A Friendly Introduction to Appell Polynomials

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1 Prologue

One of the first differentiation formulas beginning calculus students learn is

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x^n = n\, x^{n-1}.$$

A natural question is to ask whether there are any other sequences of polynomial functions $A_n(x)$ having derivative $n\, A_{n-1}(x)$. A little trial and error shows that the sequence of polynomials $p_n(x) = x^n + n\, x^{n-1}$ has this property. In fact, there are many families of polynomials that satisfy this property. Just look at the famous Euler polynomials, the first few of which are

$$E_0(x) = 1,\ E_1(x) = x - 1/2,\ E_2(x) = x^2 - x,\ E_3(x) = x^3 - \frac{3}{2}\, x^2 + \frac{1}{4}.$$

All Euler polynomials can be obtained by the (exponential) generating function

$$2e^{xz}/(e^z + 1) = \sum_{n\geqslant 0} E_n(x) z^n/n!$$

as described below.

One of the more significant sequences was introduced by Paul Appell [1] in 1880, who defined polynomials $P_n(x)$ of degree n that satisfy the relation:

$$\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} = \phi(n)\, P_{n-1}(x),$$

where $\phi(n)$ is a function of a non-negative integer n. Actually, such a sequence $\{P_n(x)\}_{n\geqslant 0}$ is equivalent to the sequence $\{A_n(x)\}_{n\geqslant 0}$ of polynomials satisfying

$$\frac{\mathrm{d}A_n(x)}{\mathrm{d}x} = n\, A_{n-1}(x) \qquad \text{or} \qquad \frac{\mathrm{d}^n A_n(x)}{\mathrm{d}x^n} = n!\, A_0(x). \tag{1.1}$$

This follows by taking

$$A_n(x) = \frac{1 \cdot 2 \cdot 3 \cdots n}{\phi(1) \cdot \phi(2) \cdots \phi(n)}\, P_n(x) = \frac{n!}{\Pi_{i=1}^n \phi(i)}\, P_n(x).$$

Thus, we can assume that $\phi(n) = n$.

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A sequence $\{A_n(x)\}_{n\geqslant 0}$, where each term $A_n(x)$ is a polynomial of degree n, satisfying the differential equation (1.1), is called a sequence of Appell polynomials or simply an Appell sequence. Paul Émile Appell (1855–1930), a French mathematician, astronomer and life-long friend of Henri Poincaré (1854–1912), was a Rector of the University of Paris. He is known for his work in projective geometry, algebraic functions, differential equations, complex analysis, and mechanics. Appell polynomials include many types [5, 13], the most famous of which are Euler and Bernoulli polynomials.

Sequences of Appell polynomials have been well studied because of their remarkable applications in number theory and mathematical analysis. In 1935 Appell polynomials were extended by Sheffer [11, 12] to a class of polynomials now bearing his name. A more general treatment is also given by Luzon and Moron in [9]. Recently, Appell polynomials have been popularized in social media such as Wikipedia and Encyclopedia Of Math (see [15] and [16]), but descriptions tend to be sketchy. In addition to an historical context, our paper presents an introduction to this topic as Appell developed it (including some of his notations), providing a showcase for this remarkable class of polynomials.

2 Terminology

We often refer to a sequence $\{A_n(x)\}_{n\geqslant 0}$ of polynomials simply as A. Integration of the differential equation in (1.1) allows one to restore coefficients of the polynomial $A_n(x)$ from the coefficients of the polynomial of degree $n-1$ up to arbitrary constants. However, this process is cumbersome. Appell's brilliant idea was to seek the sequence of polynomials satisfying the relation (1.1) in the form of the binomial convolution

$$\begin{aligned} A_n(x) &= a_n + \frac{n}{1}\,a_{n-1}\,x + \frac{n(n-1)}{1\cdot 2}\,a_{n-2}\,x^2 + \cdots + \frac{n}{1}\,a_1\,x^{n-1} + a_0\,x^n \\ &= \sum_{k\geqslant 0} \frac{n^{\underline{k}}}{k!}\,a_k\,x^{n-k} = \sum_{k\geqslant 0} \binom{n}{k} a_k\,x^{n-k} \qquad (n = 0, 1, 2, \ldots), \end{aligned} \tag{2.1}$$

where a_0, a_1 , . . . , a_n are arbitrary numbers. Here $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$ is kth falling factorial and $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ is the binomial coefficient, which is assumed to be zero for $k > n$. In general, *binomial convolution* of two sequences $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\} = \{\alpha_n\}_{n\geqslant 0}$ and $\boldsymbol{\beta} = \{\beta_k\}_{k\geqslant 0}$ is a sequence denoted by

$$\boldsymbol{\alpha} \star \boldsymbol{\beta} = \boldsymbol{\beta} \star \boldsymbol{\alpha} \qquad \text{with general term} \qquad (\boldsymbol{\alpha} \star \boldsymbol{\beta})_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k \beta_{n-k} = \sum_{j=0}^{n} \binom{n}{n-j} \alpha_{n-j} \beta_j.$$

Formula (2.1) can then be written concisely as

$$A_n(x) = \big(\{a_i\} \star \{x^j\}\big)_n\,, \tag{2.2}$$

which is the n-th term in the binomial convolution of two sequences, namely the arbitrary sequence of numbers $\{a_i\}$ and the sequence of monomials $\{x^j\}$. It defines Appell polynomials because

$$A_n'(x) = \sum_{k} \binom{n}{k} a_k\,(n-k)\,x^{n-k-1} = n \sum_{k} \binom{n-1}{k} a_k\,x^{n-1-k} = n\,A_{n-1}(x). \tag{2.3}$$

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Thus, for any sequence of numbers $\{a_k\}_{k\geqslant 0}$, there corresponds a unique sequence of Appell polynomials (2.1), and vice versa. In particular, $A_k(0) = a_k$, and by changing the index of summation setting $j = n - k$, we have $A_n(x) = \sum_{j=0}^{n} \binom{n}{n-j} A_{n-j}(0)\, x^j$.

Paul Appell recognized that generating functions would be particularly fruitful in describing his sequences. Recall that an arbitrary sequence of numbers $\{a_k\}_{k\geqslant 0}$ is usually associated with two generating functions:

$$a(z) = \sum_{k\geqslant 0} a_k z^k,$$

called the ordinary generating function, and

$$\hat{a}(z) = \sum_{k\geqslant 0} a_k \frac{z^k}{k!},$$

called the exponential generating function. Binomial convolution is useful in this context: computation shows that if $\hat{b}(z) = \sum_{k\geqslant 0} b_k \frac{z^k}{k!}$, then

$$\hat{a}(z)\, \hat{b}(z) \text{ generates the sequence } \{a_k\} \star \{b_n\}, \tag{2.4}$$

that is, the series product $\hat{a}(z)\, \hat{b}(z)$ generates the binomial convolution of their sequences of coefficients (see [6, 14]). Although convergence plays no role in the generating-function method, which deals with formal sums, when convergence occurs and infinite sums are known, this additional information may be beneficial in actual calculations. It is often useful to have a closed form formula for an infinite sum. An ordinary generating function converges only when the coefficients of the sequence grow no faster than polynomial growth. On the other hand, exponential generating functions can converge for sequences with exponential growth. Therefore, calculus of exponential generating functions is wider in scope than that of ordinary generating functions. That is why we use mainly exponential generating functions below.

Generating functions are defined similarly for more general sequences. The next set of equations presents an exponential generating function for the sequence $\{A_n(x)\}_{n\geqslant 0}$ of Appell polynomials and reduces it to a convenient closed form:

$$\begin{aligned} F_A(x,z) &= \sum_{n\geqslant 0} A_n(x)\, \frac{z^n}{n!} = \sum_{n\geqslant 0} \sum_{k=0}^{n} \binom{n}{k} a_k\, x^{n-k}\, \frac{z^n}{n!} = \sum_{k\geqslant 0} a_k\, x^{-k} \sum_{n\geqslant k} \binom{n}{k} \frac{(xz)^n}{n!} \\ &= \sum_{k\geqslant 0} a_k\, \frac{x^{-k}}{k!} \sum_{n\geqslant k} \frac{(xz)^n}{(n-k)!} = \sum_{k\geqslant 0} a_k\, \frac{x^{-k}}{k!}\, (xz)^k\, e^{xz} = e^{xz}\, \hat{a}(z), \end{aligned} \tag{2.5}$$

where $\hat{a}(z)$ is the exponential generating function for the sequence $\{A_n\}_{n\geqslant 0}$ used to define $A_n(x)$ in Eq. (2.2). Thus the exponential generating function for any sequence of Appell polynomials contains a factor e^{xz} and is in fact characterized by this property. Formula (2.5) shows that Appell polynomials can be defined directly without any reference to Eq. (1.1); just multiply an arbitrary exponential generating function of numbers, $\sum_{k\geqslant 0} a_k \frac{z^k}{k!}$, by e^{xz} and extract coefficients. In what follows, we refer to $F_A(x,z)$ or more briefly F_A as the generating function of Appell polynomials $\{A_n(x)\}_{n\geqslant 0}$ despite that it is an exponential generating function.

The importance of generating functions is based on the correspondence between operations on sequences and their generating functions. In this paper, we are interested in relations between number sequences and corresponding Appell polynomial sequences especially as they reflect operations on generating functions. These sequences of numbers and polynomials have generating functions $\hat{a}(z)$ and $F_A(x,z) = \hat{a}(z)e^{xz}$, respectively. Following Paul Appell, we will often refer to either of these as generating the polynomial sequence and trust that the context will alleviate any confusion.

A knowledge of the generating function $F_A(x,z)$ for a sequence of Appell polynomials allows one to determine the original sequence of numbers $\{a_n\}_{n\geqslant 0}$ from which Appell polynomials were constructed, that is,

$$a_n = n!\,[z^n]\,F_A(x,z)\,e^{-xz},$$

where the notation $[z^n]b(z)$ stands for the nth coefficient from the Maclaurin series expansion of a function $b(z)$: $[z^n]\sum_{k\geqslant 0} b_k z^k = b_n$. To illustrate (2.5), suppose we are given the Appell sequence

$$1, x, x^2, \ldots, x^n, \ldots.$$

Its (exponential) generating function is

$$F_A(x,z) = \sum_{n\geqslant 0} \frac{x^n}{n!}\,z^n = e^{xz}.$$

Therefore, $\hat{a}(z)$ is just a constant series, that is, $\hat{a}(z) = 1$. This function generates the sequence $\{a_n\}_{n\geqslant 0} = \{1, 0, 0, \ldots\}$ of zeroes except for the first term. Likewise, the Appell sequence $\{x^n + nx^{n-1}\}_{n\geqslant 1}$ corresponds to the sequence $\{1, 1, 0, 0, 0, \ldots\}$, generated by the function $\hat{a}(z) = 1 + z$; the function $\hat{a}(z) = 1 + z + 2z^2/2!$ corresponds to the sequence $\{1, 1, 2, 0, 0, \ldots\}$ and generates the Appell sequence $\{x^n + nx^{n-1} + n(n-1)x^{n-2}\}_{n\geqslant 2}$, and so on. Extending this idea, consider the generating function $\hat{a}(z) = \sum_{k\geqslant 0} z^k = 1/(1-z)$ and compare with $\hat{a}(z) = \sum_{k\geqslant 0} a_k \frac{z^k}{k!}$. This yields the number sequence $\{a_k = k!\}_{k\geqslant 0}$ and the corresponding Appell polynomial sequence $\{A_n(x) = \sum_{k\geqslant 0} n^{\underline{k}}\, x^{n-k}\}_{n\geqslant 0}$.

As another example, the Appell sequence $\{(x+r)^n\}_{n\geqslant 0}$ corresponds to the sequence $\{r^n\}_{n\geqslant 0}$, generated by the function $\hat{a}(z) = e^{rz}$ (because $\hat{a}(z)e^{xz} = e^{(x+r)z}$). In this connection, we have the sequence $\{F_n\}_{n\geqslant 0}$ of Fibonacci numbers that satisfy the recurrence $F_{n+2} = F_{n+1} + F_n$ subject to the initial conditions $F_0 = 0$, $F_1 = 1$. Its (exponential) generating function is known to be [6]

$$\hat{a}(z) = \sum_{n\geqslant 1} \frac{z^n}{n!}\,F_n = \frac{1}{\sqrt{5}}\left[e^{\phi z} - e^{-z/\phi}\right],$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Multiplying the preceding by e^{xz}, we see that the Appell polynomials for the sequence of Fibonacci numbers are

$$A_n(x) = \frac{1}{\sqrt{5}}\left[(x+\phi)^n - \left(x - \phi^{-1}\right)^n\right], \qquad n = 0, 1, 2, \ldots.$$

3 Algebraic Operations

How are Appell polynomial sequences multiplied? For given generating functions $\hat{a}(z) = \sum_{n\geqslant 0} a_n \frac{z^n}{n!}$ and $\hat{b}(z) = \sum_{k\geqslant 0} b_k \frac{z^k}{k!}$ of sequences A and B, respectively, the answer is found with the product $\hat{a}(z)\,\hat{b}(z)$, or more specifically, the function $F_{AB}(x,z) = \hat{a}(z)\,\hat{b}(z)\,e^{xz}$. To this end, Paul Appell replaced the term x^{n-k} in $A_n(x) = \sum_{k\geqslant 0}\binom{n}{k} a_k\, x^{n-k}$ by the polynomial $B_{n-k}(x)$ for each k. The resulting polynomials are written $(AB)_n(x)$ or $A_n(B(x))$. This new product sequence of polynomials,

$$(A\,B)_0\,, (A\,B)_1\,, \ldots, (A\,B)_n\,, \ldots,$$

which is denoted by AB, has the derivative property expressed by (1.1):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} A_n(B(x)) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k\geqslant 0} \binom{n}{k} a_k\, B_{n-k}(x) = \sum_{k\geqslant 0} \binom{n}{k} a_k \frac{\mathrm{d}}{\mathrm{d}x} B_{n-k}(x) \\
&= \sum_{k\geqslant 0} \binom{n}{k} a_k\, (n-k)\, B_{n-k-1}(x) = \sum_{k\geqslant 0} \frac{n!}{k!\,(n-k-1)!} a_k\, B_{n-k-1}(x) \\
&= n \sum_{k\geqslant 0} \frac{(n-1)!}{k!\,(n-k-1)!} a_k\, B_{n-k-1}(x) = n \sum_{k\geqslant 0} \binom{n-1}{k} a_k\, B_{n-k-1}(x) \\
&= n\, A_{n-1}(B(x)).
\end{aligned}$$

Since the Appell product is given by

$$(AB)_n(x) = \sum_{k\geqslant 0} \binom{n}{k} a_k\, B_{n-k}(x),$$

the generating function F_{AB} for the product sequence is obtained by multiplying the series

$$F_B(x,z) = B_0 + \frac{z}{1} B_1(x) + \frac{z^2}{1\cdot 2} B_2(x) + \cdots + \frac{z^n}{1\cdot 2\cdots n} B_n(x) + \cdots\,,$$

by $\hat{a}(z)$; thus

$$F_{AB} = \hat{a}(z)\, F_B(x,z).$$

Replacing $F_B(x,z)$ by $\hat{b}(z)e^{xz}$, we have

$$F_{AB} = \hat{a}(z)\hat{b}(z)\, e^{xz}. \tag{3.1}$$

Thus the product of two Appell polynomial sequences is obtained from the product of their number-sequence generating functions.

Switching the roles of A and B, we see that $F_{AB} = F_{BA}$ because the polynomial sequences AB and BA have the same generating function $\hat{a}(z)\,\hat{b}(z)\,e^{xz}$. Let us display the Appell polynomials generated by Eq. (3.1):

$$AB = \{A_n(0)\} \star \{B_k(x)\} \quad\Longrightarrow\quad (AB)_n\,(x) = \sum_{k=0}^{n} \binom{n}{n-k} A_k(0)\, B_{n-k}(x), \tag{3.2}$$

(where $A_j(0) = a_j$). Of course,

$$(A\,B)_n = (B\,A)_n = (\{A_i(0)\} \star \{B_k(x)\})_n = (\{A_i(x)\} \star \{B_k(0)\})_n\,.$$

If you take A and B to be identical, you get $(AA)_n$ or $(A^2)_n$ that has generating function F_{A^2}, after replacing x^k by A_k in the polynomials A.

To illustrate, let us find the Appell polynomials that correspond to the product $\hat{a}^2(z) = (1-z)^2$. The function $\hat{a}(z) = 1 - z$ generates the sequences

$$\{a_n\}_{n\geqslant 0} = \{1, -1, 0, 0, 0, \dots\} \text{ and } \{A_n(x) = x^n - nx^{n-1}\}. \tag{3.3}$$

Applying (3.2) we have, for $n \geqslant 1$

$$\begin{aligned}(A^2)_n(x) = (\{A_i(x)\} \star \{a_k\})_n &= \sum_{k=0}^{n} \binom{n}{n-k} A_{n-k}(x)\, a_k \\ &= \binom{n}{n} A_n(x) - \binom{n}{n-1} A_{n-1}(x) = x^n - 2n\, x^{n-1} + n(n-1)\, x^{n-2}.\end{aligned} \tag{3.4}$$

These polynomials can also be obtained by expanding the generating function $\hat{a}^2(z)e^{xz} = (1-z)^2 e^{xz}$ into a Maclaurin series with respect to z and extracting coefficients of z^n. The reader may wish to work with $\hat{a}^3(z) = (1-z)^3$.

The famous Euler and Bernoulli polynomial sequences are generated by $2\,e^{xz}/(e^z+1)$ and $z\,e^{xz}/(e^z-1)$ respectively. Computation with higher powers of $2/(e^z+1)$ and $z/(e^z-1)$ has led to the study of generalized Euler and Bernoulli polynomials [6, 8].

Having defined multiplication of polynomials based on their generating functions, Paul Appell considered operations analogous to division. Given polynomial sequences A and B, he wanted to find a polynomial sequence C such that

$$(A\,C)_n = B_n, \qquad n = 1, 2, \dots.$$

The generating function of these polynomials $C_n(x)$ will be the quotient of generating functions B by A and is designated $F_{\frac{B}{A}}$, so we have

$$(A\,C)_n = \left(A\,\frac{B}{A}\right)_n = B_n.$$

Finding polynomials generated by $\frac{B}{A}$ amounts to finding the inverse generating function $\frac{1}{A}$ of A.

4 Appell Polynomial Inverses

Surprisingly, one of the most useful ways of finding Appell polynomials $A_n(x)$ with generating function $\hat{a}(z)$ is through the polynomials $B_n(x)$ generated by the inverse function $\hat{b}(z) = 1/\hat{a}(z)$. In this case we will refer to A and B as *polynomial inverses* and write $B = \frac{1}{A}$ or A^{-1}. In some practical applications, the original Appell approach that assigns the Appell polynomial (2.1) to a given sequence of numbers $\{a_k\}$ is not productive. It may happen that we know a sequence of Appell polynomials $\{A_n(x)\}$, but do not have an explicit formula for the corresponding sequence of numbers a_k. This is true of some of the most famous examples of Appell polynomials including the Bernoulli and Euler polynomials, for which it is hard to determine the generating sequence of numbers according to (2.1). Fortunately, in these cases the sequence $\{b_n\}$ for the inverse generating function $\hat{b}(z)$ is known, from which polynomials $A_n(x)$ and the corresponding number sequence can be determined, as illustrated at the end of this section.

Generally, if $a_0 \neq 0$ then the inverse of the generating function $\hat{a}(z) = \sum_{k\geqslant 0} a_k \frac{z^k}{k!}$ is a function $\hat{b}(z)$ such that $\hat{a}(z)\hat{b}(z) = 1$. Thus

$$\hat{b}(z) = \frac{1}{\hat{a}(z)} = \sum_{k\geqslant 0} b_k \frac{z^k}{k!}, \qquad \text{where } b_0 = \frac{1}{a_0}.$$

The equation

$$\hat{a}(z)\,\hat{b}(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}\frac{z^n}{n!} = 1$$

yields

$$\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k} = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n > 0. \end{cases}$$

The values b_n can be solved for recursively:

$$\begin{cases} b_0 = a_0^{-1}, & n = 0, \\ b_n = -a_0^{-1}\sum_{k=1}^{n}\binom{n}{k} a_k b_{n-k}, & n = 1, 2, \ldots. \end{cases} \tag{4.1}$$

Similarly, the sequence $\{a_n\}$ can be expressed in terms of the values b_k.

Recalling the fact that the constant function $\hat{f}(z) = 1$ generates the trivial sequence of Appell polynomials $\{x^n\}$, the convolution equation $\hat{a}(z)\,\hat{b}(z) = 1$, when translated to polynomials, becomes

$$(A\,B)_n = x^n \qquad \text{or} \qquad \sum_{k=0}^{n}\binom{n}{k} A_{n-k}(0)\,B_k(x) = x^n, \quad n = 0, 1, 2, \ldots. \tag{4.2}$$

In simple cases finding inverses is straight-forward. For example, consider the constant sequence of numbers $a_n = 1$ that corresponds to the sequence of Appell polynomials

$$A_n(x) = \sum_{k\geqslant 0}\binom{n}{k} x^{n-k} = x^n\left(1 + x^{-1}\right)^n = (1 + x)^n.$$

The generating function is $F_A(x, z) = e^{(1+x)z}$. Since $\hat{a}(z) = e^z$, its inverse is $\hat{b}(z) = e^{-z}$ and the generating function for the inverse polynomials $B_k(x)$ is $F_B = e^{-z}e^{xz} = e^{z(x-1)}$. Thus, $B_k(x) = (x-1)^k$, $k = 0, 1, 2, \ldots$. The corresponding number sequence becomes $\{b_k = (-1)^k\}_{k\geqslant 0}$.

A more complicated situation presents itself with the example of polynomials $\{A_n(x)\}$ having generating function $\hat{a}(z) = (1 - z)$ mentioned earlier (see (3.3):

$$A_n(x) = x^n - n\,x^{n-1}.$$

To find polynomial inverses for A, we replace x^i by $B_i(x)$ in $A_n(x)$ and seek polynomials B such that

$$(A\,B)_n = B_n - n\,B_{n-1} = x^n, \qquad B_0 = 1, \qquad n = 1, 2, \ldots. \tag{4.3}$$

Solving this recurrence relation of the first order, we obtain

$$B_n(x) = 1 \cdot 2 \cdots n \left(1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \cdots + \frac{x^n}{1 \cdot 2 \cdots n}\right) = n!\,e_n(x), \tag{4.4}$$

where $e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is the incomplete exponential function. Formula (4.4) can be easily verified by substituting B_n back into (4.3). The corresponding generating function for the sequence of Appell polynomials $\{B_n(x) = n!\,e_n(x)\}_{n\geqslant 0}$ is $(1 - z)^{-1}e^{xz}$.

Some examples may require solving the full-history recurrence (4.1). Alternatively, one can express the Appell polynomials $A_k(x)$ in terms of the sequence $\{b_n\}$ that define the inverse polynomials $B_n(x)$ as follows. The polynomials $B_n(x)$ are generated by

$$F_B = \hat{b}(z)e^{xz} = \frac{1}{\hat{a}(z)}\,e^{xz} = \sum_{n\geqslant 0} B_n(x)\,\frac{z^n}{n!},$$

where $B_n(x) = (\{b_k\} \star \{x^i\})_n$. Thus,

$$e^{xz} = \hat{a}(z)\hat{b}(z)\,e^{xz} \implies \sum_{n\geqslant 0} \frac{x^n z^n}{n!} = \sum_{n\geqslant 0} A_n(x)\,\frac{z^n}{n!} \sum_{k\geqslant 0} \frac{z^k}{k!}\,b_k.$$

On equating coefficients of z^n, we are led to the following convolution equation

$$\{b_k\} \star \{A_n(x)\} = \{x^i\}, \qquad b_k = B_k(0), \tag{4.5}$$

which yields the infinite system of equations:

$$\begin{cases} A_0(x)\,b_0 = 1, & n = 0, \\ A_0(x)\,b_1 + A_1(x)\,b_0 = x, & n = 1, \\ A_0(x)\,b_2 + \binom{2}{1}\,A_1(x)\,b_1 + A_2(x)\,b_0 = x^2, & n = 2, \\ \cdots & \\ A_0(x)\,b_n + \binom{n}{1}\,A_1(x)\,b_{n-1} + \binom{n}{2}A_2(x)\,b_{n-2} + \cdots + A_n(x)\,b_0 = x^n, & \vdots \\ \vdots \qquad . & \end{cases} \tag{4.6}$$

This is a linear system which can be truncated at any finite step resulting in a system with a triangular matrix of order $n + 1$. So it can be solved recursively, which allows one to determine the Appell polynomials $\{A_n(x)\}$ through the sequence $\{b_k\}$ and vise versa.

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To illustrate, consider the sequence of Euler numbers generated by $\hat{a}(z) = 2\left(1+e^z\right)^{-1}$, for which finding the sequence $\{a_n\}$ is a challenging problem. But the inverse of $\hat{a}(z)$ is $\hat{b}(z) = (1+e^z)/2 = 1 + {}^1\!/_2 \sum_{k\geqslant 1} z^k/k!$ which gives us the values b_k to put into (4.6). Substituting $b_0 = 1$, $b_k = 1/2$ for $k = 1, 2 \cdots n$, we obtain

$$A_n(x) = E_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ 1 & {}^1\!/_2 & {}^1\!/_2 & \cdots & \cdots & {}^1\!/_2 & {}^1\!/_2 \\ 0 & 1 & \binom{2}{1}{}^1\!/_2 & \cdots & \cdots & \binom{n-1}{1}{}^1\!/_2 & \binom{n}{1}{}^1\!/_2 \\ 0 & 0 & 1 & \cdots & \cdots & \binom{n-1}{2}{}^1\!/_2 & \binom{n}{2}{}^1\!/_2 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \binom{n}{n-1}{}^1\!/_2 \end{vmatrix}.$$

Expanding the determinant with respect to the first row will produce the representation of $E_n(x)$ where every coefficient is a determinant of a square $n \times n$ matrix. The values $n = 1, 2, 3$ yield the Euler polynomials mentioned in the Prologue.

The constant coefficients give the elusive sequence of numbers a_n that define the Euler polynomials as an Appell sequence. They are

$$a_n = E_n(0) = -\frac{(-1)^{n-1}}{2} \begin{vmatrix} 2 & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 2 & 2 & 3 & 4 & 5 & \cdots & \cdots & n-1 & n \\ 0 & 2 & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \cdots & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ 0 & 0 & 2 & \binom{4}{3} & \binom{5}{3} & \cdots & \cdots & \binom{n-1}{3} & \binom{n}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \binom{n-1}{n-2} & \binom{n}{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & \binom{n}{n-1} \end{vmatrix}.$$

For larger systems, a computer algebra system can be useful. It is also possible to use Cramer's Rule to solve (4.6) and produce a formula for the numbers a_n as a full history recurrence.(see [3, 6], for example).

The reader may want to experiment finding the first few Bernoulli polynomials. The numbers a_n, generated by $z/(e^z - 1)$, defining the Bernoulli polynomials according to Eq. (2.2) do not have a closed form formula. Express the inverse of this generating function as a Maclaurin series to find values b_k.

5 Appell-Derivatives

Since Appell polynomials are motivated by a natural property of derivatives, it is appropriate that we end our presentation with a look at Appell's exploits with differentiation. Not surprisingly, he wanted to find the polynomials generated by the derivative, $\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}$ of a given function $\hat{a}(z)$. The resulting sequence is denoted by $\{(dA)_n\}$. Starting with

$$\hat{a}(z)\, e^{xz} = A_0 + \frac{z}{1}\, A_1(x) + \frac{z^2}{1\cdot 2}\, A_2(x) + \cdots + \frac{z^n}{1\cdot 2\cdots n}\, A_n(x) + \cdots\,, \tag{5.1}$$

differentiate both sides with respect to z while suppressing x, to obtain

$$\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}\, e^{xz} + x\,\hat{a}(z)\, e^{xz} = A_1 + \frac{z}{1}\, A_2 + \cdots + \frac{z^{n-1}}{1\cdot 2\cdots (n-1)}\, A_n + \cdots\,.$$

Now replace $\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}\, e^{xz}$ in the preceding by its expansion

$$\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}\, e^{xz} = (\mathrm{d}A)_0 + \frac{z}{1}\,(\mathrm{d}A)_1 + \frac{z^2}{1\cdot 2}\,(\mathrm{d}A)_2 + \cdots + \frac{z^n}{1\cdot 2\cdots n}\,(\mathrm{d}A)_n + \cdots\,, \tag{5.2}$$

and equate the coefficients of z^n in both members of the resulting equation. This produces the relation

$$(\mathrm{d}A)_n = A_{n+1}(x) - xA_n(x), \qquad (\mathrm{d}A)_0 = a_1, \tag{5.3}$$

which gives a polynomial of degree n in x generated by $\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}$. It is not hard to verify that $\{(dA)_n\}$ is an Appell sequence.

As an illustration, let us return to an example in the preceding section. Relabeling B_n, let $A_n(x) = n!e_n(x)$ be the polynomials in (4.4) generated by $\hat{a}(z) = (1-z)^{-1}$. The derivative of this function with respect to z is $(1-z)^{-2}$; (5.3) leads us to the Appell polynomials

$$(dA)_n = (n+1)!\, e_{n+1}(x) - x\, n!\, e_n(x).$$

Note that these are the inverse polynomials for those in (3.4), generated by $(1-z)^2$.

Concluding Remarks

Our presentation has drawn from the first five of twenty sections comprising the paper [1] which enables a flavor for Appell polynomials. As mentioned in the prologue, every Appell sequence is a Sheffer sequence and is referred to as a generalized Appell sequence (see the web site [17]). A more general discussion of Sheffer polynomials can be found in [10].

Variations of Appell's approach have also been considered. For example, in 2007, Biazar and Shafiof [2] introduced a new algorithm in the calculation of Adomian polynomials; it is based on the formula $\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} = (n+1)\, P_{n+1}(x)$. Di Bucchianico and Loeb [5] summarize and document more than five hundred references related to Appell polynomial sequences.

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