3G(0)}{3G(0)} = 1 \end{aligned}$$

by using (4.6) and (4.7). Thus, $S(z)$ is continuous at $z = 0$. □

Let $F(z) = re^{i\eta(z)}$ where $\eta(z) = \arg F(z)$. Therefore $\eta(z)$ is an odd function in $(-\pi, \pi) \setminus \{0\}$, because $re^{-i\eta(z)} = \overline{F(z)} = F(-z) = re^{i\eta(-z)}$, i.e. $\eta(-z) = -\eta(z)$.

Theorem 4.3. *The Levinson type formula of L_λ*

$$\frac{1}{2\pi} \left\{ 2\eta_0(\pi) + \left[\eta_\pi(-\pi) - \eta_\pi(\pi) \right] + \left[\eta_\infty(\pi) - \eta_\infty(-\pi) \right] \right\} + C(\beta_1) = p \tag{4.8}$$

can be written where

$$\begin{aligned} \eta_R(z) &= \eta(z + iR), \\ C(\beta_2) &= \begin{cases} 2 & ; \quad \beta_2 \neq 0 \\ 1 & ; \quad \beta_2 = 0 \end{cases} \end{aligned}$$

and p is the number of zeros of the Jost function $F(z)$ in B_0.

Proof. Let $r_1 > 0$ and $r_2 > 0$ be in the neighborhood of 0 and π, respectively. Also, let $r_1 < r_2 < \pi$ and $r_3 > \pi > r_1 + r_2$. We consider the following path

$$\Gamma^+_{r_1,r_2,r_3} : C_1^- \cup C_2 \cup C_3^- \cup C_4 \cup C_5^+ \cup C_6 \cup C_7^- \cup C_8$$

stated in Figure 4.1, where

C_1^- is half circle with radius r_1 and center 0,
C_2 is line segment oriented the points $(r_1, 0)$ to $(r_2, 0)$,
C_3^- is quarter circle with radius $\pi - r_2$ and center π,
C_4 is line segment oriented the points (π, r_2) to $\left(\pi, \sqrt{r_3^2 - \pi^2}\right)$,
C_5^+ is piece of half circle inside B with radius r_3 and center 0,
C_6 is line segment oriented the points $\left(-\pi, \sqrt{r_3^2 - \pi^2}\right)$ to $(-\pi, r_2)$,
C_7^- is quarter circle with radius $\pi - r_2$ and center $-\pi$,
C_8 is line segment oriented the points $(-r_2, 0)$ to $(-r_1, 0)$.

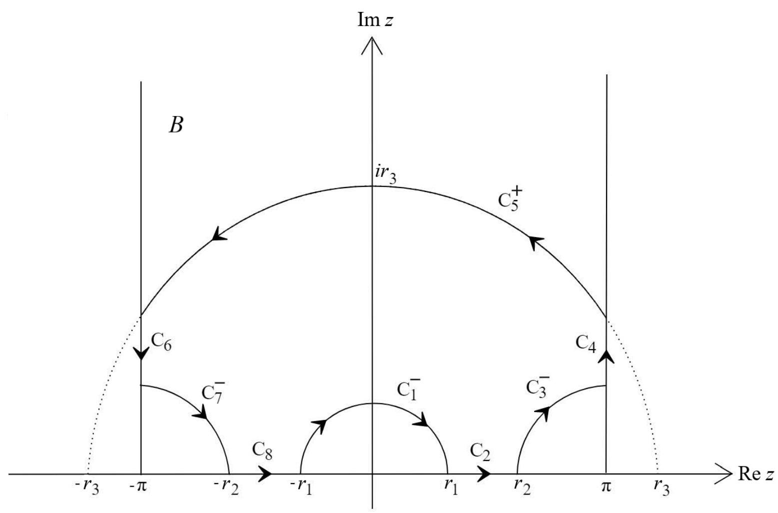

Figure 4.1

The Jost function $F(z)$ is analytic inside the curve $\Gamma^+_{r_1,r_2,r_3}$ and continuous on the boundary of it. Moreover, $\Gamma^+_{r_1,r_2,r_3}$ does not include the zeros of $F(z)$. By using the argument principle, it can be found that

$$\begin{aligned} p &= \frac{1}{2\pi}\Lambda\Gamma^+_{r_1,r_2,r_3} \arg F(z) = \frac{1}{2\pi}\Lambda\Gamma^+_{r_1,r_2,r_3}\eta(z) \\ &= \frac{1}{2\pi}\Lambda C_1^-\eta(z) + \frac{1}{2\pi}\Lambda C_3^-\eta(z) + \frac{1}{2\pi}\Lambda C_5^+\eta(z) + \frac{1}{2\pi}\Lambda C_7^-\eta(z) \\ &\quad + \frac{1}{2\pi}\Lambda\left(C_4 \cup C_6\right)\eta(z) + \frac{1}{2\pi}\Lambda\left(C_2 \cup C_8\right)\eta(z) \end{aligned} \tag{4.9}$$

where $\Lambda\Gamma$ is the argument change on Γ. From (2.5), it is obtained that

$$\arg F(z) = \eta(z) = \begin{cases} -2\operatorname{Re} z & ; \quad \beta_2 \neq 0 \\ -\operatorname{Re} z & ; \quad \beta_2 = 0 \end{cases},$$

and for the last equality,

$$\begin{aligned}
C(\beta_2) &= \frac{1}{2\pi}\lim_{r_3\to\infty}\Lambda C_5^+\eta(z) \\
&= \frac{1}{2\pi}\lim_{r_3\to\infty}\begin{cases} -2\left(-\pi-\pi\right) & ; \quad \beta_2\neq 0 \\ -\left(-\pi-\pi\right) & ; \quad \beta_2 = 0\end{cases} \\
&= \frac{1}{2\pi}\begin{cases} 4\pi \ ; & \beta_2\neq 0 \\ 2\pi \ ; & \beta_2 = 0\end{cases} \\
&= \begin{cases} 2 \ ; & \beta_2\neq 0 \\ 1 \ ; & \beta_2 = 0\end{cases} .
\end{aligned} \tag{4.10}$$

On the other hand, it can be written that

$$F(z)\approx\begin{cases} F(0) & ; \quad F(0)\neq 0 \\ 3\alpha_1\left[\left(\gamma_0+2\gamma_1+4\gamma_2\right)a_0+\left(\beta_0+2\beta_1+4\beta_2\right)\right] & ; \quad F(0)=0\end{cases}$$

from (4.6) and (4.7) for $z\in B$ and $|z|\to 0$. For this reason,

$$\begin{aligned}
\frac{1}{2\pi}\lim_{r_1\to\infty}\Lambda C_1^-\eta(z) &= \begin{cases} 0 \ ; & F(z)\neq 0 \\ 0 \ ; & F(z)=0\end{cases} \\
&= 0.
\end{aligned} \tag{4.11}$$

In addition,

$$F(z)=F(\pi) \quad ; \quad |z|\to\pi$$

and then

$$\frac{1}{2\pi}\lim_{r_2\to\pi}\Lambda\left(C_3^-\cup C_7^-\right)\eta(z)=\frac{1}{2\pi}\left(0+0\right)=0. \tag{4.12}$$

Finally,

$$\begin{aligned}
\frac{1}{2\pi}\lim_{\substack{r_2\to\pi\\ r_3\to\infty}}\Lambda\left(C_4\cup C_6\right)\eta(z) &= \frac{1}{2\pi}\lim_{\substack{r_2\to\pi\\ r_3\to\infty}}\left\{\left[\eta\left(\pi+i\sqrt{r_3^2-\pi^2}\right)-\eta(\pi+ir_2)\right]\right. \\
&\quad \left.+\left[\eta(-\pi+ir_2)-\eta\left(-\pi+i\sqrt{r_3^2-\pi^2}\right)\right]\right\} \\
&= \frac{1}{2\pi}\left\{\left[\eta\left(\pi+i\infty\right)-\eta\left(-\pi+i\infty\right)\right]\right. \\
&\quad \left.+\left[\eta(\left(-1+i\right)\pi)-\eta(\left(1+i\right)\pi)\right]\right\} \\
&= \frac{1}{2\pi}\left\{\left[\eta_\infty(\pi)-\eta_\infty(-\pi)\right]\right. \\
&\quad \left.+\left[\eta_\pi(-\pi)-\eta_\pi(\pi)\right]\right\}
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
\frac{1}{2\pi}\lim_{\substack{r_1\to 0\\ r_2\to\pi}}\Lambda\left(C_8\cup C_2\right)\eta(z) &= \frac{1}{2\pi}\lim_{\substack{r_1\to 0\\ r_2\to\pi}}\left\{\left[\eta(-r_1)-\eta(-r_2)\right]+\left[\eta(r_2)-\eta(r_1)\right]\right\} \\
&= \frac{1}{\pi}\eta(\pi) \\
&= \frac{1}{\pi}\eta_0(\pi)
\end{aligned} \tag{4.14}$$

because $\eta(-\pi)=-\eta(\pi)$. Thus, we reach to (4.8) from (4.9)-(4.14). □

Let $a_0 \neq c_4 \frac{(2\beta_1-\beta_0-4\beta_2)}{(\gamma_0-2\gamma_1+4\gamma_2)}$ and

$$S(\pi)=\begin{cases} 1 & ; \quad F(\pi)\neq 0 \\ \frac{\overline{F'(\pi)}}{F'(\pi)} & ; \quad F(\pi)=0 \end{cases}$$

where $c_4 = \frac{1+\sum_{m=1}^{\infty}(1-m)A_{0m}(-1)^m}{\sum_{m=1}^{\infty} mA_{1m}(-1)^m}$ and $S(\pi) = S(-\pi)$ since $F(\pi) = F(-\pi)$. Therefore, the scattering function $S(z)$ is also continuous in $\{-\pi, \pi\}$ and the following corollary can be obtained:

Corollary 4.4. *Under the condition (2.6),*

$$\frac{1}{4\pi i}\left\{2S_0(\pi)+[S_\pi(-\pi)-S_\pi(\pi)]+[S_\infty(\pi)-S_\infty(-\pi)]\right\}=C(\beta_2)-p \qquad (4.15)$$

is another representation of the Levinson type formula of L_λ where

$$S_R(z) = \ln S(z+iR).$$

Proof. From the definition of $F(z)$, it is clear that

$$S(z)=\frac{\overline{F(z)}}{F(z)}=\frac{re^{-i\eta(z)}}{re^{i\eta(z)}}=e^{-2i\eta(z)}.$$

So,

$$\eta(z) = -\frac{\ln S(z)}{2i} \qquad (4.16)$$

and then (4.15) is obtained by using Theorem 4.3 and (4.16). □

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Conflict of Interest

This work does not have any conflicts of interest.

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