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DECOMPOSING A FIXED POINT PROBLEM INTO MULTIPLE FIXED POINT PROBLEMS

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ABSTRACT. We decompose an operator associated to a right focal boundary value problem, whose fixed points are solutions of the boundary value problem, into multiple fixed point problems. We provide conditions for the original boundary value problem to have a solution that can be found by iteration using the decomposition.

1. Introduction

A standard approach to showing the existence of solutions to boundary value problems, and iterating to find solutions of boundary value problems, is to convert the boundary value problem to a fixed point problem. Consider the second order right focal boundary value problem given by

$$y''(t)+g(y(t))=0, \quad t\in(0,1), \tag{1}$$

$$y(0)=y'(1)=0, \tag{2}$$

where $g:[0,\infty)\to[0,\infty)$ is differentiable. The Green's function for (1), (2) is given by

$$G(t,s)=\min\{t,s\};$$

and every solution of (1), (2) is a fixed point of the operator $H:C[0,1]\to C^2[0,1]$ defined by

$$Hy(t)=\int_0^1 G(t,s)g(y(s))\,ds, \tag{3}$$

where the norm $||\cdot||$ on $C[0,1]$ is the usual supremum norm. There are many different results in the literature giving conditions and techniques to verify the existence of solutions as well as iterative techniques for the right focal boundary value problem (1), (2). See [1, 2, 3, 4, 8, 9] for some interesting approaches and techniques that are currently in the literature. Converting the operator fixed point problem to a real valued fixed point problem is significanlty different than any of the arguments currently in the literature. If we let

$$P=\{y\in C[0,1]\ :\ y(0)=0 \text{ and } y \text{ is non-decreasing}\},$$

then it is a trivial exercise to verify that $H:P\to P$, and that verification of the existence of solutions, or the finding and iterating to solutions of the boundary value problem (1), (2), has been converted to

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finding fixed points of the operator H since for any $y \in P$ and $t \in (0,1)$,

$$(Hy)'(t) = \int_t^1 g(y(s))\,ds,$$
$$(Hy)''(t) = -g(y(t)),$$

and

$$(Hy)(0) = 0 = (Hy)'(1).$$

The operator H is a completely continuous operator, thus if there is an $R > 0$ with

$$P_R = \{y \in P \;:\; \|y\| \le R\} \tag{4}$$

such that

$$H : P_R \to P_R,$$

then H has a fixed point in P_R by Schauder's Fixed Point Theorem [12].

Lemma 1. *Let $R \in \mathbb{R}$. If $g : [0,R] \to [0,2R]$, then*

$$H : P_R \to P_R,$$

and H has a fixed point in P_R which is a solution of (1), (2).

Proof. Letting $y \in P_R$, where P_R is given in (4), it follows that

$$\begin{aligned} \|Hy\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)g(y(s))\,ds \right| \\ &= \int_0^1 G(1,s)g(y(s))\,ds \\ &= \int_0^1 s g(y(s))\,ds \\ &\le 2R\int_0^1 s\,ds = R. \end{aligned}$$

Therefore $H : P_R \to P_R$ and P_R is a closed, convex subset of the Banach space of $E = C[0,1]$ with the sup norm, hence by Schauder's fixed point theorem (see [13] for a modern statement and proof of this classical result), H has a fixed point in P_R. Furthermore, since any fixed point of H is a solution of (1), (2), we have verified the existence of at least one solution in P_R. □

One can look at alternative types of sets in which the operator H is invariant, such as the Leggett-Williams [11] functional wedges using concave and convex functionals to have less restrictive conditions in showing existence of solutions to boundary value problems or as is the purpose of this manuscript to develop an iterative scheme converging to a solution. There are many types of existence of solutions arguments, however there is a limited collection of iterative techniques which converge to actual solutions. In this paper we will outline a new iterative technique converting a boundary value problem into a fixed point of a real valued function problem. Functional wedges are the foundation of Leggett-Williams [11] arguments. The beauty of the Leggett-Williams arguments is in showing that there is a fixed point in the underlying set even though the operator is not necessarily invariant on this

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set, but in our argument we need the operator to be invariant on the functional wedge so we can verify that our sequence of iterates remains in the underlying set. For $y \in P$ let

$$\alpha(y) = \min_{t\in[\frac{1}{4},1]} |y(t)| = y\left(\frac{1}{4}\right), \tag{5}$$

and for $0 < r < R$ define the functional wedge $P(\alpha, r, R)$ by

$$P(\alpha, r, R) = \{y \in P \,:\, r \le \alpha(y) \text{ and } \|y\| \le R\}, \tag{6}$$

which is a closed, convex subset of P.

Lemma 2. *Let $r, R \in \mathbb{R}$ with $0 < r < \frac{3R}{8}$, and suppose*

$$g : [0, R] \to [0, 2R] \text{ with } g(y) \ge \frac{16r}{3} \text{ for } y \in [r, R].$$

Then

$$H : P(\alpha, r, R) \to P(\alpha, r, R),$$

and H has a fixed point in $P(\alpha, r, R)$ which is a solution of (1), (2), for $P(\alpha, r, R)$ given in (6).

Proof. Given $R > 0$, we must have $0 < r < \frac{3R}{8}$ under the assumption that $\frac{16r}{3} \le g(y) \le 2R$. Let $y \in P(\alpha, r, R)$ as defined in (6). Thus by Lemma 1 we know $Hy \in P_R$. Since Hy is non-decreasing, and using (5), we have

$$\begin{aligned}
\alpha(Hy) &= \min_{t\in[\frac{1}{4},1]} \left| \int_0^1 G(t,s) g(y(s))\, ds \right| \\
&= \int_0^1 G\left(\frac{1}{4}, s\right) g(y(s))\, ds \\
&\ge \int_{\frac{1}{4}}^1 G\left(\frac{1}{4}, s\right) g(y(s))\, ds \\
&= \left(\frac{1}{4}\right) \int_{\frac{1}{4}}^1 g(y(s))\, ds \\
&\ge \left(\frac{1}{4}\right) \int_{\frac{1}{4}}^1 \left(\frac{16r}{3}\right) ds \\
&= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{16r}{3}\right) = r.
\end{aligned}$$

Therefore $H : P(\alpha, r, R) \to P(\alpha, r, R)$, and $P(\alpha, r, R)$ is a closed, convex subset of the Banach space $E = C[0,1]$ with the sup norm. Hence by Schauder's fixed point theorem, H has a fixed point in $P(\alpha, r, R)$, and since any fixed point of H is a solution of (1), (2), we have verified the existence of at least one solution in $P(\alpha, r, R)$. □

Note that one may want to define the concave functional α on a different interval which would lead to different bounds that the nonlinear function g would need to meet in order to be able to apply the main techniques that follow.

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2. Preliminaries

For $r, R \in \mathbb{R}$ with $0 < r < \frac{3R}{8}$, let

$$Q = \left\{ y \in C\left[\frac{1}{4}, 1\right] \;:\; y \text{ is non-negative and non-decreasing} \right\},$$

which is a cone in the Banach space $B_u = C\left[\frac{1}{4}, 1\right]$ with the sup norm, that is, for $y \in B_u$ let

$$\|y\|_u = \max_{t \in \left[\frac{1}{4}, 1\right]} |y(t)|.$$

Furthermore, let

$$S = \left\{ y \in C\left[0, \frac{1}{4}\right] \;:\; y \text{ is non-negative, non-decreasing and } y(0) = 0 \right\},$$

which is a cone in the Banach space $B_v = C\left[0, \frac{1}{4}\right]$ with the sup norm, that is, for $y \in B_v$ let

$$\|y\|_v = \max_{t \in \left[0, \frac{1}{4}\right]} |y(t)|.$$

Let

$$Q[r,R] = \left\{ y \in Q \;:\; r \le y(t) \le R \text{ for all } t \in \left[\frac{1}{4}, 1\right] \right\}$$

and

$$S_R = \left\{ y \in S \;:\; y(t) \le R \text{ for all } t \in \left[0, \frac{1}{4}\right] \right\}.$$

Our decomposition will involve operators $A_l : S \to S$ defined by

$$A_l y(t) = \int_0^{\frac{1}{4}} G(t,s) g(y(s))\, ds + tl \tag{7}$$

for each non-negative real number l, and operators $D_m : Q \to Q$ defined by

$$D_m y(t) = m + \int_{\frac{1}{4}}^{1} G(t,s) g(y(s))\, ds \tag{8}$$

for each non-negative real number m.

Lemma 3. *Let $R \in \mathbb{R}$, $l \in \left[0, \frac{3R}{2}\right]$, $g : [0,R] \to [0,2R]$ be differentiable, $a_{l,0} \equiv 0$, and define the recursive sequence*

$$a_{l,n+1} = A_l a_{l,n}$$

for A_l given in (7). If $\tau \in (0,32)$ such that

$$|g'(a)| \le \tau < 32$$

for all $a \in [0,R]$, then

$$a_{l,n} \to a_{l*} \in S_R.$$

Moreover,

$$a_{l*} = A_l a_{l*}$$

and

$$a''_{l*}(t) = -g(a_{l*}(t))$$

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for all $t \in \left(0, \frac{1}{4}\right)$, with $a_{l*}(0) = 0$. Furthermore, for $k_a = \frac{\tau}{32}$ we have that

$$\|a_{l*} - a_{l,n}\|_v \leq \left(\frac{k_a^n}{1-k_a}\right)\|a_{l,1} - a_{l,0}\|_v \leq \frac{Rk_a^n}{1-k_a}.$$

Proof. Let $y \in S_R$ and $l \in \left[0, \frac{3R}{2}\right]$, following a similar argument as in Lemma 1, we have

$$\begin{aligned}\|A_l(y)\|_v &= \max_{t\in[0,\frac{1}{4}]}\left|\int_0^{\frac{1}{4}} G(t,s)g(y(s))\,ds + lt\right| \\ &= \int_0^{\frac{1}{4}} G\left(\frac{1}{4},s\right)g(y(s))\,ds + \frac{l}{4} \\ &\leq \int_0^{\frac{1}{4}} 2Rs\,ds + \frac{3R}{8} = \frac{7R}{16}\end{aligned}$$

thus $A_l : S_R \to S_R$. Let $y,z \in S_R$ thus for each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $y(s)$ and $z(s)$ such that

$$g(y(s)) - g(z(s)) = g'(w(s))(y(s) - z(s))$$

by the mean value theorem (note that we assumed that g was a differentiable function). Hence

$$\begin{aligned}\|A_l y - A_l z\|_v &= \max_{t\in[0,\frac{1}{4}]}\left|\int_0^{\frac{1}{4}} G(t,s)g(y(s))\,ds + lt - \int_0^{\frac{1}{4}} G(t,s)g(z(s))\,ds - lt\right| \\ &\leq \max_{t\in[0,\frac{1}{4}]}\int_0^{\frac{1}{4}} G(t,s)\left|g(y(s)) - g(z(s))\right|\,ds \\ &\leq \int_0^{\frac{1}{4}} s\left|g'(w(s))(y(s)-z(s))\right|\,ds \\ &\leq \tau\int_0^{\frac{1}{4}} s\|y-z\|_v\,ds = \frac{\tau\|y-z\|_v}{32},\end{aligned}$$

thus $A_l : S_R \to S_R$ is contractive with constant $k_a = \frac{\tau}{32} < 1$. Let $a_{l,0} \equiv 0$, and define the recursive sequence

$$a_{l,n+1} = A_l a_{l,n}.$$

We have that $\{a_{l,k}\}_{k=0}^{\infty} \subset S_R$ since $A_l : S_R \to S_R$. Since A_l is contractive on S_R, by the Banach Fixed Point Theorem [5] there is a unique $a_{l*} \in S_R$ such that $a_{l,n} \to a_{l*}$. Note that we are technically applying Banachs Corollary of the Banach Contraction Principle, see Granas-Dugundji [6] for details concerning the corollary and see [13] for a modern, unified treatment of the Banach Contraction Principle with its corollary embedded in the statement of the principle. Thus

$$a_{l*}(t) = \int_0^{\frac{1}{4}} G(t,s)g(a_{l*}(s))\,ds + tl,\ t \in \left[0, \frac{1}{4}\right].$$

Clearly

$$a_{l*}(0) = 0$$

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since $G(0,s)=0$ for all $s\in\left[0,\frac{1}{4}\right]$, and for $t\in\left(0,\frac{1}{4}\right)$ we have

$$a'_{l*}(t)=\int_t^{\frac{1}{4}} g(a_{l*}(s))\,ds+l \tag{9}$$

and

$$a''_{l*}(t)=-g(a_{l*}(t)).$$

Also, for any natural numbers n and j by mathematical induction we have

$$\|a_{l,n+j+1}-a_{l,n+j}\|_v\le k_a\|a_{l,n+j}-a_{l,n+j-1}\|_v\le\cdots\le k_a^j\|a_{l,n+1}-a_{l,n}\|_v$$

hence, for any natural numbers n and p, applying the triangle inequality, we have

$$\begin{aligned}
\|a_{l,n+p}-a_{l,n}\|_v &\le \sum_{j=0}^{p-1}\|a_{l,n+j+1}-a_{l,n+j}\|_v\\
&\le \sum_{j=0}^{p-1}k_a^j\|a_{l,n+1}-a_{l,n}\|_v\\
&\le \sum_{j=0}^{\infty}k_a^j\|a_{l,n+1}-a_{l,n}\|_v\\
&= \left(\frac{1}{1-k_a}\right)\|a_{l,n+1}-a_{l,n}\|_v\\
&\le \left(\frac{k_a^n}{1-k_a}\right)\|a_{l,1}-a_{l,0}\|_v.
\end{aligned}$$

Hence letting $p\to\infty$ we have that

$$\|a_{l*}-a_{l,n}\|_v\le\left(\frac{k_a^n}{1-k_a}\right)\|a_{l,1}-a_{l,0}\|_v\le\frac{Rk_a^n}{1-k_a}.$$

This ends the proof. □

Lemma 4. *Let $r,R\in\mathbb{R}$ with $0<r<\frac{3R}{8}$, $m\in\left[0,\frac{R}{16}\right]$, $g:[0,R]\to[0,2R]$ be differentiable, $\frac{16r}{3}\le g(y)$ for all $y\in[r,R]$, $b_{m,0}\equiv r$, and define the recursive sequence*

$$b_{m,n+1}=D_m b_{m,n}$$

for D_m given in (8). If $\mu\in\left(0,\frac{32}{15}\right)$ such that

$$|g'(b)|\le\mu<\frac{32}{15}$$

for all $b\in[r,R]$, then

$$b_{m,n}\to b_{m*}\in Q[r,R].$$

Moreover,

$$b_{m*}=D_m b_{m*}$$

and

$$b''_{m*}(t) = -g(b_{m*}(t))$$

for all $t \in \left(\frac{1}{4}, 1\right)$ *and* $b'_{m*}(1) = 0$*. Furthermore, for* $k_b = \frac{15\mu}{32}$ *we have that*

$$\|b_{m*} - b_{m,n}\|_u \leq \left(\frac{k_b^n}{1-k_b}\right) \|b_{m,1} - b_{m,0}\|_u \leq \frac{Rk_b^n}{1-k_b}.$$

Proof. Let $y \in Q[r,R]$ and $m \in \left[0, \frac{R}{16}\right]$, thus following a similar argument as in Lemma 2, we have

$$\begin{aligned}
\alpha(D_m y) &= \min_{t\in[\frac{1}{4},1]} \left| m + \int_{\frac{1}{4}}^{1} G(t,s) g(y(s))\, ds \right| \\
&= m + \int_{\frac{1}{4}}^{1} G\left(\frac{1}{4}, s\right) g(y(s))\, ds \\
&= m + \left(\frac{1}{4}\right) \int_{\frac{1}{4}}^{1} g(y(s))\, ds \\
&\geq m + \left(\frac{1}{4}\right) \int_{\frac{1}{4}}^{1} \left(\frac{16r}{3}\right) ds \\
&= m + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{16r}{3}\right) = m + r > r,
\end{aligned}$$

and following a similar argument as in Lemma 1 we have

$$\begin{aligned}
\|D_m y\|_u &= \max_{t\in[\frac{1}{4},1]} \left| m + \int_{\frac{1}{4}}^{1} G(t,s) g(y(s))\, ds \right| \\
&= m + \int_{\frac{1}{4}}^{1} G(1,s)\, g(y(s))\, ds \\
&= m + \int_{\frac{1}{4}}^{1} s g(y(s))\, ds \\
&\leq m + \int_{\frac{1}{4}}^{1} 2Rs\, ds \\
&= m + \frac{15R}{16} \leq R
\end{aligned}$$

thus $D_m : Q[r,R] \to Q[r,R]$. Let $y, w \in Q[r,R]$, for each $s \in \left[\frac{1}{4}, 1\right]$, let $w(s)$ be between $y(s)$ and $z(s)$ such that

$$g(y(s)) - g(w(s)) = g'(z(s))(y(s) - w(s))$$

by the mean value theorem. Hence

$$\begin{aligned}\|D_m y - D_m z\|_u &= \max_{t\in[\frac{1}{4},1]} \left|\int_{\frac{1}{4}}^{1} G(t,s)g(y(s))\,ds - \int_{\frac{1}{4}}^{1} G(t,s)g(z(s))\,ds\right| \\ &\le \max_{t\in[\frac{1}{4},1]} \int_{\frac{1}{4}}^{1} G(t,s)\left|g(y(s)) - g(z(s))\right|\,ds \\ &\le \int_{\frac{1}{4}}^{1} s\left|g'(w(s))(y(s)-z(s))\right|\,ds \\ &\le \mu\int_{\frac{1}{4}}^{1} s\|y-z\|_u\,ds = \frac{15\mu\|y-z\|_u}{32},\end{aligned}$$

thus $D_m : Q[r,R] \to Q[r,R]$ is contractive with constant $k_b = \frac{15\mu}{32} < 1$. Let $b_{m,0} \equiv r$, and define the recursive sequence

$$b_{m,n+1} = D_m b_{m,n}.$$

We have that $\{b_{m,n}\}_{n=0}^{\infty} \subset Q[r,R]$ since $D_m : Q[r,R] \to Q[r,R]$. Since D_m is contractive on $Q[r,R]$, by the Banach Fixed Point Theorem [5] there is a unique $b_{m*} \in Q[r,R]$ such that $b_{m,n} \to b_{m*}$. Thus

$$b_{m*}(t) = m + \int_{\frac{1}{4}}^{1} G(t,s)g(b_{m*}(s))\,ds,\ t \in \left[\frac{1}{4},1\right].$$

Since

$$b'_{m*}(t) = \int_{t}^{1} g(b_{m*}(s))\,ds \tag{10}$$

clearly $b'_{m*}(1) = 0$ and

$$b''_{m*}(t) = -g(b_{m*}(t)).$$

Just like in Lemma 3, for any natural numbers n and j by mathematical induction we have

$$\|b_{m,n+j+1} - b_{m,n+j}\|_u \le k_b\|b_{m,n+j} - b_{m,n+j-1}\|_u \le \cdots \le k_b^j\|b_{m,n+1} - b_{m,n}\|_u$$

hence, for any natural numbers n and p, applying the triangle inequality, we have

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$$\begin{aligned}
\|b_{m,n+p}-b_{m,n}\|_u &\le \sum_{j=0}^{p-1}\|b_{m,n+j+1}-b_{m,n+j}\|_u\\
&\le \sum_{j=0}^{p-1}k_b^j\|b_{m,n+1}-b_{m,n}\|_u\\
&\le \sum_{j=0}^{\infty}k_b^j\|b_{m,n+1}-b_{m,n}\|_u\\
&= \left(\frac{1}{1-k_b}\right)\|b_{m,n+1}-b_{m,n}\|_u\\
&\le \left(\frac{k_b^n}{1-k_b}\right)\|b_{m,1}-b_{m,0}\|_u.
\end{aligned}$$

Hence letting $p\to\infty$ we have that

$$\|b_{m*}-b_{m,n}\|_u \le \left(\frac{k_b^n}{1-k_b}\right)\|b_{m,1}-b_{m,0}\|_u \le \frac{Rk_b^n}{1-k_b}.$$

This ends the proof. □

For $l\in\left[0,\frac{3R}{2}\right]$ and a natural number p let

$$m_l=\int_0^{\frac14}G\left(\frac14,s\right)g(a_{l*}(s))\,ds=\int_0^{\frac14}sg(a_{l*}(s))\,ds,$$

$$m_{l,p}=\int_0^{\frac14}G\left(\frac14,s\right)g(a_{l,p}(s))\,ds=\int_0^{\frac14}sg(a_{l,p}(s))\,ds,$$

and define the real valued function h by

(11) $$h(l)=\int_{\frac14}^1 g(b_{m_l*}(s))\,ds.$$

Note that m_l is a quantity that is the result of a limiting process, whereas $m_{l,p}$ is a real number that can be calculated through iteration. In the following lemma we provide a bound on $\|b_{m_l*}-b_{m_{l,p}*}\|_u$ which is one of the error bounds we will need to calculate a bound on the error of our approximate solution of our boundary value problem. In Theorem 4 we will need to approximate $h(l)$ by

$$\int_{\frac14}^1 g(b_{m_{l,p}*}(s))\,ds$$

so we will define the function

(12) $$h(l,p)=\int_{\frac14}^1 g(b_{m_{l,p}*}(s))\,ds.$$

Lemma 5. *Let $r,R\in\mathbb{R}$ with $0<r<\frac{3R}{8}$, $m_l\in\left[0,\frac{R}{16}\right]$, $\mu\in\left(0,\frac{32}{15}\right)$, and $\tau\in(0,32)$ such that*

(A1) $g:[0,R]\to[0,2R]$ is differentiable;

(A2) $\frac{16r}{3} \leq g(y)$ *for all* $y \in [r,R]$*;*
(A3) $|g'(a)| \leq \tau < 32$ *for all* $a \in [0,r]$*;*
(A4) $|g'(b)| \leq \mu < \frac{32}{15}$ *for all* $b \in [r,R]$.

For $k_a = \frac{\tau}{32}$ *and a natural number* p*,*

$$\|b_{m_l*} - b_{m_{l,p}*}\|_u \leq \frac{\tau R k_a^p}{(32-15\mu)(1-k_a)}.$$

Proof. For each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $a_{l*}(s)$ and $a_{l,p}(s)$ such that

$$g(a_{l*}(s)) - g(a_{l,p}(s)) = g'(w(s))(a_{l*}(s) - a_{l,p}(s))$$

by the mean value theorem, thus from Lemma 3 we have

$$\begin{aligned}
|m_l - m_{l,p}| &= \left| \int_0^{\frac{1}{4}} s g(a_{l*}(s))\, ds - \int_0^{\frac{1}{4}} s g(a_{l,p}(s))\, ds \right| \\
&\leq \int_0^{\frac{1}{4}} s \left| g(a_{l*}(s)) - g(a_{l,p}(s)) \right|\, ds \\
&\leq \int_0^{\frac{1}{4}} s \left| g'(w(s))(a_{l*}(s) - a_{l,p}(s)) \right|\, ds \\
&\leq \tau \int_0^{\frac{1}{4}} s \|a_{l*} - a_{l,p}\|_v\, ds \\
&= \frac{\tau \|a_{l*} - a_{l,p}\|_v}{32} \\
&\leq \frac{\tau R k_a^p}{32(1-k_a)}.
\end{aligned}$$

By Lemma 4 there exist $b_{m_l*}, b_{m_{l,p}*} \in Q[r,R]$ such that

$$b_{m_l*} = D_{m_l} b_{m_l*} \quad \text{and} \quad b_{m_{l,p}*} = D_{m_{l,p}} b_{m_{l,p}*}.$$

For each $s \in \left[\frac{1}{4}, 1\right]$, let $z(s)$ be between $b_{m_l*}(s)$ and $b_{m_{l,p}*}(s)$ such that

$$g(b_{m_l*}(s)) - g(b_{m_{l,p}*}(s)) = g'(z(s))(b_{m_l*}(s) - b_{m_{l,p}*}(s))$$

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by the mean value theorem, hence

$$\begin{aligned}\|b_{m_l*}-b_{m_{l,p}*}\|_u &= \max_{t\in[\frac{1}{4},1]}\left|m_l+\int_{\frac{1}{4}}^{1}G(t,s)g(b_{m_l*}(s))\,ds-m_{l,p}-\int_{\frac{1}{4}}^{1}G(t,s)g(b_{m_{l,p}*}(s))\,ds\right|\\ &\leq |m_l-m_{l,p}|+\max_{t\in[\frac{1}{4},1]}\int_{\frac{1}{4}}^{1}G(t,s)\left|g(b_{m_l*}(s))-g(b_{m_{l,p}*}(s))\right|\,ds\\ &\leq |m_l-m_{l,p}|+\int_{\frac{1}{4}}^{1}s\left|g'(z(s))(b_{m_l*}(s)-b_{m_{l,p}*}(s))\right|\,ds\\ &\leq |m_l-m_{l,p}|+\mu\int_{\frac{1}{4}}^{1}s\|b_{m_l*}-b_{m_{l,p}*}\|_u\,ds\\ &= |m_l-m_{l,p}|+\frac{15\mu\|b_{m_l*}-b_{m_{l,p}*}\|_u}{32}\\ &\leq \frac{\tau Rk_a^p}{32(1-k_a)}+\frac{15\mu\|b_{m_l*}-b_{m_{l,p}*}\|_u}{32}.\end{aligned}$$

Therefore

$$\|b_{m_l*}-b_{m_{l,p}*}\|_u\leq\frac{\tau Rk_a^p}{(32-15\mu)(1-k_a)}.$$

This ends the proof. □

In what follows we convert an operator fixed point problem into a real valued function fixed point problem.

Theorem 1. *If $\theta\in\left[0,\frac{3R}{2}\right]$ and $\theta=h(\theta)$, then*

$$y_*(t)=\begin{cases}a_{\theta*}(t) & 0\leq t\leq\frac{1}{4}\\ b_{m_\theta*}(t) & \frac{1}{4}\leq t\leq 1\end{cases}$$

is a solution of (1), (2).

Proof. Since

$$\theta=h(\theta)=\int_{\frac{1}{4}}^{1}g(b_{m_\theta*}(s))\,ds$$

and

$$m_\theta=\int_0^{\frac{1}{4}}G\left(\frac{1}{4},s\right)g(a_{\theta*}(s))\,ds=\int_0^{\frac{1}{4}}sg(a_{\theta*}(s))\,ds,$$

we have that

$$
\begin{aligned}
y_*(t) &= \begin{cases} \int_0^{\frac{1}{4}} G(t,s)g(a_{\theta*}(s))\,ds + t\theta & 0 \le t \le \frac{1}{4} \\ m_\theta + \int_{\frac{1}{4}}^1 G(t,s)g(b_{m_\theta *}(s))\,ds & \frac{1}{4} \le t \le 1 \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} G(t,s)g(a_{\theta*}(s))\,ds + t\int_{\frac{1}{4}}^1 g(b_{m_\theta *}(s))\,ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{4}} G(\frac{1}{4},s)g(a_{\theta*}(s)) + \int_{\frac{1}{4}}^1 G(t,s)g(b_{m_\theta *}(s))\,ds & \frac{1}{4} \le t \le 1 \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} G(t,s)g(y_*(s))\,ds + \int_{\frac{1}{4}}^1 G(t,s)g(y_*(s))\,ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{4}} G(t,s)g(y_*(s)) + \int_{\frac{1}{4}}^1 G(t,s)g(y_*(s))\,ds & \frac{1}{4} \le t \le 1 \end{cases} \\
&= \begin{cases} \int_0^1 G(t,s)g(y_*(s))\,ds & 0 \le t \le \frac{1}{4} \\ \int_0^1 G(t,s)g(y_*(s))\,ds & \frac{1}{4} \le t \le 1 \end{cases} \\
&= Hy_*(t).
\end{aligned}
$$

Therefore y_* is a fixed point of the operator H and thus a solution of the boundary value problem (1), (2). This ends the proof. □

3. Main results

Now that we have converted our operator fixed point problem into a real valued fixed point problem we need to show that our real valued fixed point problem is going to have a fixed point and the first step to showing that is to show that the function h is uniformly continuous so we can apply the intermediate value theorem and a bisection method.

Lemma 6. *Let $r, R \in \mathbb{R}$ with $0 < r < \frac{3R}{8}$, $\tau \in (0,32)$, $\mu \in \left(0, \frac{32}{15}\right)$, and suppose that*

- $(A1)$ *$g : [0,R] \to [0,2R]$ is differentiable;*
- $(A2)$ *$\frac{16r}{3} \le g(y)$ for all $y \in [r,R]$;*
- $(A3)$ *$|g'(a)| \le \tau < 32$ for all $a \in [0,r]$;*
- $(A4)$ *$|g'(b)| \le \mu < \frac{32}{15}$ for all $b \in [r,R]$.*

Then the function h given in (11) is uniformly continuous on $\left[0, \frac{3R}{2}\right]$.

Proof. If we let $l, j \in \left[0, \frac{3R}{2}\right]$, then by Lemma 3 there exist $a_{l*}, a_{j*} \in S_R$ such that

$$a_{l*} = A_l a_{l*} \quad \text{and} \quad a_{j*} = A_j a_{j*}.$$

For each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $a_{l*}(s)$ and $a_{j*}(s)$ such that

$$g(a_{l*}(s)) - g(a_{j*}(s)) = g'(w(s))(a_{l*}(s) - a_{j*}(s))$$

by the mean value theorem, thus

$$\begin{aligned}
\|a_{l*}-a_{j*}\|_v &= \max_{t\in[0,\frac{1}{4}]}\left|\int_0^{\frac{1}{4}} G(t,s)g(a_{l*}(s))\,ds+tl-\int_0^{\frac{1}{4}} G(t,s)g(a_{j*}(s))\,ds-tj\right| \\
&\leq \max_{t\in[0,\frac{1}{4}]}\int_0^{\frac{1}{4}} G(t,s)\left|g(a_{l*}(s))-g(a_{j*}(s))\right|\,ds+\frac{|l-j|}{4} \\
&\leq \int_0^{\frac{1}{4}} s\left|g'(w(s))(a_{l*}(s)-a_{j*}(s))\right|\,ds+\frac{|l-j|}{4} \\
&\leq \tau\int_0^{\frac{1}{4}} s\|a_{l*}-a_{j*}\|_v\,ds+\frac{|l-j|}{4} \\
&= \frac{\tau\|a_{l*}-a_{j*}\|_v}{32}+\frac{|l-j|}{4}.
\end{aligned}$$

Therefore

$$\|a_{l*}-a_{j*}\|_v\leq\frac{8|l-j|}{32-\tau},$$

and for

$$m_l=\int_0^{\frac{1}{4}} sg(a_{l*}(s))\,ds \quad\text{and}\quad m_j=\int_0^{\frac{1}{4}} sg(a_{j*}(s))\,ds$$

we have

$$\begin{aligned}
|m_l-m_j| &= \left|\int_0^{\frac{1}{4}} sg(a_{l*}(s))\,ds-\int_0^{\frac{1}{4}} sg(a_{j*}(s))\,ds\right| \\
&\leq \int_0^{\frac{1}{4}} s\left|g(a_{l*}(s))-g(a_{j*}(s))\right|\,ds \\
&\leq \int_0^{\frac{1}{4}} s\left|g'(w(s))(a_{l*}(s)-a_{j*}(s))\right|\,ds \\
&\leq \tau\int_0^{\frac{1}{4}} s\|a_{l*}-a_{j*}\|_v\,ds \\
&= \frac{\tau\|a_{l*}-a_{j*}\|_v}{32} \\
&\leq \frac{\tau|l-j|}{4(32-\tau)}.
\end{aligned}$$

By Lemma 4 there exist $b_{m_l*},b_{m_j*}\in Q[r,R]$ such that

$$b_{m_l*}=D_{m_l}b_{m_l*}\quad\text{and}\quad b_{m_j*}=D_{m_j}b_{m_j*}.$$

For each $s\in\left[\frac{1}{4},1\right]$, let $z(s)$ be between $b_{m_l*}(s)$ and $b_{m_j*}(s)$ such that

$$g(b_{m_l*}(s))-g(b_{m_j*}(s))=g'(z(s))(b_{m_l*}(s)-b_{m_j*}(s))$$

by the mean value theorem, hence

$$\begin{aligned}
\|b_{m_l*}-b_{m_j*}\|_u &= \max_{t\in[\frac14,1]}\left|m_l+\int_{\frac14}^1 G(t,s)g(b_{m_l*}(s))\,ds-m_j-\int_{\frac14}^1 G(t,s)g(b_{m_j*}(s))\,ds\right| \\
&\le |m_l-m_j|+\max_{t\in[\frac14,1]}\int_{\frac14}^1 G(t,s)\left|g(b_{m_l*}(s))-g(b_{m_j*}(s))\right|\,ds \\
&\le |m_l-m_j|+\int_{\frac14}^1 s\left|g'(z(s))(b_{m_l*}(s)-b_{m_j*}(s))\right|\,ds \\
&\le |m_l-m_j|+\mu\int_{\frac14}^1 s\|b_{m_l*}-b_{m_j*}\|_u\,ds \\
&= |m_l-m_j|+\frac{15\mu\|b_{m_l*}-b_{m_j*}\|_u}{32} \\
&\le \frac{\tau|l-j|}{4(32-\tau)}+\frac{15\mu\|b_{m_l*}-b_{m_j*}\|_u}{32}.
\end{aligned}$$

Therefore

$$\|b_{m_l*}-b_{m_j*}\|_u\le\frac{8\tau|l-j|}{(32-\tau)(32-15\mu)},$$

and

$$\begin{aligned}
|h(l)-h(j)| &= \left|\int_{\frac14}^1 g(b_{m_l*}(s))\,ds-\int_{\frac14}^1 g(b_{m_j*}(s))\,ds\right| \\
&\le \frac{15\mu\|b_{m_l*}-b_{m_j*}\|_u}{32} \\
&\le \frac{15\mu\tau|l-j|}{4(32-\tau)(32-15\mu)}.
\end{aligned}$$

Therefore h is uniformly continuous on $\left[0,\frac{3R}{2}\right]$. This ends the proof. □

In the following Theorem we show how to apply the bisection method to the real valued fixed point problem now that we have that h is continuous.

Theorem 2. *Let $r,R\in\mathbb{R}$ with $0<r<\frac{3R}{8}$, $\tau\in(0,32)$, $\mu\in(0,\frac{32}{15})$, and suppose that*

(A1) *$g:[0,R]\to[0,2R]$ is differentiable;*
(A2) *$\frac{16r}{3}\le g(y)$ for all $y\in[r,R]$;*
(A3) *$|g'(a)|\le\tau<32$ for all $a\in[0,r]$;*
(A4) *$|g'(b)|\le\mu<\frac{32}{15}$ for all $b\in[r,R]$.*

Then there exists a $\theta\in\left[0,\frac{3R}{2}\right]$ such that $h(\theta)=\theta$ for h in (11), and thus

$$y_*(t)=\begin{cases} a_{\theta*}(t) & 0\le t\le\frac14 \\ b_{m_\theta*}(t) & \frac14\le t\le 1\end{cases}$$

is a solution of (1), (2). Moreover, there is a sequence $\{\theta_n\}_{n=0}^{\infty}\subseteq\left[0,\frac{3R}{2}\right]$ such that

$$\theta_n\to\theta$$

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with

$$|\theta - \theta_n| \leq \frac{3R}{2^{n+2}}.$$

Proof. If we let $l \in \left[0, \frac{3R}{2}\right]$, then

$$h(l) = \int_{\frac{1}{4}}^{1} g(b_{m_l*}(s))\, ds \geq \int_{\frac{1}{4}}^{1} \frac{16r}{3}\, ds = 4r \geq 0$$

and

$$h(l) = \int_{\frac{1}{4}}^{1} g(b_{m_l*}(s))\, ds \leq \int_{\frac{1}{4}}^{1} 2R\, ds = \frac{3R}{2}.$$

Hence $h : \left[0, \frac{3R}{2}\right] \to \left[0, \frac{3R}{2}\right]$ is a continuous real valued function. By the intermediate value theorem applied to

$$f(x) = h(x) - x,$$

there exists a $\theta \in \left[0, \frac{3R}{2}\right]$ such that $f(\theta) = 0$, which implies that

$$h(\theta) = \theta$$

and by Lemma 1

$$y_*(t) = \begin{cases} a_{\theta*}(t) & 0 \leq t \leq \frac{1}{4} \\ b_{m_\theta*}(t) & \frac{1}{4} \leq t \leq 1 \end{cases}$$

is a solution of (1), (2). Let

$$c_0 = 0,\ d_0 = \frac{3R}{2} \text{ and } \ \theta_0 = \frac{c_0 + d_0}{2}$$

then recursively define the sequences $\{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ by

$$c_{n+1} = \theta_n, d_{n+1} = d_n \text{ and } \theta_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $h(\theta_n) \geq \theta_n$ and

$$c_{n+1} = c_n, d_{n+1} = \theta_n \text{ and } \theta_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $h(\theta_n) < \theta_n$. Observe that for each natural number n that

$$h(c_n) \geq c_n \ \text{ and } \ h(d_n) \leq d_n$$

thus by the intermediate value theorem there is $\theta \in [c_n, d_n]$ such that $h(\theta) = \theta$. By induction we have that

$$d_n - c_n = \frac{d_{n-1} - c_{n-1}}{2} = \frac{d_0 - c_0}{2^n} = \frac{3R}{2^{n+1}}$$

and since θ_n is the midpoint of the interval $[c_n, d_n]$ and $\theta \in [c_n, d_n]$ we have that

$$|\theta - \theta_n| \leq \frac{3R}{2^{n+2}}.$$

This ends the proof. □

Below we summarize the previous results that under some less restrictive conditions than what is in the literature currently regarding the bounds on the derivative to apply Banachs Theorem, there is an iterative process that converges to a solution of boundary value problem (1), (2).

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Theorem 3. *Let $r, R \in \mathbb{R}$ with $0 < r < \frac{3R}{8}$, $\tau \in (0,32)$, $\mu \in (0, \frac{32}{15})$, and suppose that*

(A1) *$g : [0,R] \to [0,2R]$ is differentiable;*
(A2) *$\frac{16r}{3} \le g(y)$ for all $y \in [r,R]$;*
(A3) *$|g'(a)| \le \tau < 32$ for all $a \in [0,r]$;*
(A4) *$|g'(b)| \le \mu < \frac{32}{15}$ for all $b \in [r,R]$.*

Then there exists an iterative scheme converging to a solution of (1), (2).

Proof. For natural numbers n and p let

$$y_{n,p}(t) = \begin{cases} a_{\theta_n,p}(t) & 0 \le t \le \frac{1}{4} \\ b_{m_{\theta_n,p},p}(t) & \frac{1}{4} \le t \le 1. \end{cases}$$

From the work in Lemma 6 we have

$$\|a_{\theta *} - a_{\theta_n *}\|_v \le \frac{8|\theta - \theta_n|}{32 - \tau}$$

and from the work on Lemma 3 we have

$$\|a_{\theta_n *} - a_{\theta_n,p}\|_v \le \left(\frac{k_a^p}{1-k_a}\right) \|a_{\theta_n,1} - a_{\theta_n,0}\|_v \le \frac{Rk_a^p}{1-k_a}$$

thus we have

$$\begin{aligned} \|a_{\theta *} - a_{\theta_n,p}\|_v &\le \|a_{\theta *} - a_{\theta_n *}\|_v + \|a_{\theta_n *} - a_{\theta_n,p}\|_v \\ &\le \frac{8|\theta - \theta_n|}{32 - \tau} + \frac{Rk_a^p}{1-k_a}. \end{aligned}$$

From the work in Lemma 6 we have

$$\|b_{m_{\theta *}} - b_{m_{\theta_n *}}\|_u \le \frac{8\tau|\theta - \theta_n|}{(32-\tau)(32-15\mu)}$$

and from the work in Lemma 5 we have

$$\|b_{m_{\theta_n *}} - b_{m_{\theta_n,p} *}\|_u \le \frac{\tau R k_a^p}{(32-15\mu)(1-k_a)}$$

and from the work in Lemma 4 we have

$$\|b_{m_{\theta_n,p} *} - b_{m_{\theta_n,p},p}\|_u \le \frac{Rk_b^p}{1-k_b}$$

thus we have

$$\begin{aligned} \|b_{m_{\theta *}} - b_{m_{\theta_n,p},p}\|_u &\le \|b_{m_{\theta *}} - b_{m_{\theta_n *}}\|_u + \|b_{m_{\theta_n *}} - b_{m_{\theta_n,p} *}\|_u + \|b_{m_{\theta_n,p} *} - b_{m_{\theta_n,p},p}\|_u \\ &\le \frac{8\tau|\theta - \theta_n|}{(32-\tau)(32-15\mu)} + \frac{\tau R k_a^p}{(32-15\mu)(1-k_a)} + \frac{Rk_b^p}{1-k_b}. \end{aligned}$$

Therefore

$$\|y_* - y_{n,p}\| \le \max\{\|a_{\theta *} - a_{\theta_n,p}\|_v, \|b_{m_{\theta *}} - b_{m_{\theta_n,p},p}\|_u\}.$$

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For $\varepsilon_n = \frac{1}{n}$ let N_n be a natural number such that

$$\max\left\{\frac{8\tau|\theta-\theta_{N_n}|}{(32-\tau)(32-15\mu)}, \frac{8|\theta-\theta_{N_n}|}{32-\tau}\right\} < \frac{\varepsilon_n}{2}$$

and let P_n be a natural number such that

$$\max\left\{\frac{\tau R k_a^{P_n}}{(32-15\mu)(1-k_a)} + \frac{R k_b^{P_n}}{1-k_b}, \frac{R k_a^{p}}{1-k_a}\right\} < \frac{\varepsilon_n}{2}.$$

For every natural number n define

$$z_n = y_{N_n,P_n}$$

thus

$$\|y_* - z_n\| \le \max\{\|a_{\theta*} - a_{\theta_{N_n,P_n}}\|_v, \|b_{m_{\theta*}} - b_{m_{\theta_{N_n,P_n,P_n}}}\|_u\} < \varepsilon_n$$

so $\{z_n\}$ is a sequence of functions that converges to y_* a solution of (1), (2).
This ends the proof. □

It is not a trivial exercise to provide an approximation of θ where $h(\theta) = \theta$ since for each whole number n to determine c_{n+1}, d_{n+1} and θ_{n+1} we need to determine if $h(\theta_n) \ge \theta_n$ or if $h(\theta_n) < \theta_n$.

For $l \in \left[0, \frac{3R}{2}\right]$ and a natural number p we have

$$m_l = \int_0^{\frac{1}{4}} G\left(\frac{1}{4}, s\right) g(a_{l*}(s))\, ds = \int_0^{\frac{1}{4}} s g(a_{l*}(s))\, ds,$$

$$m_{l,p} = \int_0^{\frac{1}{4}} G\left(\frac{1}{4}, s\right) g(a_{l,p}(s))\, ds = \int_0^{\frac{1}{4}} s g(a_{l,p}(s))\, ds.$$

For $l \in \left[0, \frac{3R}{2}\right]$ the real valued function h is defined by

$$h(l) = \int_{\frac{1}{4}}^{1} g(b_{m_l*}(s))\, ds$$

is approximated by

$$h(l,p) = \int_{\frac{1}{4}}^{1} g(b_{m_{l,p}*}(s))\, ds$$

and since we need to approximate $h(l,p)$ by

$$\int_{\frac{1}{4}}^{1} g(b_{m_{l,p,p}}(s))\, ds$$

we will define a new real valued function by

$$h(l,p,p) = \int_{\frac{1}{4}}^{1} g(b_{m_{l,p,p}}(s))\, ds. \tag{13}$$

The following Lemma is essential for finding the sequence $\{\theta_n\}$.

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Lemma 7. *Let n be a whole number and p be a natural number and suppose that*

$$|h(\theta_n) - h(\theta_n, p, p)| \leq |h(\theta_n, p, p) - \theta_n|$$

then

$$\text{if } h(\theta_n, p, p) \geq \theta_n \text{ then } h(\theta_n) \geq \theta_n$$

and

$$\text{if } h(\theta_n, p, p) \leq \theta_n \text{ then } h(\theta_n) \leq \theta_n.$$

Proof. Either $h(\theta_n, p, p) \geq \theta_n$ or $h(\theta_n, p, p) \leq \theta_n$.
Claim 1: if $h(\theta_n, p, p) \geq \theta_n$ then $h(\theta_n) \geq \theta_n$. Since

$$\theta_n - h(\theta_n, p, p) \leq h(\theta_n) - h(\theta_n, p, p) \leq h(\theta_n, p, p) - \theta_n$$

we have $\theta_n < h(\theta_n)$.
Claim 2: if $h(\theta_n, p, p) < \theta_n$ then $h(\theta_n) < \theta_n$. Since

$$h(\theta_n, p, p) - \theta_n \leq h(\theta_n) - h(\theta_n, p, p) \leq \theta_n - h(\theta_n, p, p)$$

we have $h(\theta_n) \leq \theta_n$.
This ends the proof. □

For every whole number n and every natural number p we have that

$$m_{\theta_n} = \int_0^{\frac{1}{4}} sg(a_{\theta_n *}(s))\, ds \text{ and } m_{\theta_n, p} = \int_0^{\frac{1}{4}} sg(a_{\theta_n, p}(s))\, ds$$

as well as

$$h(\theta_n) = \int_{\frac{1}{4}}^1 g(b_{m_{\theta_n} *}(s))\, ds,\ h(\theta_n, p) = \int_{\frac{1}{4}}^1 g(b_{m_{\theta_n, p} *}(s))\, ds \text{ and } h(\theta_n, p, p) = \int_{\frac{1}{4}}^1 g(b_{m_{\theta_n, p, p}}(s))\, ds.$$

Theorem 4. *Let n be a whole number and p be a natural number then*

$$|h(\theta_n) - h(\theta_n, p, p)| \leq \frac{(64 - 15\mu)\tau R k_a^p}{8(32 - 15\mu)(1 - k_a)} + \frac{4 R k_b^{p+1}}{1 - k_b}.$$

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Proof. From Lemma 4 we have

$$
\begin{aligned}
|h(\theta_n,p)-h(\theta_n,p,p)| &= \left|\int_{\frac{1}{4}}^{1} g(b_{m_{\theta_n,p}*}(s))\,ds-\int_{\frac{1}{4}}^{1} g(b_{m_{\theta_n,p,p}}(s))\,ds\right| \\
&= 4\left|\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p}*}(s))\,ds-\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p,p}}(s))\,ds\right| \\
&= 4\left|m_{\theta_n,p}+\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p}*}(s))\,ds\right. \\
&- \left.\left(m_{\theta_n,p}-\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p,p}}(s))\,ds\right)\right| \\
&= 4\left|b_{m_{\theta_n,p}*}(1/4)-b_{m_{\theta_n,p,p+1}}(1/4)\right| \\
&\leq 4\left\|b_{m_{\theta_n,p}*}-b_{m_{\theta_n,p,p+1}}\right\|_u \\
&\leq \frac{4Rk_b^{p+1}}{1-k_b}
\end{aligned}
$$

and from Lemma 5 we have

$$
\begin{aligned}
|h(\theta_n)-h(\theta_n,p)| &= \left|\int_{\frac{1}{4}}^{1} g(b_{m_{\theta_n}*}(s))\,ds-\int_{\frac{1}{4}}^{1} g(b_{m_{\theta_n,p}*}(s))\,ds\right| \\
&= 4\left|\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n}*}(s))\,ds-\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p}*}(s))\,ds\right| \\
&= 4\left|m_{\theta_n}+\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n}*}(s))\,ds\right. \\
&- \left.\left(m_{\theta_n,p}+\int_{\frac{1}{4}}^{1} G\left(\frac{1}{4},s\right) g(b_{m_{\theta_n,p}*}(s))\,ds\right)-(m_{\theta_n}-m_{\theta_n,p})\right| \\
&= 4\left|b_{m_{\theta_n}*}(1/4)-b_{m_{\theta_n,p}*}(1/4)-(m_{\theta_n}-m_{\theta_n,p})\right| \\
&\leq 4\left\|b_{m_{\theta_n}*}-b_{m_{\theta_n,p}*}\right\|_u+4|m_{\theta_n}-m_{\theta_n,p}| \\
&\leq \frac{4\tau Rk_a^p}{(32-15\mu)(1-k_a)}+\frac{\tau Rk_a^p}{8(1-k_a)}=\frac{(64-15\mu)\tau Rk_a^p}{8(32-15\mu)(1-k_a)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|h(\theta_n)-h(\theta_n,p,p)| &\leq |h(\theta_n)-h(\theta_n,p)|+|h(\theta_n,p)-h(\theta_n,p,p)| \\
&\leq \frac{(64-15\mu)\tau Rk_a^p}{8(32-15\mu)(1-k_a)}+\frac{4Rk_b^{p+1}}{1-k_b}.
\end{aligned}
$$

This ends the proof. □

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Note that for every whole number n we have that

$$\lim_{p\to\infty} |h(\theta_n) - h(\theta_n, p, p)| = 0.$$

Remark 1. *The iterative technique presented in this paper can be applied to the operator corresponding to a right focal boundary value problem when the standard Banach fixed point theorem and the monotone iterative techniques don't apply thus expanding the collection of problems in which iteration can be applied. This technique is not nearly as easy to apply as other iterative techniques. Creating the sequence $\{\theta_n\}$ requires iteration at every stage before one can iterate to approximate an actual solution. There are lots of research opportunities related to this technique, but none greater than a comparison with other techniques and the creation of computer code which can be used to apply the technique.*

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