On Matrix Polynomial in Two Variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$

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Abstract

In this paper, we introduce a matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$ and obtain some properties including the Kampé de Fériet matrix series representation, generating matrix functions, integral representations, finite summation formulas and recurrence relations. Some special cases have also been obtained.

Keywords: Matrix polynomial, Kampé de Fériet matrix series, Generating matrix function, Generalized hypergeometric matrix function.

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1 Introduction

Higher transcendental functions and classical polynomials of matrices are important topic in the literature of matrix analysis. A large piece of mathematics and its applications (both theoretical and practical) has been cut across the subject of orthogonal polynomials. The property of orthogonality, generating matrix functions, Rodrigues formula, a relation between different orthogonal matrix polynomials, matrix differential equation, a three-term matrix recurrence relation holds the theoretical examples, while, statistics, group representation theory, scattering theory, differential equations, Fourier series expansions, interpolation, quadrature and splines, embrace the practical ones (see [1, 2, 4, 6, 7, 9, 20, 21]). Orthogonal Latest innovations in matrix versions for the classical families of orthogonal polynomials such as extended Laguerre, Jacobi, Bessel, Hermite, Gegenbauer, Chebyshev polynomials and some other special functions have been introduced by many authors (see [5, 11, 12, 13, 17, 19]).

In 1994, Jódar et al. [11] introduced the Laguerre matrix polynomials $L_n^{(M,\lambda)}(x)$ and defined as,

$$L_n^{(M,\lambda)}(x) = \sum_{p=0}^{n} \frac{(-1)^p \lambda^p}{p!(n-p)!}(M+I)_n[(M+I)_p]^{-1}x^p. \tag{1.1}$$

where λ is a complex number with $\Re(\lambda) > 0$ and M is a matrix in $\mathbb{C}^{r\times r}$ with $M + nI$ invertible for all $n \geq 1$.

Jatav and Shukla [10] gave the generalization of the Laguerre matrix polynomials $L_n^{(M,\lambda)}(x)$ due to Jódar et al. [11], which is defined as,

$$L_n^{(M,\delta,\lambda)}(x)=\frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^p}{p!}\Gamma^{-1}(M+(\delta p+1)I). \tag{1.2}$$

where $\delta\in\mathbb{Z}^+$, $\lambda\in\mathbb{C}$, $\Re(\lambda)>0$ and $\Re(\rho)>-1$, $\forall\rho\in\sigma(M)$, and $\sigma(M)$ denotes the set of all eigenvalues of $M\in\mathbb{C}^{r\times r}$.

In this paper, we define a matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$ which unifies the Laguerre matrix polynomials $L_n^{(M,\lambda)}(x)$ [11] and the matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$ [10].

The present work is systematized as: In section 2, we discuss the basic definitions and results which are needed in the sequel to the study. In section 3, we introduce a matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$ and represent it in form of the Kampé de Fériet matrix series. In section 4, we establish various interesting generating matrix relations. In section 5, we obtain integral representations. Section 6 deals with summation formulas and recurrence relations. The conclusion and future scope have been discussed in section 7.

2 Preliminaries

Let $\mathbb{C}^{r\times r}$ denote the vector space of all r-square matrices with complex entries. For $M\in\mathbb{C}^{r\times r}$, its spectrum $\sigma(M)$ denotes the set of all eigenvalues of M and $\alpha(M)=\max[\Re(\gamma):\gamma\in\sigma(M)]$, $\beta(M)=\min[\Re(\gamma):\gamma\in\sigma(M)]$. Any square matrix $M\in\mathbb{C}^{r\times r}$ is a positive stable, if $\Re(\gamma)>0$ for all $\gamma\in\sigma(M)$. A matrix norm is a vector norm on $\mathbb{C}^{r\times r}$. If $||M||$ denotes the norm of the matrix M, then the operator norm corresponding to the 2-norm for vectors is

$$||M||=\sup_{x\neq 0}\frac{||Mx||_2}{||x||_2}=\max\{\sqrt{\lambda}:\lambda\in\sigma(M^*M)\},$$

where for any vector x in r^{th} complex plane, $||x||_2=(x^*x)^{1/2}$ is the Euclidean norm of x and M^* denotes the transposed conjugate of M. The identity matrix and zero matrix in $\mathbb{C}^{r\times r}$ are symbolized by I and $\mathbb{O}$ respectively. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z and defined in an open set Ω in complex plane, then from the properties of the matrix functional calculus [8], it follows $f(M)g(N)=g(N)f(M)$, where M,N are commuting matrices in $\mathbb{C}^{r\times r}$ with $\sigma(M)\subset\Omega$ and $\sigma(N)\subset\Omega$ such that $MN=NM$.
The reciprocal gamma function denoted by $\Gamma^{-1}(z)=\frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. For $M\in\mathbb{C}^{r\times r}$, image of $\Gamma^{-1}(z)$ acting on M and denoted by $\Gamma^{-1}(M)$ which is a well defined matrix.

If $M\in\mathbb{C}^{r\times r}$ and $M+nI$ is invertible matrix for all non-negative integer n, then the matrix version of the Pochhammer symbol $(M)_n$ is defined (Jódar and Cortés [13]) as,

$$(M)_n=\Gamma(M+nI)\Gamma^{-1}(M)=\begin{cases} M(M+I)(M+2I)...(M+(n-1)I) & (n\in\mathbb{N})\\ I & (n=0).\end{cases} \tag{2.1}$$

For any matrix $M \in \mathbb{C}^{r\times r}$, one can get the relation due to Jódar and Cortés [13],

$$(1-z)^{-M} = \sum_{n\geq 0} \frac{(M)_n}{n!} z^n, \quad |z|<1. \tag{2.2}$$

If M and N are commuting matrices in $\mathbb{C}^{r\times r}$ and $M+nI$, $N+nI$, $M+N+nI$ are invertible for all non-negative integer n, then (Jódar and Cortés [14]),

$$B(M,N) = \Gamma(M)\Gamma(N)\Gamma^{-1}(M+N). \tag{2.3}$$

Jódar and Cortés [14] defined the Beta matrix function as,

$$B(M,N) = \int_0^1 x^{M-I}(1-x)^{N-I}dx, \tag{2.4}$$

where M, N are positive stable matrices in $\mathbb{C}^{r\times r}$.
Jódar and Cortés [14] defined the Gamma matrix function as,

$$\Gamma(M) = \int_0^\infty e^{-x}x^{M-I}dx, \quad x^{M-I} = \exp\left((M-I)\ln x\right). \tag{2.5}$$

where M is a positive stable matrix in $\mathbb{C}^{r\times r}$.

Dwivedi and Sahai [3] discussed the generalized hypergeometric matrix function as,

$${}_pF_q\left[\begin{matrix}A_1, A_2, ..., A_p \\ B_1, B_2, ..., B_q\end{matrix} \;;\; z\right] = \sum_{n\geq 0} \frac{(A_1)_n(A_2)_n...(A_p)_n[(B_1)_n]^{-1}[(B_2)_n]^{-1}...[(B_q)_n]^{-1}z^n}{n!}, \tag{2.6}$$

where $A_i, B_j \in \mathbb{C}^{r\times r}$, $1\leq i\leq p$, $1\leq j\leq q$, $p,q\in\mathbb{N}$ and B_j+kI are invertible for all integers $k\geq 0$. It is easy to prove the convergence of (2.6) for $|z|<1$.

Dwivedi and Sahai [3] also defined the Kampé de Fériet matrix series as,

$$\begin{aligned}&F^{s_1,r_1,t_1}_{s_2,r_2,t_2}\left[\begin{matrix}A, B, C \\ D, E, F\end{matrix} \;;\; x, y\right] \\ &= \sum_{m,n\geq 0} \prod_{j=1}^{s_1}(A_j)_{m+n} \prod_{j=1}^{r_1}(B_j)_m \prod_{j=1}^{t_1}(C_j)_n \prod_{j=1}^{s_2}[(D_j)_{m+n}]^{-1} \prod_{j=1}^{r_2}[(E_j)_m]^{-1} \prod_{j=1}^{t_2}[(F_j)_n]^{-1} \frac{x^m y^n}{m!n!},\end{aligned} \tag{2.7}$$

where A_j, B_j, C_j, D_j, E_j and F_j be commuting matrices in $\mathbb{C}^{r\times r}$ and D_j+kI, E_j+kI and F_j+kI are invertible for all integers $k\geq 0$.

To study such a matrix polynomial, we need the following Lemma due to Khan and Shukla [16].

Lemma 2.1. *If $M(k,j,n)\in\mathbb{C}^{r\times r}$, then the following series relations are satisfied:*

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} M(k,j,n) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n-j} M(k,j,n-j-k)$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n-j} M(k,j,n) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} M(k,j,n+j+k).$$

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3 The matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$

Definition 3.1. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition*

$$\Re(\rho) > -1,\ \Re(\gamma) > -1,\quad \forall \rho \in \sigma(M),\ \forall \gamma \in \sigma(N) \tag{3.1}$$

and $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then the matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$ defines as:

$$\begin{aligned} L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2} \\ &\times \sum_{q=0}^{n}\sum_{p=0}^{n-q} \frac{(-nI)_{p+q}\lambda^p\eta^q x^p y^q}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I). \end{aligned} \tag{3.2}$$

Remark 3.1. *On taking $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (3.2), this reduces to matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$ (Jatav and Shukla [10]),*

$$L_n^{(M,\mathbb{O},\delta,1,\lambda,1)}(x,0) = \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^p}{p!}\Gamma^{-1}(M+(\delta p+1)I) = L_n^{(M,\delta,\lambda)}(x). \tag{3.3}$$

Remark 3.2. *On taking $N = \mathbb{O}$, $y = 0$, $\eta = 1$, $\delta = 1$ and $\xi = 1$ in (3.2), this reduces to Laguerre matrix polynomial $L_n^{(M,\lambda)}(x)$ (Jódar et al. [11]),*

$$L_n^{(M,\mathbb{O},1,1,\lambda,1)}(x,0) = \sum_{p=0}^{n}\frac{(-1)^p\lambda^p}{p!(n-p)!}(M+I)_n[(M+I)_p]^{-1}x^p = L_n^{(M,\lambda)}(x). \tag{3.4}$$

3.1 The Kampé de Fériet matrix series representation

The matrix polynomial in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)$ can be represented in the form of the Kampé de Fériet matrix series as,

$$L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) = \frac{(M+I)_{\delta n}(N+I)_{\xi n}}{(n!)^2}\, F^{1,0,0}_{0,\delta,\xi}\left[\begin{matrix} -nI \\ \Delta(\delta; M+I), \Delta(\xi; N+I) \end{matrix}\ ;\ \frac{\lambda x}{\delta^\delta}, \frac{\eta y}{\xi^\xi}\right], \tag{3.5}$$

where $F^{1,0,0}_{0,\delta,\xi}(\cdot)$ is called the Kampé de Fériet matrix series and $\Delta(\xi; N+I)$ is the array of ξ parameters:

$$\left(\frac{N+I}{\xi}\right), \left(\frac{N+2I}{\xi}\right), \left(\frac{N+3I}{\xi}\right), \ldots, \left(\frac{N+\xi I}{\xi}\right).$$

Remark 3.3. *If we take $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (3.5), then this yields the corresponding result due to Jatav and Shukla [10],*

$$L_n^{(M,\mathbb{O},\delta,1,\lambda,1)}(x,0) = \frac{(M+I)_{\delta n}}{n!}\, {}_1F_\delta\left[\begin{matrix} -nI \\ \Delta(\delta; M+I) \end{matrix}\ ;\ \frac{\lambda x}{\delta^\delta}\right] = L_n^{(M,\delta,\lambda)}(x). \tag{3.6}$$

Remark 3.4. *If we take $N = \mathbb{O}$, $y = 0$, $\eta = 1$, $\delta = 1$ and $\xi = 1$ in (3.5), this yields the corresponding result due to Shehata [18],*

$$L_n^{(M,\mathbb{O},1,1,\lambda,1)}(x,0) = \frac{(M+I)_n}{n!}\, {}_1F_1\left[\begin{matrix} -nI \\ M+I \end{matrix} ;\ \lambda x\right] = L_n^{(M,\lambda)}(x). \tag{3.7}$$

4 Generating matrix functions

In this section, we establish new generating matrix functions of the matrix polynomial (3.2).

Theorem 4.1. *Let $M, N, P \in \mathbb{C}^{r\times r}$ be commuting matrices such that M, N satisfy the spectral condition (3.1) and P is positive stable matrix, $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$ and $|\omega| < 1$, $|\frac{-\lambda\omega x}{1-\omega}| < 1$, $|\frac{-\eta\omega y}{1-\omega}| < 1$. Then the following generating matrix relation holds:*

$$\begin{aligned}&\sum_{n=0}^{\infty}(P)_n\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\ &= \Gamma^{-1}(M+I)\Gamma^{-1}(N+I)e^{\omega}\, F^{1,0,0}_{0,\delta,\xi}\left[\begin{matrix} P \\ \Delta(\delta; M+I), \Delta(\xi; N+I)\end{matrix} ;\ \frac{-\lambda\omega x}{1-\omega}, \frac{-\eta\omega y}{1-\omega}\right].\end{aligned} \tag{4.1}$$

Proof. Consider the left hand side of (4.1) and employing (3.2), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}(P)_n\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\ &= \sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-n)_{p+q}\lambda^p\eta^q x^p y^q}{p!q!n!}(P)_n\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)\omega^n,\end{aligned}$$

on applying the Lemma 2.1, this gives

$$\begin{aligned}&\sum_{n=0}^{\infty}(P)_n\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\ &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}\lambda^p\eta^q x^p y^q}{p!q!n!}(P)_{n+p+q}\Gamma^{-1}(M+(\delta p+1)I)\\ &\times \Gamma^{-1}(N+(\xi q+1)I)\omega^{n+p+q}\\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}\lambda^p\eta^q x^p y^q(P)_{p+q}}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)\\ &\times \sum_{n=0}^{\infty}\frac{\omega^n}{n!}(P+(p+q))_n\\ &= \Gamma^{-1}(M+I)\Gamma^{-1}(N+I)(1-\omega)^{-P}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\left(\frac{-\lambda\omega x}{1-\omega}\right)^p\left(\frac{-\eta\omega y}{1-\omega}\right)^q(P)_{p+q}}{p!q!}\\ &\times \prod_{m=1}^{\delta}\left(\frac{M+mI}{\delta}\right)_p^{-1}\prod_{r=1}^{\xi}\left(\frac{N+rI}{\xi}\right)_q^{-1},\end{aligned}$$

on using (2.7), this establishes the result (4.1). ∎

Theorem 4.2. *Let $M, N, P \in \mathbb{C}^{r\times r}$ be commuting matrices such that M, N satisfy the spectral condition (3.1) and P is positive stable matrix, $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$ and $|\omega| < 1$, $|\frac{-\lambda\omega x}{1-\omega}| < 1$, $|\frac{-\eta\omega y}{1-\omega}| < 1$. Then the following generating matrix relation holds:*

$$\begin{aligned}&\sum_{n=0}^{\infty}(P)_n\left[(M+I)_{\delta n}\right]^{-1}\left[(N+I)_{\xi n}\right]^{-1}L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\&= e^{\omega}\,F^{1,0,0}_{0,\delta,\xi}\left[\begin{matrix}P\\ \Delta(\delta;M+I),\Delta(\xi;N+I)\end{matrix}\;;\;\frac{-\lambda\omega x}{1-\omega},\frac{-\eta\omega y}{1-\omega}\right].\end{aligned}\tag{4.2}$$

Proof. Proof of Theorem 4.2 is similar as given in Theorem 4.1. □

Remark 4.1. *If we set $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (4.2), then this yields the corresponding result due to Jatav and Shukla [10],*

$$\sum_{n=0}^{\infty}(P)_n\left[(M+I)_{\delta n}\right]^{-1}L_n^{(M,\delta,\lambda)}(x)\omega^n = (1-\omega)^{-N}\,{}_1F_{\delta}\left[\begin{matrix}N\\ \Delta(\delta;M+I)\end{matrix}\;;\;\frac{-\omega\lambda x}{(1-\omega)\delta^{\delta}}\right].\tag{4.3}$$

Remark 4.2. *If we set $N = \mathbb{O}$, $y = 0$, $\eta = 1$, $\delta = 1$ and $\xi = 1$ in (4.2), then this yields the corresponding result due to Jatav and Shukla [10],*

$$\sum_{n=0}^{\infty}(P)_n\left[(M+I)_{n}\right]^{-1}L_n^{(M,\lambda)}(x)\omega^n = (1-\omega)^{-N}\,{}_1F_1\left[\begin{matrix}N\\ M+I\end{matrix}\;;\;\frac{-\omega\lambda x}{(1-\omega)}\right].\tag{4.4}$$

Theorem 4.3. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition (3.1), $\lambda, \eta \in \mathbb{C}$, $\delta, \xi \in \mathbb{Z}^+$, $\Re(\lambda) > 0$, $\Re(\eta) > 0$ and $|\frac{-\lambda\omega x}{\delta^{\delta}}| < 1$, $|\frac{-\eta\omega y}{\xi^{\xi}}| < 1$. Then the following generating matrix relation holds:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\&=\Gamma^{-1}(M+I)\Gamma^{-1}(N+I)\,e^{\omega}\,{}_0F_{\delta}\left[\begin{matrix}-\\ \Delta(\delta;M+I)\end{matrix}\;;\;-\frac{\lambda\omega x}{\delta^{\delta}}\right]{}_0F_{\xi}\left[\begin{matrix}-\\ \Delta(\xi;N+I)\end{matrix}\;;\;-\frac{\eta\omega y}{\xi^{\xi}}\right].\end{aligned}\tag{4.5}$$

Proof. From left hand side of (4.5) and employing (3.2), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!\\&=\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}\lambda^p\eta^q x^p y^q}{p!q!n!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)\,\omega^n,\end{aligned}$$

on using the Lemma 2.1, we obtain

$$\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!$$

$$= \Gamma^{-1}(M+I)\Gamma^{-1}(N+I)\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{p+q}\lambda^p\eta^q x^p y^q}{p!q!n!}\left[(M+I)_{\delta p}\right]^{-1}\left[(N+I)_{\xi q}\right]^{-1}\omega^{n+p+q}$$

$$= \Gamma^{-1}(M+I)\Gamma^{-1}(N+I)\, e^{\omega}\sum_{p=0}^{\infty}\frac{(-\lambda\omega x)^p}{p!\delta^{\delta p}}\prod_{m=1}^{\delta}\left(\frac{M+mI}{\delta}\right)_p^{-1}\sum_{q=0}^{\infty}\frac{(-\eta\omega y)^q}{q!\xi^{\xi q}}\prod_{r=1}^{\xi}\left(\frac{N+rI}{\xi}\right)_q^{-1},$$

on applying (2.7), this completes the proof of Theorem 4.3. □

Theorem 4.4. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition (3.1), $\lambda, \eta \in \mathbb{C}$, $\delta, \xi \in \mathbb{Z}^+$, $\Re(\lambda) > 0$, $\Re(\eta) > 0$ and $|\frac{-\lambda\omega x}{\delta^\delta}| < 1$, $|\frac{-\eta\omega y}{\xi^\xi}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty}\left[(M+I)_{\delta n}\right]^{-1}\left[(N+I)_{\xi n}\right]^{-1} L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\,\omega^n n!$$
$$= e^{\omega}\,{}_0F_{\delta}\left[\begin{matrix}-\\ \Delta(\delta; M+I)\end{matrix}\;;\; -\frac{\lambda\omega x}{\delta^\delta}\right]{}_0F_{\xi}\left[\begin{matrix}-\\ \Delta(\xi; N+I)\end{matrix}\;;\; -\frac{\eta\omega y}{\xi^\xi}\right]. \tag{4.6}$$

Proof. Proof of Theorem 4.4 is similar as Theorem 4.3. □

Remark 4.3. *If we take $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (4.6), this yields the corresponding result due to Jatav and Shukla [10],*

$$\sum_{n=0}^{\infty}\left[(M+I)_{\delta n}\right]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n = e^{\omega}\,{}_0F_{\delta}\left[\begin{matrix}-\\ \Delta(\delta; M+I)\end{matrix}\;;\; \frac{-\omega\lambda x}{\delta^\delta}\right]. \tag{4.7}$$

Remark 4.4. *If we take $N = \mathbb{O}$, $y = 0$, $\eta = 1$, $\delta = 1$ and $\xi = 1$ in (4.6), then this yields the corresponding result due to Jódar and Sastre [15],*

$$\sum_{n=0}^{\infty}\left[(M+I)_{n}\right]^{-1} L_n^{(M,\lambda)}(x)\omega^n = e^{\omega}\,{}_0F_{1}\left[\begin{matrix}-\\ M+I\end{matrix}\;;\; -\omega\lambda x\right]. \tag{4.8}$$

Theorem 4.5. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition (3.1), $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$ and $|\frac{-(\omega+\sigma)\lambda x}{\delta^\delta}| < 1$, $|\frac{-(\omega+\sigma)\eta y}{\xi^\xi}| < 1$. Then the following double series generating matrix relation holds:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{[(m+n)!]^2}{n!m!}\Gamma^{-1}(M+(\delta(n+m)+1)I)\Gamma^{-1}(N+(\xi(n+m)+1)I)$$
$$\times L_{n+m}^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\omega^m\sigma^n$$
$$= \Gamma^{-1}(M+I)\Gamma^{-1}(N+I)\, e^{\sigma+\omega}\,{}_0F_{\delta}\left[\begin{matrix}-\\ \Delta(\delta; M+I)\end{matrix}\;;\; -\frac{(\omega+\sigma)\lambda x}{\delta^\delta}\right]$$
$$\times\,{}_0F_{\xi}\left[\begin{matrix}-\\ \Delta(\xi; N+I)\end{matrix}\;;\; -\frac{(\omega+\sigma)\eta y}{\xi^\xi}\right]. \tag{4.9}$$

Proof. On denoting the left hand side of (4.9) by Ł$_1$ and using the Lemma 2.1, we get

$$Ł_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(n!)^2}{m!(n-m)!}\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)$$

$$\times L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)\omega^m\sigma^{n-m}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\Gamma^{-1}(M+(\delta n+1)I)\Gamma^{-1}(N+(\xi n+1)I)L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)(\sigma+\omega)^n n!.$$

Now using the Theorem 4.3, this immediately leads to the proof. □

5 Integral representations

In this section, we establish new integral representations of the matrix polynomial (3.2).

Theorem 5.1. *Let M, M', N and N' be commuting matrices in $\mathbb{C}^{r\times r}$ such that $M, N, M+M', N+N'$ satisfy the spectral condition (3.1) and M', N' are positive stable matrices, $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then*

$$\int_0^r\int_0^s x^M(s-x)^{M'-1}y^N(t-y)^{N'-1}L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x^\delta,y^\xi)dxdy$$
$$= s^{M+M'}t^{N+N'}\Gamma(M')\Gamma(N')\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)$$
$$\times\Gamma^{-1}(M+M'+(\delta n+1)I)\Gamma^{-1}(N+N'+(\xi n+1)I)\,L_n^{(M+M',N+N',\delta,\xi,\lambda,\eta)}(s^\delta,t^\xi). \tag{5.1}$$

Proof. On denoting the left hand side of (5.1) by $Ł_2$,

$$Ł_2 = \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}$$
$$\times\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}\lambda^p\eta^q}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)$$
$$\times\left(\int_0^s x^{M+\delta pI}(s-x)^{M'-I}dx\right)\left(\int_0^t y^{N+\delta qI}(t-y)^{M'-I}dy\right),$$

on making substitution $s-x = s(1-\sigma)$ and $t-y = t(1-\omega)$ in the above equation, we can easily arrive at the desired result (5.1). □

Further, one can also express the result (5.1) in the Kampé de Fériet matrix series form as:

$$\int_0^r\int_0^s x^M(s-x)^{M'-1}y^N(t-y)^{N'-1}L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x^\delta,y^\xi)dxdy$$
$$= s^{M+M'}t^{N+N'}\Gamma(M')\Gamma(N')\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)\Gamma^{-1}(M+M'+I)$$
$$\times\Gamma^{-1}(N+N'+I)\,F^{1,0,0}_{0,\delta,\xi}\left[\begin{matrix}-nI\\ \Delta(\delta;M+M'+I),\Delta(\xi;N+N'+I)\end{matrix};\ \frac{\lambda s^\delta}{\delta^\delta},\frac{\eta t^\xi}{\xi^\xi}\right]. \tag{5.2}$$

Remark 5.1. *On setting $N = N' = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (5.1), this yields the corresponding result due to Jatav and Shukla [10],*

$$\int_0^s (s-x)^{M'-I}x^M L_n^{(M,\delta,\lambda)}(x^\delta)dx = s^{M+M'}\Gamma(M')\Gamma(M+I+\delta nI)$$
$$\times\Gamma^{-1}(M+M'+I+\delta nI)L_n^{(M+M',\delta,\lambda)}(s^\delta). \tag{5.3}$$

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Remark 5.2. *On setting $N = N' = \mathbb{O}$, $y = 0$, $\eta = 1$, $\xi = 1$ and $\delta = 1$ in (5.1), this yields the corresponding result due to Jatav and Shukla [10],*

$$\int_0^s (s-x)^{M'-I} x^M L_n^{(M,\lambda)}(x)dx = s^{M+M'}\Gamma(M')\Gamma(M+(n+1)I) \times \Gamma^{-1}(M+M'+(1+n)I)L_n^{(M+M',\lambda)}(s). \tag{5.4}$$

Theorem 5.2. *Let M and N be commuting matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (3.1) and $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then*

$$\begin{aligned}&\int_0^\infty \int_0^\infty e^{-x-y} x^M y^N \, L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)dxdy \\ &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2} \\ &\times F^{1,1,1}_{0,\delta,\xi}\left[\begin{matrix}-nI, M+I, N+I \\ \Delta(\delta; M+I), \Delta(\xi; N+I)\end{matrix} \; ; \; \frac{\lambda}{\delta^\delta}, \frac{\eta}{\xi^\xi}\right].\end{aligned} \tag{5.5}$$

Proof. On denoting the left hand side of (5.5) by $Ł_3$,

$$\begin{aligned}Ł_3 &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2} \\ &\times \sum_{q=0}^{n}\sum_{p=0}^{n-q} \frac{(-nI)_{p+q}\lambda^p\eta^q}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I) \\ &\times \left(\int_0^\infty e^{-x}x^{M+pI}dx\right)\left(\int_0^\infty e^{-y}y^{N+qI}dy\right) \\ &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2} \\ &\times \sum_{q=0}^{n}\sum_{p=0}^{n-q} \frac{(-nI)_{p+q}\lambda^p\eta^q}{p!q!}(M+I)_p(N+I)_q \prod_{m=1}^{\delta}\left(\frac{M+mI}{\delta}\right)_p^{-1} \prod_{k=1}^{\xi}\left(\frac{N+kI}{\xi}\right)_q^{-1},\end{aligned}$$

which establishes the result (5.5). ∎

Remark 5.3. *On putting $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (5.5), then this yields the corresponding result due to Jatav and Shukla [10],*

$$\int_0^\infty x^M e^{-x} L_n^{(M,\delta,\lambda)}(x)dx = \frac{\Gamma(M+(\delta n+1)I)}{n!} \, {}_2F_\delta\left[\begin{matrix}-nI, M+I \\ \Delta(\delta; M+I)\end{matrix} \; ; \; \frac{\lambda}{\delta^\delta}\right]. \tag{5.6}$$

Theorem 5.3. *Let M and N be commuting matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (3.1) and $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then*

$$\begin{aligned}&\int_0^\infty \int_0^\infty e^{-x-y} x^M y^N \, L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x^\delta, y^\xi)dxdy \\ &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}(1-\lambda-\eta)^n.\end{aligned} \tag{5.7}$$

Proof. On denoting the left hand side of (5.7) by Ł$_4$,

$$
\begin{aligned}
Ł_4 &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2} \\
&\times \sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}\lambda^p\eta^q}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I) \\
&\times \left(\int_0^\infty e^{-x}x^{M+\delta pI}dx\right)\left(\int_0^\infty e^{-y}y^{N+\xi qI}dy\right) \\
&= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}\lambda^p\eta^q}{p!q!},
\end{aligned}
$$

this immediately establishes the result (5.7). □

Corollary 5.1. *Let M be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (3.1) and $\delta \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Then*

$$\int_0^\infty e^{-x}x^M\, L_n^{(M,\delta,\lambda)}(x^\delta)dx = \frac{\Gamma(M+(\delta n+1)I)}{n!}(1-\lambda)^n. \tag{5.8}$$

Corollary 5.2. *Let M be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (3.1) and $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Then*

$$\int_0^\infty e^{-x}x^M\, L_n^{(M,\lambda)}(x)dx = \frac{\Gamma(M+(n+1)I)}{n!}(1-\lambda)^n. \tag{5.9}$$

6 Summation formulas and recurrence relations

In this section, we establish finite summation formulas and differential recurrence relations of the matrix polynomial (3.2).

Theorem 6.1. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition (3.1) and $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then the following summation formulas hold:*

$$
\begin{aligned}
\sum_{u=0}^{k}\binom{k}{u}x^u\left[(M+I-kI)_u\right]^{-1}D_x^uL_n^{(M,N,\delta,\xi,\lambda,\eta)}(x^\delta,y) &= (M+I)_{\delta n}\left[(M+I-kI)_{\delta n}\right]^{-1} \\
&\times L_n^{(M-kI,N,\delta,\xi,\lambda,\eta)}(x,y)
\end{aligned} \tag{6.1}
$$

$$
\begin{aligned}
\sum_{u=0}^{k}\binom{k}{u}y^u\left[(N+I-kI)_u\right]^{-1}D_y^uL_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y^\xi) &= (N+I)_{\xi n}\left[(N+I-kI)_{\xi n}\right]^{-1} \\
&\times L_n^{(M,N-kI,\delta,\xi,\lambda,\eta)}(x,y)
\end{aligned} \tag{6.2}
$$

$$\sum_{u=0}^{n}\frac{(\sigma)^u}{u!}D_x^uL_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) = L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x+\sigma,y) \tag{6.3}$$

$$\sum_{u=0}^{n}\frac{(\sigma)^u}{u!}D_y^uL_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) = L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y+\sigma) \tag{6.4}$$

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$$\sum_{u=0}^{n}\frac{(-x)^u}{u!}D_x^u L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)=\frac{\Gamma(M+(\delta n+1)I)}{n!}L_n^{(N,\xi,\eta)}(y) \quad (6.5)$$

$$\sum_{u=0}^{n}\frac{(-y)^u}{u!}D_y^u L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)=\frac{\Gamma(N+(\xi n+1)I)}{n!}L_n^{(M,\delta,\lambda)}(x) \quad (6.6)$$

$$\sum_{t=0}^{n}\sum_{u=0}^{n}\frac{(\omega)^t(\sigma)^u}{u!t!}D_x^u D_y^t L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)=L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x+\sigma,y+\omega) \quad (6.7)$$

$$\sum_{t=0}^{n}\sum_{u=0}^{n}\frac{(\omega)^t(-x)^u}{u!}D_x^u D_y^t L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)=\frac{\Gamma(M+(\delta n+1)I)}{n!}L_n^{(N,\xi,\eta)}(y+\omega) \quad (6.8)$$

$$\sum_{t=0}^{n}\sum_{u=0}^{n}\frac{(-y)^t(\sigma)^u}{u!}D_x^u D_y^t L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y)=\frac{\Gamma(N+(\xi n+1)I)}{n!}L_n^{(M,\delta,\lambda)}(x+\sigma). \quad (6.9)$$

Proof. On denoting the left hand side of (6.1) by Ł$_5$,

$$\begin{aligned}
Ł_5 &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}\\
&\times \sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}x^{\delta p}y^q\lambda^p\eta^q}{p!q!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)\\
&\times \sum_{u=0}^{k}\frac{(-kI)_u(\delta p)_u}{u!}\left[(M+I-kI)_u\right]^{-1}\\
&= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)\Gamma(M+I-kI)\Gamma^{-1}(M+I)}{(n!)^2}\\
&\times \sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}x^{\delta p}y^q\lambda^p\eta^q}{p!q!}\Gamma^{-1}(M+I-kI+\delta pI)\Gamma^{-1}(N+(\xi q+1)I),
\end{aligned}$$

which yields the result (6.1). Similarly, we can establish the result (6.2).

On denoting the left hand side of (6.3) by Ł$_6$,

$$\begin{aligned}
Ł_6 &= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}x^p y^q\lambda^p\eta^q}{p!q!}\\
&\times \Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I)\sum_{u=0}^{n}\frac{\left(\frac{-\sigma}{x}\right)^u(-p)_u}{u!}\\
&= \frac{\Gamma(M+(\delta n+1)I)\Gamma(N+(\xi n+1)I)}{(n!)^2}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\frac{(-nI)_{p+q}(x+\sigma)^p y^q\lambda^p\eta^q}{p!q!}\\
&\times \Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I),
\end{aligned}$$

on applying (3.2), this leads to form of the result (6.3). Similarly, one can easily prove the desire result (6.4). On putting $\sigma=-x$ in (6.3) , we get the result (6.5) and on putting

$\sigma = -y$ in (6.4), we arrive at (6.6).

Indicating the left hand side of (6.7) by Ł$_7$,

$$
\begin{aligned}
Ł_7 = & \frac{\Gamma(M+\delta n+1)\Gamma(N+\xi n+1)}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} x^p y^q \lambda^p \eta^q}{p!q!} \\
& \times \Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I) \left(\sum_{t=0}^{n} \sum_{u=0}^{n} \frac{(-p)_u (-q)_t \left(\frac{-\sigma}{x}\right)^u \left(\frac{-\omega}{y}\right)^t}{u!t!} \right) \\
= & \frac{\Gamma(M+\delta n+1)\Gamma(N+\xi n+1)}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} (x+\sigma)^p (y+\omega)^q \lambda^p \eta^q}{p!q!} \\
& \times \Gamma^{-1}(M+(\delta p+1)I)\Gamma^{-1}(N+(\xi q+1)I),
\end{aligned}
$$

use of (3.2) establishes the result (6.7). On setting $\sigma = -x$ in (6.7), this yields (6.8) and on putting $\omega = -y$ in (6.7), we can easily obtain (6.9). □

Remark 6.1. *If we set $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (6.1) and (6.3), then we get the corresponding results due to Jatav and Shukla [10],*

$$
\begin{aligned}
\sum_{u=0}^{k} \binom{k}{u} x^u \left[(M+I-kI)_u\right]^{-1} D_x^u L_n^{(M,\delta,\lambda)}(x^\delta) = & (M+I)_{\delta n} \left[(M+I-kI)_{\delta n}\right]^{-1} \\
& \times L_n^{(M-kI,\delta,\lambda)}(x)
\end{aligned}
\tag{6.10}
$$

and

$$
\sum_{u=0}^{n} \frac{(\sigma)^u}{u!} D_x^u L_n^{(M,\delta,\lambda)}(x) = L_n^{(M,\delta,\lambda)}(x+\sigma). \tag{6.11}
$$

Theorem 6.2. *Let $M, N \in \mathbb{C}^{r\times r}$ be commuting matrices satisfying the spectral condition (3.1) and $\delta, \xi \in \mathbb{Z}^+$, $\lambda, \eta \in \mathbb{C}$ with $\Re(\lambda) > 0$, $\Re(\eta) > 0$. Then the following differential recurrence relations hold:*

$$
ML_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) + x\delta D_x L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) = (M+\delta nI)L_n^{(M-I,N,\delta,\xi,\lambda,\eta)}(x,y) \tag{6.12}
$$

$$
NL_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) + y\xi D_y L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x,y) = (N+\xi nI)L_n^{(M,N-I,\delta,\xi,\lambda,\eta)}(x,y). \tag{6.13}
$$

Proof. Consider the right hand side of (6.12),

$$
\begin{aligned}
& (M+\delta nI)L_n^{(M-I,N,\delta,\xi,\lambda,\eta)}(x,y) \\
& = \frac{(M+\delta nI)(M)_{\delta n}(N+I)_{\xi n}}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} x^p y^q \lambda^p \eta^q}{p!q!} \left[(M)_{\delta p}\right]^{-1} \left[(N+I)_{\xi q}\right]^{-1} \\
& = \frac{(M+I)_{\delta n}(N+I)_{\xi n}}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} x^p y^q \lambda^p \eta^q}{p!q!} (M+\delta pI)(M+I)_{\delta p}^{-1}(N+I)_{\xi q}^{-1}
\end{aligned}
$$

$$= \frac{M(M+I)_{\delta n}(N+I)_{\xi n}}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} x^p y^q \lambda^p \eta^q}{p!q!} (M+I)_{\delta p}^{-1} (N+I)_{\xi q}^{-1}$$

$$+ \frac{x\delta(M+I)_{\delta n}(N+I)_{\xi n}}{(n!)^2} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} p x^{p-1} y^q \lambda^p \eta^q}{p!q!} (M+I)_{\delta p}^{-1} (N+I)_{\xi q}^{-1},$$

this immediately leads to form of the result (6.12). Similarly, we can prove the result (6.13). □

Remark 6.2. *If we take $N = \mathbb{O}$, $y = 0$, $\eta = 1$ and $\xi = 1$ in (6.12), then this yields the corresponding result due to Jatav and Shukla [10],*

$$ML_n^{(M,\delta,\lambda)}(x) + x\delta \frac{d}{dx} L_n^{(M,\delta,\lambda)}(x) = (M + \delta n I) L_n^{(M-I,\delta,\lambda)}(x). \tag{6.14}$$

7 Concluding Remarks

In this paper, we introduced the matrix polynomials in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x, y)$ which unifies a number of matrix polynomials and discussed various properties including the Kampé de Fériet matrix series representation, generating matrix functions, integral representations, finite summation formulas and recurrence relations with their several interesting special cases have been obtained. These results can play a significant role in the Wireless Communications and Signal Processing, Combinatorial Problems, Theory of Special Functions, Operator Theory, Matrix Analysis Theory, Mathematical Physics, Fractional Calculus and Statistics.

One can find orthogonality and q-analogues of the matrix polynomials in two variables $L_n^{(M,N,\delta,\xi,\lambda,\eta)}(x, y)$.

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