Solvability of quartic integral equation in Hölder space

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Abstract

In this paper, an existence theorem is proved for the solution of a nonlinear integral equation of Volterra type in Hölder space using the technique of relative compactness combined with fixed point theorem. Our main tool is the Schauder fixed point theorem. We provide a suitable example to show the efficiency of our main result.

Keywords: Relative compactness, Hölder function space, Fixed point theorem (FPT), Volterra Integral equation.

Mathematics Subject Classification 2020: 26B35, 45D05, 47H10.

1. Introduction

The study of existence of solution to nonlinear integral equation (IE) is an essential area as they appeared in different branch of science. With the help of integral equation, we are able to define different physical phenomena, see [3]. The nonlinear integral equations have huge application in science, engineering and other fields. In particular, quadratic integral equation appears in the theory of radiative transfer, neutron transport, kinetic theory of gases, viscoelasticity and so on, for instance see [6, 7, 9, 16, 17] and the references cited therein.

The purpose of this paper is to enquires the existence of solution of quartic integral equation of Volterra type

$$\xi_1(t) = z(t) + \xi_1(t)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu)) + g(\xi_1(\nu))]\xi_1(\nu)d\nu \tag{1}$$

for $t, \nu \in [0,1]$. We will study the Eq. (1) in the space of Hölder functions by using Schauder Fixed Point Theorem (FPT).

A similar type of quadratic IE of Volterra type studied in [22] by Salem in the space of $C[0,1]$. Caballero et al. [4, 5] solved quadratic IE of Fredholm type in Hölder space. It is worth mentioning that up to now the work [19] studied by Özdemir is the only paper of quartic integral equation of mixed Volterra-Fredholm type in Hölder space. Recently many researchers worked on different types of integral equations in function spaces, for references one can see [2, 8, 10, 11, 12, 13, 14, 15, 18, 20, 21, 23] and references therein.

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2. Preliminaries

By $C[a,b]$ and $C_\omega[a,b]$ we denote space of all continuous functions and space of functions with tempered increments (see [1]) on $[a,b]$. For fixed $0<\alpha\le 1$, we write $H_\alpha[a,b]$ to denote real functions space on $[a,b]$ satisfying the Hölder condition, and defined as

$$|\xi_1(t)-\xi_1(s)|\le H_{\xi_1}|t-s|^\alpha,$$

for all $t,s\in[a,b]$. By $H_{\xi_1}>0$ we denote the least possible constant for which the above inequality satisfied. More precisely,

$$H_{\xi_1}=\sup\left\{\frac{|\xi_1(t)-\xi_1(s)|}{|t-s|^\alpha}:t,s\in[a,b],t\neq s\right\}.$$

The space $H_\alpha[a,b]$ can be equipped with the norm

$$\|\xi_1\|_\alpha=|\xi_1(a)|+\sup\left\{\frac{|\xi_1(t)-\xi_1(s)|}{|t-s|^\alpha}:t,s\in[a,b],t\neq s\right\}.$$

Also, the following inequality satisfied [1]

$$\|\xi_1\|_\infty\le\max\{1,(b-a)^\alpha\}\|\xi_1\|_\alpha$$

$$\|\xi_1\|_\alpha\le\max\{1,(b-a)^{\varpi-\alpha}\}\|\xi_1\|_\varpi,$$

where $0<\alpha<\varpi\le 1$.

Now, a sufficient condition on relative compactness is given below (readers can also see [1]).

Theorem 2.1. [4] *A bounded subset U in $H_\varpi[a,b]$, is relatively compact in $H_\alpha[a,b]$, where $0<\alpha<\varpi\le 1$.*

3. Main results

We consider the Eq. (1) in $H_\alpha[0,1],\alpha$ is a fixed number such that $\alpha\in[0,1],0<\alpha<\varpi\le 1$.

We will formulate following assumption to study the Eq. (1).

(i) $z\in H_\alpha[0,1]$. More precisely,

$$|z(t)-z(s)|\le Z_\varpi|t-s|^\varpi$$

for all $t,s\in[0,1]$ and $Z_\varpi>0$.

(ii) $h:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists function $H:\mathbb{R}_+\to\mathbb{R}_+$ with nondecreasing on $\mathbb{R}_+$, also the inequality

$$|h(t,\nu,\xi_1)-h(s,\nu,\xi_2)|\le\gamma|t-s|+H(|\xi_1-\xi_2|)$$

holds for all $t, \nu \in [0,1],\ \xi_1, \xi_2 \in \mathbb{R}$, where $\gamma>0$ is a constant.

(iii) $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that, there exist $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ being nondecreasing on $\mathbb{R}_+$ and $F(t), G(t) \to 0$ as $t \to 0$, also the inequality

$$|f(\xi_1) - f(\xi_2)| \leq F(|\xi_1 - \xi_2|)$$
$$|g(\xi_1) - g(\xi_2)| \leq G(|\xi_1 - \xi_2|)$$

holds for all $\xi_1, \xi_2 \in \mathbb{R}$.

Let us define $\bar{F}, \bar{G}, \bar{H}$ as

$$\bar{F} = \sup\{|f(\xi_1(t)) : t \in [0,1]|\}$$
$$\bar{G} = \sup\{|g(\xi_1(t)) : t \in [0,1]|\}$$
$$\bar{H} = \sup\{|h(t,\nu,0)| : t, \nu \in [0,1]\}.$$

(iv) The following inequality holds:

$$(\bar{F} + \bar{G})[2(H(r) + \bar{H}) + \gamma](|z(0)| + z_\varpi) < \frac{1}{4}.$$

Theorem 3.1. *Assume that hypotheses $(i) - (iv)$ are satisfied. Then there exists at least one solution of the Eq. (1) in the Hölder space $H_\alpha[0,1],\ 0<\alpha<\varpi \leq 1$.*

Proof. Let us consider the operator P on $H_\varpi[0,1]$ for an arbitrary fixed function $\xi_1 \in H_\varpi[0,1]$.

For arbitrary $t, s \in [0,1]$ with $t \neq s$. To show $P\xi_1 \in H_\varpi[0,1]$.

$$
\begin{aligned}
&\frac{|(P\xi_1)(t)-(P\xi_1)(s)|}{|t-s|^{\varpi}}\\
\leq&\frac{|z(t)-z(s)|}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(t)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_1(s)\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu|}{|t-s|^{\varpi}}\\
\leq&\frac{|z(t)-z(s)|}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(t)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_1(s)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu|}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(s)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_1(s)\int_0^t h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu|}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(s)\int_0^t h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-|\xi_1(s)\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu|}{|t-s|^{\varpi}}\\
\leq&\frac{|z(t)-z(s)|}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(t)-\xi_1(s)|\int_0^t|h(t,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(s)|\int_0^t|h(t,\nu,\xi_1(\nu))-h(s,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu}{|t-s|^{\varpi}}\\
&+\frac{|\xi_1(s)|\int_s^t|h(s,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu}{|t-s|^{\varpi}}\\
\leq&Z_{\varpi}+\|\xi_1\|_{\varpi}\int_0^t(H(\|\xi_1\|_{\infty})+\bar{H})(\bar{F}+\bar{G})\,\|\xi_1\|_{\infty}\,d\nu+\|\xi_1\|_{\infty}\,\gamma|t-s|^{1-\varpi}\int_0^t(\bar{F}+\bar{G})\,\|\xi_1\|_{\infty}\,d\nu\\
&+\|\xi_1\|_{\infty}\,|t-s|^{-\varpi}\int_s^t(H(\|\xi_1\|_{\infty})+\bar{H})(\bar{F}+\bar{G})\,\|\xi_1\|_{\infty}\,d\nu\\
\leq&Z_{\varpi}+\|\xi_1\|_{\varpi}^2\,(H(\|\xi_1\|_{\varpi})+\bar{H})(\bar{F}+\bar{G})t+\|\xi_1\|_{\varpi}^2\,\gamma|t-s|^{1-\varpi}(\bar{F}+\bar{G})t\\
&+\|\xi_1\|_{\varpi}^2\,|t-s|^{1-\varpi}(H(\|\xi_1\|_{\varpi})+\bar{H})(\bar{F}+\bar{G})\\
\leq&Z_{\varpi}+2\,\|\xi_1\|_{\varpi}^2\,(H(\|\xi_1\|_{\varpi})+\bar{H})(\bar{F}+\bar{G})+\|\xi_1\|_{\varpi}^2\,\gamma(\bar{F}+\bar{G})\\
\leq&Z_{\varpi}+\|\xi_1\|_{\varpi}^2\,(\bar{F}+\bar{G})\left[2(H(\|\xi_1\|_{\varpi})+\bar{H})+\gamma\right].
\end{aligned}
\tag{2}
$$

Also,

$$|(P\xi_1)(0)| \leq |z(0)|. \tag{3}$$

Combining Eq. (2) and Eq. (3), we obtain

$$\|P\xi_1\|_{\varpi} \leq |z(0)| + Z_{\varpi} + \|\xi_1\|_{\varpi}^2 (\bar{F} + \bar{G}) \left[2(H(\|\xi_1\|_{\varpi}) + \bar{H}) + \gamma\right] < \infty. \tag{4}$$

This shows the operator P transforms $H_{\varpi}[0,1]$ into itself.

The inequality

$$|z(0)| + Z_{\varpi} + r_0^2(\bar{F} + \bar{G}) \left[2(H(r_0) + \bar{H}) + \gamma\right] < r_0.$$

is satisfied for values between the numbers $r_1 = \frac{1-\sqrt{1-4\theta(|z(0)|+Z_{\varpi})}}{2\theta}$ and $r_2 = \frac{1+\sqrt{1-4\theta(|z(0)|+Z_{\varpi})}}{2\theta}$, where $\theta = (\bar{F} + \bar{G}) \left[2(H(r_0) + \bar{H}) + \gamma\right]$.

Clearly, $r_1, r_2 > 0$ by assumption (iv). Also, from Eq.(4), it follows that P transforms the ball $B_{r_0}^{\varpi} = \{\xi_1 \in H_{\varpi}[0,1] : \|\xi_1\|_{\varpi} \leq r_0\}$ into itself, where $r_1 \leq r_0 \leq r_2$. By Theorem 2.1, the set $B_{r_0}^{\varpi}$ is relatively compact in $H_{\alpha}[0,1]$, for any $0 < \alpha < \varpi \leq 1$. Moreover, $B_{r_0}^{\varpi}$ is a compact subset in $H_{\alpha}[0,1]$.

In the sequel, we will prove that P is continuous on the ball $B_{r_0}^{\varpi}$ with respect to the norm$\|.\|_{\alpha}$, where $0 < \alpha < \varpi \leq 1$. Let us fix $\xi_1 \in B_{r_0}$ and $\epsilon > 0$. Suppose that $y \in B_{r_0}$ and $\|\xi_1 - \xi_2\|_{\alpha} < \delta$. Then we obtain

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$$
\begin{aligned}
&\frac{|(P\xi_1)(t)-(P\xi_2)(t)-[(P\xi_1)(s)-(P\xi_2)(s)]|}{|t-s|^{\alpha}}\\
=&\bigg|\xi_1(t)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_2(\nu)d\nu\\
&-\bigg[\xi_1(s)\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_2(\nu)d\nu\bigg]\bigg||t-s|^{-\alpha}\\
=&\bigg|(\xi_1(t)-\xi_2(t))\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu+\xi_2(t)\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu+\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_1(\nu)d\nu+\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_2(\nu)d\nu-\bigg[(\xi_1(s)-\xi_2(s))\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_1(\nu)d\nu-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))]\xi_2(\nu)d\nu\bigg]\bigg||t-s|^{-\alpha}\\
=&\bigg|(\xi_1(t)-\xi_2(t))\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(t)\int_0^t [h(t,\nu,\xi_1(\nu))-h(t,\nu,\xi_1(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))-f(\xi_2(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(t)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\\
&-\bigg[(\xi_1(s)-\xi_2(s))\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^s [h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&\quad+\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))-f(\xi_2(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&\quad+\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\bigg]\bigg||t-s|^{-\alpha}
\end{aligned}
$$

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$$
\begin{aligned}
=\Big|&[\xi_1(t)-\xi_2(t)-(\xi_1(s)-\xi_2(s))]\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+(\xi_1(s)-\xi_2(s))\int_0^t h(t,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-(\xi_1(s)-\xi_2(s))\int_0^t h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+(\xi_1(s)-\xi_2(s))\int_0^t h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+(\xi_2(t)-\xi_2(s))\int_0^t [h(t,\nu,\xi_1(\nu))-h(t,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^t [h(t,\nu,\xi_1(\nu))-h(t,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(s)\int_0^t [h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^t [h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&+(\xi_2(t)-\xi_2(s))\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(s)\int_0^t h(s,\nu,\xi_2(\nu))[f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&+\xi_2(s)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&+(\xi_2(t)-\xi_2(s))\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\\
&+\xi_2(s)\int_0^t h(t,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\\
&-\xi_2(s)\int_0^t h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\\
&+\xi_2(s)\int_0^t h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\\
&-(\xi_1(s)-\xi_2(s))\int_0^s h(s,\nu,\xi_1(\nu))[f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(s)\int_0^s [h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu))][f(\xi_1(\nu))+g(\xi_1(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))]\xi_1(\nu)d\nu\\
&-\xi_2(s)\int_0^s h(s,\nu,\xi_2(\nu))[f(\xi_2(\nu))+g(\xi_2(\nu))](\xi_1(\nu)-\xi_2(\nu))d\nu\Big||t-s|^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{|\xi_1(t)-\xi_2(t)-(\xi_1(s)-\xi_2(s))|}{|t-s|^\alpha}\int_0^t |h(t,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + \frac{|\xi_1(s)-\xi_2(s)|}{|t-s|^\alpha}\int_0^t |h(t,\nu,\xi_1(\nu))-h(s,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + \frac{|\xi_1(s)-\xi_2(s)|}{|t-s|^\alpha}\int_s^t |h(s,\nu,\xi_1(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + \frac{|\xi_2(t)-\xi_2(s)|}{|t-s|^\alpha}\int_0^t |h(t,\nu,\xi_1(\nu))-h(t,\nu,\xi_2(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_0^t |h(t,\nu,\xi_1(\nu))-h(t,\nu,\xi_2(\nu))-(h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu)))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_s^t |h(s,\nu,\xi_1(\nu))-h(s,\nu,\xi_2(\nu))||f(\xi_1(\nu))+g(\xi_1(\nu))||\xi_1(\nu)|d\nu \\
& + \frac{|\xi_2(t)-\xi_2(s)|}{|t-s|^\alpha}\int_0^t |h(t,\nu,\xi_2(\nu))||f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))||\xi_1(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_0^t |h(t,\nu,\xi_2(\nu))-h(s,\nu,\xi_2(\nu))||f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))||\xi_1(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_s^t |h(s,\nu,\xi_2(\nu))||f(\xi_1(\nu))-f(\xi_2(\nu))+g(\xi_1(\nu))-g(\xi_2(\nu))||\xi_1(\nu)|d\nu \\
& + \frac{|\xi_2(t)-\xi_2(s)|}{|t-s|^\alpha}\int_0^t |h(t,\nu,\xi_2(\nu))||f(\xi_2(\nu))+g(\xi_2(\nu))||\xi_1(\nu)-\xi_2(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_0^t |h(t,\nu,\xi_2(\nu))-h(s,\nu,\xi_2(\nu))||f(\xi_2(\nu))+g(\xi_2(\nu))||\xi_1(\nu)-\xi_2(\nu)|d\nu \\
& + |\xi_2(s)||t-s|^{-\alpha}\int_s^t |h(s,\nu,\xi_2(\nu))||f(\xi_2(\nu))+g(\xi_2(\nu))||\xi_1(\nu)-\xi_2(\nu)|d\nu \\
\leq & ||\xi_1-\xi_2||_\alpha(H(||\xi_1||_\infty)+\bar{H})(\bar{F}+\bar{G})||\xi_1||_\infty t + ||\xi_1-\xi_2||_\infty|t-s|^{-\alpha}\gamma|t-s|(\bar{F}+\bar{G})||\xi_1||_\infty t \\
& + ||\xi_1-\xi_2||_\infty|t-s|^{-\alpha}(H(||\xi_1||_\infty)+\bar{H})(\bar{F}+\bar{G})||\xi_1||_\infty|t-s| \\
& + ||\xi_2||_\alpha H(||\xi_1-\xi_2||_\infty)(\bar{F}+\bar{G})||\xi_1||_\infty t + ||\xi_2||_\infty|t-s|^{-\alpha}\times 0\times(\bar{F}+\bar{G})||\xi_1||_\infty t \\
& + ||\xi_2||_\infty|t-s|^{-\alpha}H(||\xi_1-\xi_2||_\infty)(\bar{F}+\bar{G})||\xi_1||_\infty|t-s| \\
& + ||\xi_2||_\alpha(H(||\xi_2||_\infty)+\bar{H})(F(||\xi_1-\xi_2||_\infty)+G(||\xi_1-\xi_2||_\infty))||\xi_1||_\infty t \\
& + ||\xi_2||_\infty|t-s|^{-\alpha}\gamma|t-s|(F(||\xi_1-\xi_2||_\infty)+G(||\xi_1-\xi_2||_\infty))||\xi_1||_\infty t \\
& + ||\xi_2||_\infty|t-s|^{-\alpha}(H(||\xi_2||_\infty)+\bar{H})(F(||\xi_1-\xi_2||_\infty)+G(||\xi_1-\xi_2||_\infty))||\xi_1||_\infty|t-s| \\
& + ||\xi_2||_\alpha(H(||\xi_2||_\infty)+\bar{H})(\bar{F}+\bar{G})||\xi_1-\xi_2||_\infty t + ||\xi_2||_\infty|t-s|^{-\alpha}\gamma|t-s|(\bar{F}+\bar{G})||\xi_1-\xi_2||_\infty t \\
& + ||\xi_2||_\infty|t-s|^{-\alpha}(H(||y||_\infty)+\bar{H})(\bar{F}+\bar{G})||\xi_1-\xi_2||_\infty|t-s|.
\end{aligned}
$$

Since, $||\xi_1||_\infty \leq ||\xi_1||_\alpha \leq ||\xi_1||_\varpi,\ 0<\alpha<\varpi \leq 1,$ therefore replacing

$$||\xi_1||_\infty, ||\xi_1||_\alpha \text{ by } ||\xi_1||_\varpi,$$

$$||\xi_2||_\infty, ||\xi_2||_\alpha \text{ by } ||\xi_2||_\varpi$$

and

$$||\xi_1 - \xi_2||_\infty \text{ by } ||\xi_1 - \xi_2||_\alpha$$

and by putting their appropriate values, we get

$$\begin{aligned}
&\frac{|(P\xi_1)(t) - (P\xi_2)(t) - [(P\xi_1)(s) - (P\xi_2)(s)]|}{|t-s|^\alpha} \\
\leq &\delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) + \delta r_0 \gamma(\bar{F} + \bar{G}) + \delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) + r_0^2 H(\delta)(\bar{F} + \bar{G}) \\
&+ r_0^2 H(\delta)(\bar{F} + \bar{G}) + r_0^2(H(r_0) + \bar{H})(F(\delta) + G(\delta)) + r_0^2\gamma(F(\delta) + G(\delta)) \\
&+ r_0^2(H(r_0) + \bar{H})(F(\delta) + G(\delta)) + \delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) + \delta r_0\gamma(\bar{F} + \bar{G}) \\
&+ \delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) \\
\leq &4\delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) + 2\delta r_0\gamma(\bar{F} + \bar{G}) + 2r_0^2 H(\delta)(\bar{F} + \bar{G}) \\
&+ 2r_0^2(H(r_0) + \bar{H})(F(\delta) + G(\delta)) + r_0^2\gamma(F(\delta) + G(\delta)).
\end{aligned} \tag{5}$$

Now

$$|(P\xi_1)(0) - (P\xi_2)(0)| = 0. \tag{6}$$

Combining Eq.(5) and Eq,(6),

$$\begin{aligned}
||P\xi_1 - P\xi_2|| \leq &4\delta r_0(H(r_0) + \bar{H})(\bar{F} + \bar{G}) + 2\delta r_0\gamma(\bar{F} + \bar{G}) + 2r_0^2 H(\delta)(\bar{F} + \bar{G}) \\
&+ 2r_0^2(H(r_0) + \bar{H})(F(\delta) + G(\delta)) + r_0^2\gamma(F(\delta) + G(\delta)).
\end{aligned}$$

This proves the continuity of the operator P at $\xi_1 \in H_\alpha[0,1]$. since $B_{r_0}^\varpi$ is compact in $H_\alpha[0,1]$, by the classical Schauder fixed point theorem [23], we complete the proof. □

We provide an example which illustrates the efficiency of our results.

Example 3.1. Consider the following nonlinear intergal equation

$$\xi_1(t) = \frac{\tan^{-1}\sqrt{t}}{100} + \xi_1(t)\int_0^t \left(\frac{e^t - \nu^2}{10} + \sqrt[10]{\xi_1^9 + \frac{e^t}{t^2+2}}\right)\left(\sin\xi_1(\nu) + \frac{1}{1+\xi_1^2(\nu)}\right)\xi_1(\nu)d\nu. \tag{7}$$

Comparing Eq. (7) with Eq. (1), we have

$$\begin{aligned} z(t) =& \frac{\tan^{-1}\sqrt{t}}{100} \\ h(t,\nu,\xi_1) =& \frac{e^t-\nu^2}{10} + \sqrt[10]{\xi_1^9 + \frac{e^t}{t^2+2}} \\ f(\xi_1(t)) =& \sin\xi_1(t) \\ g(\xi_1(t)) =& \frac{1}{1+\xi_1^2(t)}. \end{aligned}$$

Now,

$$\begin{aligned} |z(t)-z(s)| =& \frac{1}{100}|\tan^{-1}\sqrt{t} - \tan^{-1}\sqrt{s}| \\ \leq& \frac{1}{100}|\sqrt{t}-\sqrt{s}| \\ \leq& \frac{1}{100}|t-s|^{\frac{1}{2}}. \end{aligned}$$

Hence, $z(t)$ satisfies assumption (i) with $Z_{\frac{1}{2}} = 0.01$.
Also, $|z(0)| = 0$.

$$\begin{aligned} |h(t,\nu,\xi_1) - h(s,\nu,\xi_2)| =& \left| \frac{e^t-\nu^2}{10} + \sqrt[10]{\xi_1^9 + \frac{e^t}{t^2+2}} - \frac{e^s-\nu^2}{10} - \sqrt[10]{\xi_2^9 + \frac{e^s}{s^2+2}} \right| \\ \leq& \left| \frac{e^t-\nu^2}{10} - \frac{e^s-\nu^2}{10} \right| + \left| \left(\xi_1^9 + \frac{e^t}{t^2+2}\right)^{\frac{1}{10}} - \left(\xi_2^9 + \frac{e^t}{t^2+2}\right)^{\frac{1}{10}} \right| \\ &+ \left| \left(\xi_2^9 + \frac{e^t}{t^2+2}\right)^{\frac{1}{10}} - \left(\xi_2^9 + \frac{e^s}{s^2+2}\right)^{\frac{1}{10}} \right| \\ \leq& \frac{1}{10}|e^t - e^s| + |\xi_1-\xi_2|^{\frac{9}{10}} + \left| \left(\frac{e^t}{t^2+2}\right)^{\frac{1}{10}} - \left(\frac{e^s}{s^2+2}\right)^{\frac{1}{10}} \right| \\ \leq& \frac{1}{10}|h_1^{'}(c_1)||t-s| + |\xi_1-\xi_2|^{\frac{9}{10}} + |h_2^{'}(c_2)||t-s| \\ \leq& \left(\frac{1}{10}|h_1^{'}(c_1)| + |h_2^{'}(c_2)|\right)|t-s| + |\xi_1-\xi_2|^{\frac{9}{10}} \\ \leq& 0.7718|t-s| + |\xi_1-\xi_2|^{\frac{9}{10}}. \end{aligned}$$

Hence, $h(t,\nu,\xi_1)$ satisfies assumption (ii) with $\gamma = 0.7718,\ H(r) = r^{\frac{9}{10}}$.

$$|f(\xi_1) - f(\xi_2)| = |\sin\xi_1 - \sin\xi_2| = \left|2\cos\frac{\xi_1+\xi_2}{2}\sin\frac{\xi_1-\xi_2}{2}\right| \leq 2\left|\sin\frac{\xi_1-\xi_2}{2}\right| \leq |\xi_1 - \xi_2|$$

$$|g(\xi_1) - g(\xi_2)| = |\frac{1}{1+\xi_1^2} - \frac{1}{1+\xi_2^2}| = \left|(1+\xi_1^2)^{-1} - (1+\xi_2^2)^{-1}\right| \leq |\xi_1 - \xi_2|.$$

Therefore, f and g satisfies assumption (iii).

$$\begin{aligned}
\bar{F} &= \sup\{|\sin\xi_1(t)| : t \in [0,1]\} = 1 \\
\bar{G} &= \sup\{|\frac{1}{1+\xi_1^2}| : t \in [0,1]\} = 1 \\
\bar{H} &= \sup\{|h(t,\nu,0)| : t,\nu \in [0,1]\} \\
&= \sup\left\{\left|\frac{e^t - \nu^2}{10} + \sqrt[10]{\frac{e^t}{t^2+2}}\right| : t,\nu \in [0,1]\right\} \\
&= 1.3.
\end{aligned}$$

The inequality of assumption (iv) has the form

$$(1+1)[2(H(r)+1.3)+0.7718](0+0.01) < \frac{1}{4}.$$

We can easily check that the above inequality is satisfied when $r = 1$. Therefore, using Theorem 3.1, we assert that Eq. (7) has at least one solution for $r = 1$ in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{5}$.

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